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OPTIMAL CLOSING OF A PAIR TRADE WITH A MODEL CONTAINING JUMPS

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Abstract. A pair trade is a portfolio consisting of a long position in one asset and a short position in another, and it is a widely used investment strategy in the financial industry. Recently, Ekström, Lindberg, and Tysk studied the problem of optimally closing a pair trading strategy when the difference of the two assets is modelled by an Ornstein-Uhlenbeck process. In the present work the model is generalized to also include jumps. More precisely, we assume that the difference between the assets is an Ornstein-Uhlenbeck type process, driven by a Lévy process of finite activity. We prove a necessary condition for optimality (a so-called verification theorem), which takes the form of a free boundary problem for an integro-differential equation. We analyze a finite element method for this problem and prove rigorous error estimates, which are used to draw conclusions from numerical simulations. In particular, we present strong evidence for the existence and uniqueness of an optimal solution.

Keywords: pairs trading, optimal stopping, Ornstein-Uhlenbeck type process, finite element method, error estimate

MSC 2010: 91G10, 65N30, 45J05

1. Introduction

A portfolio which consists of a positive position in one asset and a negative position in another is called a pair trade. Pairs trading was developed at Morgan Stanley in the late 1980's, and today it is one of the most common investment strategies in the financial industry. The idea behind pairs trading is quite intuitive: the investor

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finds two assets, for which the prices have moved together historically. When the price spread widens, the investor takes a short position in the outperforming asset (borrows it and sells it), and a long position in the underperforming one (buys it) with the hope that the spread will converge again, which would generate a profit. A main advantage of pairs trading is that the combination of a short and long position can, in principle, eliminate the exposure to the general direction of the market, the so-called market risk. For a historical account of pairs trading we refer to [6].

To model the pair spread the authors in [3] proposed a mean reverting Gaussian Markov chain, which they considered to be observed in Gaussian noise. Recently, the authors in [2] suggested the continuous time analogue, the so called mean reverting Ornstein-Uhlenbeck process. In this paper we generalize the model of the spread to also include possible jumps. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space where the following processes are defined in such a way that they are independent:

- ▷ a standard Brownian motion $W = \{W_t\}_{t\geqslant 0}$;
- ▷ a Poisson process $N^\lambda = \{N_t^\lambda\}_{t\geqslant 0}$ with intensity $\lambda > 0$;
- ▷ a sequence of independent random variables $\{X_k^\varphi\}_{k=1}^\infty$ with common continuous symmetric density φ. Moreover, the support of φ is contained in the interval $(-J, J)$ for some $J > 0$.

Define the compound Poisson process $C^{\lambda,\varphi} = \{C_t^{\lambda,\varphi}\}_{t\geqslant 0}$ in the usual way as

$$C_t^{\lambda,\varphi} = \sum_{k=1}^{N_t^\lambda} X_k^\varphi$$

and denote the filtration generated by W by $C^{\lambda,\varphi}$, and the null sets of $\mathcal{F}$ by $\mathbb{F} = \{\mathcal{F}_t\}_{t\geqslant 0}$. It is well known that this filtration satisfies the usual hypotheses (see, for example, [10]). From now on, when we say that a process is a martingale, submartingale, or supermartingale we mean that this is with respect to $\mathbb{F}$.

Let the difference $U = \{U_t\}_{t\geqslant 0}$ between the assets be the unique solution of the stochastic differential equation

$$\mathrm{d}U_t = -\mu U_t \,\mathrm{d}t + \sigma \,\mathrm{d}W_t + \,\mathrm{d}C_t^{\lambda,\varphi}, \quad t > 0, \tag{1.1}$$

where $\mu > 0$, $\sigma > 0$. (The solution of equation (1.1) is usually called a generalized Ornstein-Uhlenbeck process or an Ornstein-Uhlenbeck type process.) Sometimes we will denote the driving Lévy process in (1.1) by $Z^{\sigma,\lambda,\varphi}$, that is,

$$Z_t^{\sigma,\lambda,\varphi} = \sigma W_t + C_t^{\lambda,\varphi}, \quad t \geqslant 0.$$

As discussed in [2], there is a large risk associated with a pair trading strategy. Indeed, if the market spread ceases to be mean reverting, the investor is exposed to substantial risk. Therefore, in practice the investor typically chooses in advance a stop-loss level $a < 0$, which corresponds to the level of loss above which the investor will close the pair trade. For a given stop-loss level $a < 0$ define

$$\tau_a = \inf\{t \geqslant 0\colon\ U_t \leqslant a\},$$

the first hitting time of the region $(-\infty, a]$, and the so called optimal value function

$$V(x) = \sup_{\tau} \mathbf{E}_x[U_{\tau_a \wedge \tau}], \quad x \in \mathbb{R}, \tag{1.2}$$

where the supremum is taken over all stopping times with respect to $\mathbb{F}$, that is, for all random variables τ such that $\{\tau \leqslant t\} \in \mathcal{F}_t$ for every t, $0 \leqslant t < \infty$. Here and in the sequel $\mathbf{E}_x$ denotes the expectation conditional on $U_0 = x$. The major interest here is to characterize V, as well as to describe the stopping time where the supremum is attained. Since the drift has the opposite sign as U, we have no reason to liquidate our position as long as U is negative. On the other hand, if U is positive, then the drift is working against the investor and for large values of U the size of the drift should overcome the possible benefits from random variations.

We expect from the general optimal stopping theory (described, for example, in [9, Ch. 3]) that the optimal value function is given by $V = u$, where (u, b) is the solution of the free boundary problem

$$\begin{aligned} \mathcal{G}_U u(x) &= 0, \quad x \in (a, b), \\ u(x) &= x, \quad x \notin (a, b), \\ u'(b) &= 1. \end{aligned} \tag{1.3}$$

Here $\mathcal{G}_U$ is the infinitesimal generator of U, which is defined on the space of twice continuously differentiable functions $f\colon\ \mathbb{R} \to \mathbb{R}$ with compact support:

$$\mathcal{G}_U f(x) = \frac{\sigma^2}{2} f''(x) - \mu x f'(x) + \lambda \int_{-\infty}^{\infty} (f(x+y) - f(x))\varphi(y)\,\mathrm{d}y, \quad x \in \mathbb{R}. \tag{1.4}$$

Moreover, the stopping time when the supremum in (1.2) is attained should be

$$\tau_b = \inf\{t \geqslant 0\colon\ U_t \geqslant b\}.$$

Our first result is a so called verification theorem, which verifies that our guess above is indeed correct. In this sense, the verification theorem serves as a sufficient condition for optimality.

Theorem 1.1. *Assume that* (u,b) *is a classical solution of* (1.3) *with*

(a) $\mathcal{G}_U u(x) \leqslant 0$ *for* $x > b$,

(b) $u(x) \geqslant x$ *for* $x \in \mathbb{R}$.

Then $u(x) = V(x) = \mathbf{E}_x[U_{\tau_a \wedge \tau_b}]$ *for* $x \in \mathbb{R}$, *where* V *is given by* (1.2).

We note that we cannot be sure to close the pair trade at any of the boundaries a or b, because the spread can exhibit jumps. This is not the case in [2] and it is the major reason for the additional difficulties encountered in the present paper.

As seen from the assumptions on φ, we assume that the absolute values of the jumps of the process $\{U_t\}_{\geqslant 0}$ are bounded. The reason is that on the financial market, an asset cannot jump to arbitrarily large levels. If nothing else, the jumps are bounded by all the money in the world.

The rest of the paper is organized as follows. In Section 2 we prove Theorem 1.1 and in Section 3 we analyze a finite element method for the free boundary problem (1.3) and prove rigorous error estimates. We finally present strong numerical evidence for the existence and uniqueness of a solution of (1.3) with the properties a) and b) in Theorem 1.1.

2. Proof of Theorem 1.1

Before we start to prove Theorem 1.1 we need to recall some facts. From the general theory in [5] we get that the boundary value problem

$$\begin{aligned} \mathcal{G}_U u(x) &= 0, \quad x \in (a,b), \\ u(x) &= x, \quad x \notin (a,b), \end{aligned} \tag{2.1}$$

has a unique classical solution and that such a solution belongs to the space

$$C^2(\mathbb{R} \setminus \{a,b\}) \cap C(\mathbb{R}).$$

Moreover, the finite left and right limits of u' and u'' exist at a and b. Although these facts follow from [5], we present in Theorem 3.1 a self-contained proof for the simpler situation that we consider here. Hence, if (u,b) is a classical solution of (1.3), then necessarily

$$u \in C^2(\mathbb{R} \setminus \{a,b\}) \cap C^1(\mathbb{R} \setminus \{a\}) \cap C(\mathbb{R})$$

with finite left and right limits of u' and u'' everywhere. Hence, u can be written as a difference of two convex functions (see Problem 6.24 in [7, Ch. 3]) and so it satifies the assumption of the following generalization of Itô's formula (see [10, Ch. 4]):

Theorem 2.1 [Meyer-Itô formula]. *Let $X = \{X_t\}_{\geqslant 0}$ be a semimartingale and let f be the difference of two convex functions. Then*

$$f(X_t) = f(X_0) + \int_{0+}^{t} f'(X_{s-})\,\mathrm{d}X_s + \sum_{0<s\leqslant t} (f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s) + \frac{1}{2}\int_{-\infty}^{\infty} L_t^y(X)\,\mathrm{d}\mu(y),$$

where f' is the left derivative of f, μ is a signed measure which is the second generalized derivative of f, and $\{L_t^a(X)\}_{t\geqslant 0}$ is the local time process of X at a.

Proof of Theorem 1.1. Let u be as in Theorem 1.1 and let U be a solution of (1.1). We apply Theorem 2.1 with $f = u$ and $X = U$ and compute the terms in the Meyer-Itô formula. The second derivative measure μ of u can be split into two parts $\mu = \mu_c + \mu_d$, where the continuous part μ_c is given by $\mathrm{d}\mu_c = u''\,\mathrm{d}x$ and the discrete part $\mu_d = (u'(a+) - u'(a-))\delta_a$ is a point mass at a. Here, $u''(x)$ denotes the second derivative of u at x except at the points a and b, where it has finite right and left limits, respectively. By Corollary 1 of the Meyer-Itô formula in [10], we can now write

$$\begin{aligned}\frac{1}{2}\int_{-\infty}^{\infty} L_t^y(U)\,\mathrm{d}\mu(y) &= \frac{1}{2}\int_0^t u''(U_{s-})\,\mathrm{d}[U,U]_s^c \\ &\quad + \frac{1}{2}\int_{-\infty}^{\infty} L_t^y(U)(u'(a+) - u'(a-))\,\mathrm{d}\delta_a(y) \\ &= \frac{\sigma^2}{2}\int_0^t u''(U_{s-})\,\mathrm{d}s + \frac{1}{2}L_t^a(U)(u'(a+) - u'(a-)),\end{aligned}$$

where $[U,U]^c$ denotes the continuous part of the quadratic variation $[U,U]$.

Furthermore, by using (1.1) and writing the jump measure associated with $Z^{\sigma,\lambda,\varphi}$ as

$$N_Z(\mathrm{d}t, \mathrm{d}y) = \widetilde{N}_Z(\mathrm{d}t, \mathrm{d}y) + \lambda\,\mathrm{d}t\varphi(y)\,\mathrm{d}y,$$

where $\widetilde{N}_Z$ is a compensated Poisson random measure, we get

$$\begin{aligned}&\int_{0+}^{t} u'(U_{s-})\,\mathrm{d}U_s + \sum_{0<s\leqslant t} (u(U_s) - u(U_{s-}) - u'(U_{s-})\Delta U_s) \\ &\quad = -\mu\int_0^t U_{s-}u'(U_{s-})\,\mathrm{d}s + \sigma\int_0^t u'(U_{s-})\,\mathrm{d}W_s \\ &\quad\quad + \int_{0+}^{t}\int_{\mathbb{R}} (u(U_{s-} + y) - u(U_{s-}))\widetilde{N}_Z(\mathrm{d}s, \mathrm{d}y) \\ &\quad\quad + \lambda\int_0^t\int_{\mathbb{R}} (u(U_{s-} + y) - u(U_{s-}))\varphi(y)\,\mathrm{d}y\,\mathrm{d}s.\end{aligned}$$

Summing up, we now have, for $t \geqslant 0$,

$$
\begin{aligned}
u(U_t) = u(U_0) &+ \int_0^t \Big(\frac{1}{2}\sigma^2 u''(U_{s-}) - \mu U_{s-} u'(U_{s-})\Big)\,\mathrm{d}s \\
&+ \lambda \int_0^t \int_{\mathbb{R}} (u(U_{s-}+y) - u(U_{s-}))\varphi(y)\,\mathrm{d}y\,\mathrm{d}s \\
&+ \frac{1}{2} L_t^a(U)(u'(a+) - u'(a-)) + M_t,
\end{aligned} \tag{2.2}
$$

where

$$
M_t = \sigma \int_0^t u'(U_{s-})\,\mathrm{d}W_s + \int_{0+}^t \int_{\mathbb{R}} (u(U_{s-}+y) - u(U_{s-}))\widetilde{N}_Z(\mathrm{d}s, \mathrm{d}y),
$$

and $u'(U_{s-})$ is the left derivative. Since u is Lipschitz, has a bounded left derivative, and since the jumps density φ has a finite support, we get that $\{M_t\}_{t\geqslant 0}$ is a martingale.

Since $V(x) = \mathbf{E}_x[U_{\tau_a \wedge \tau_b}] = x = u(x)$ when $x \leqslant a$, we assume in the sequel that $U_0 = x > a$. Then $L^a_{\tau_a\wedge t}(U) = 0$ a.s. for all $t \geqslant 0$. This is essentially because the local time changes only when U_t passes a and can be proved by a straightforward calculation. Define $Y_t = u(U_{\tau_a\wedge t})$, $t \geqslant 0$. By using (2.2), the expression (1.4) for the generator of U, and (1.3), we get

$$
\begin{aligned}
Y_t &= u(x) + \int_0^{\tau_a\wedge t} \mathcal{G}_U u(U_{s-})\,\mathrm{d}s + M_{\tau_a\wedge t} \\
&= u(x) + \int_0^{\tau_a\wedge t} \mathcal{G}_U u(U_{s-}) 1_{\{U_{s-}\geqslant b\}}\,\mathrm{d}s + M_{\tau_a\wedge t}.
\end{aligned} \tag{2.3}
$$

Property (a) and the martingale property of $\{M_{\tau_a\wedge t}\}$ give that $\{Y_t\}_{t\geqslant 0}$ is a supermartingale. Furthermore, from property (b) we get that $Y_t \geqslant U_{\tau_a\wedge t}$ for $t \geqslant 0$, and since

$$
U_{\tau_a\wedge t} \geqslant a - J, \quad t \geqslant 0,
$$

we see that Y_t is bounded from below and we can apply the optional sampling theorem, which implies that $\mathbf{E}_x[Y_\tau] \leqslant \mathbf{E}_x[Y_0]$ for any stopping time τ (see [7]). Hence,

$$
\mathbf{E}_x[U_{\tau_a\wedge\tau}] \leqslant \mathbf{E}_x[Y_\tau] \leqslant \mathbf{E}_x[Y_0] = u(x).
$$

Recalling the definition of V in (1.2), we conclude that

$$
V(x) = \sup_\tau \mathbf{E}_x[U_{\tau_a\wedge\tau}] \leqslant u(x) \quad \text{for } x > a. \tag{2.4}
$$

In particular, if $x \geqslant b$, then $x \leqslant V(x) \leqslant u(x) = x$ and $\mathbf{E}_x[U_{\tau_a \wedge \tau_b}] = x$. Therefore, $u(x) = V(x) = \mathbf{E}_x[U_{\tau_a \wedge \tau_b}]$ when $x \geqslant b$.

For the case when $a < x < b$, note that from (2.3) we get

$$Y_{\tau_b \wedge t} = u(x) + M_{\tau_a \wedge \tau_b \wedge t}, \quad t \geqslant 0.$$

Hence $\{Y_{\tau_b \wedge t}\}$ is a martingale and since

$$|Y_{\tau_b \wedge t}| \leqslant \max_{[a-J, b+J]} |u|, \quad t \geqslant 0,$$

the optional sampling theorem applies and we obtain $u(x) = \mathbf{E}_x[Y_0] = \mathbf{E}_x[Y_{\tau_b}]$. Finally, $u(x) = x$ for $x \notin (a,b)$ implies $Y_{\tau_b} = U_{\tau_a \wedge \tau_b}$ and hence $u(x) = \mathbf{E}_x[U_{\tau_a \wedge \tau_b}] \leqslant V(x)$. Together with (2.4) this completes the proof. □

3. Numerical solution of the free boundary value problem

We have not been able to give a rigorous proof of the existence and uniqueness of the solution (u, b) of the free boundary value problem (1.3). We therefore resort to a numerical solution by means of the finite element method. However, at the end of this section we will show that we have strong computational evidence for both the existence and uniqueness for (1.3). In order to achieve this, we first show rigorous existence and regularity results for the boundary value problem (2.1) and rigorous convergence estimates with explicit constants for the finite element approximation.

3.1. The boundary value problem. We begin by transforming the free boundary value problem (1.3) to a problem with homogeneous boundary values. Set $v(x) = u(x) - x$ and use $\int_{-\infty}^{\infty} y\varphi(y)\,\mathrm{d}y = 0$ to get

$$\begin{aligned} &-\frac{1}{2}\sigma^2 v''(x) + \mu x v'(x) \\ &\qquad -\lambda \int_{-\infty}^{\infty} (v(x+y) - v(x))\varphi(y)\,\mathrm{d}y = -\mu x, \quad && x \in (a,b), \\ &\qquad\qquad v(x) = 0, && x \notin (a,b), \\ &\qquad\qquad v'(b) = 0. \end{aligned} \tag{3.1}$$

Introducing the operators

$$\begin{aligned} \mathcal{L}v(x) &= -\frac{1}{2}\sigma^2 v''(x) + \mu x v'(x), \\ \mathcal{I}v(x) &= \lambda \int_{-\infty}^{\infty} (v(x+y) - v(x))\varphi(y)\,\mathrm{d}y, \end{aligned}$$

our approach will be to first solve the boundary value problem

$$\begin{aligned} \mathcal{L}v - \mathcal{I}v &= f, \quad x \in (a,b), \\ v(x) &= 0, \quad x \notin (a,b), \end{aligned} \tag{3.2}$$

with $f(x) = -\mu x$, and then for fixed $a < 0$ find $b > a$ such that $v'(b) = 0$.

To solve (3.2) we follow a standard approach based on a weak formulation and Fredholm's alternative. We denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the standard scalar product and norm in $L_2(a,b)$, and we denote by $H^k(a,b)$ and $H_0^1(a,b) = \{v \in H^1(a,b)\colon v(a) = v(b) = 0\}$ the standard Sobolev spaces. We denote the derivative $Dv = \mathrm{d}v/\mathrm{d}x$. We choose $v \mapsto \|Dv\|$ to be the norm in $H_0^1(a,b)$, which is equivalent to the standard H^1-norm. We extend functions $v \in L_2(a,b)$ by zero outside (a,b) so that $\mathcal{I}v$ is properly defined. We define bilinear forms

$$\begin{aligned} A_{\mathcal{L}}(u,v) &= \int_a^b \Big(\frac{1}{2}\sigma^2 u'(x)v'(x) + \mu x u'(x) v(x)\Big)\,\mathrm{d}x, \quad u,v \in H_0^1(a,b), \\ A_{\mathcal{I}}(u,v) &= \int_a^b \mathcal{I}u(x)v(x)\,\mathrm{d}x, \quad u,v \in L_2(a,b), \\ A(u,v) &= A_{\mathcal{L}}(u,v) - A_{\mathcal{I}}(u,v). \end{aligned} \tag{3.3}$$

Since $\int_{-\infty}^{\infty} \varphi(y)\,\mathrm{d}y = 1$, $\varphi(-y) = \varphi(y)$, and $v(x) = 0$ for $x \notin (a,b)$, we also have

$$\mathcal{I}v(x) = \lambda \int_a^b \varphi(x-y)v(y)\,\mathrm{d}y - \lambda v(x), \quad v \in L_2(a,b). \tag{3.4}$$

The convolution operator $\mathcal{I}_1 v(x) = \int_{-\infty}^{\infty} \varphi(x-y)v(y)\,\mathrm{d}y$ is bounded in $L_2(a,b)$ with the constant $c = \int_{-\infty}^{\infty} \varphi(y)\,\mathrm{d}y = 1$ by Young's inequality. Hence,

$$\|\mathcal{I}v\| \leqslant 2\lambda\|v\|, \quad v \in L_2(a,b), \tag{3.5}$$

$$\|D\mathcal{I}v\| \leqslant 2\lambda\|Dv\|, \quad v \in H_0^1(a,b), \tag{3.6}$$

and

$$-A_{\mathcal{I}}(v,v) \geqslant \lambda(\|v\|^2 - \|\mathcal{I}_1 v\|\|v\|) \geqslant 0, \quad v \in L_2(a,b).$$

Hence,

$$\begin{aligned} |A(u,v)| &\leqslant \frac{1}{2}\sigma^2\|Du\|\|Dv\| + \mu\max(|a|,|b|)\|Du\|\|v\| + 2\lambda\|u\|\|v\| \\ &\leqslant c_1\|Du\|\|Dv\|, \quad u,v \in H_0^1(a,b), \\ c_1 &= \frac{1}{2}\sigma^2 + c_2(\mu\max(|a|,|b|) + 2\lambda c_2), \end{aligned}$$

where we also used Poincaré's inequality

$$\|v\| \leqslant c_2\|Dv\|, \quad v \in H_0^1(a,b), \quad c_2 = (b-a)/\pi. \tag{3.7}$$

By integration by parts we obtain

$$A_{\mathcal{L}}(v,v) = \frac{1}{2}\sigma^2\|Dv\|^2 - \frac{1}{2}\mu\|v\|^2, \quad v \in H_0^1(a,b),$$

so that $A(\cdot,\cdot)$ is bounded and coercive on $H_0^1(a,b)$:

$$|A(u,v)| \leqslant c_1\|Du\|\|Dv\|, \quad u,v \in H_0^1(a,b), \tag{3.8}$$

$$A(v,v) \geqslant \frac{1}{2}\sigma^2\|Dv\|^2 - \frac{1}{2}\mu\|v\|^2, \quad v \in H_0^1(a,b). \tag{3.9}$$

We say that $v \in H_0^1(a,b)$ is a weak solution of (3.2) if

$$A(v,\varphi) = (f,\varphi) \quad \forall \varphi \in H_0^1(a,b). \tag{3.10}$$

We also use the adjoint problem: find $w \in H_0^1(a,b)$ such that

$$A(\varphi,w) = (\varphi,g) \quad \forall \varphi \in H_0^1(a,b), \tag{3.11}$$

where $g \in L_2(a,b)$ is given. The strong form is (note that $\mathcal{I}$ is self-adjoint in $L_2(a,b)$)

$$\begin{aligned} \mathcal{L}^*w(x) - \mathcal{I}w(x) &= g(x), \quad && x \in (a,b), \\ w(x) &= 0, && x \notin (a,b), \end{aligned} \tag{3.12}$$

where

$$\mathcal{L}^*w(x) = -\frac{1}{2}\sigma^2 w''(x) - \mu x w'(x) - \mu w(x).$$

We can now prove the existence and uniqueness of a classical solution of (3.2). In principle this follows from the general theory in [5], but we present a self-contained proof, with explicit constants, for the simpler situation that we consider here. The theorem also provides results necessary for the analysis of the finite element method.

Theorem 3.1. *The boundary value problem* (3.2) *has a unique weak solution* $v \in H_0^1(a,b)$ *for every* $f \in L_2(a,b)$. *The solution belongs to* $H^2(a,b)$ *and there is a constant* c_3 *such that*

$$\|D^2 v\| \leqslant c_3 \|f\|. \tag{3.13}$$

Moreover, if $f(x) = -\mu x$, *then the solution is classical,* $v \in C^2([a,b])$. *Similarly, the adjoint problem* (3.12) *has a unique weak solution* $w \in H_0^1(a,b)$ *for each* $g \in L_2(a,b)$, *which belongs to* $H^2(a,b)$ *and*

$$\|D^2 w\| \leqslant c_3 \|g\|. \tag{3.14}$$

Proof. The proof is a standard argument as presented, for example, in [4, Ch. 6] for elliptic PDEs. The only difference is that the lowest order term in $A(\cdot,\cdot)$ is defined by means of an integral operator, but the crucial properties (3.8), (3.9) are the same.

We first show that the weak solutions are regular. We use a regularity result for elliptic problems (see [4, p. 323]): If v is a weak solution of

$$\mathcal{L}v(x) = g(x),\ x \in (a,b); \quad v(a) = v(b) = 0,$$

and if $g \in H^k(a,b)$ for some $k \geqslant 0$, then $v \in H^{k+2}(a,b)$. A weak solution $v \in H_0^1(a,b)$ of (3.2) satisfies this with $g = f + \mathcal{I}v$, where by (3.5), (3.6) $\mathcal{I}v \in H^1(a,b)$. For $f \in L_2(a,b)$ we conclude that $v \in H^2(a,b)$. If $f \in H^1(a,b)$, then we have $v \in H^3(a,b)$ and by Sobolev's embedding $v \in C^2([a,b])$. In particular, a weak solution is classical when $f(x) = 0$ and $f(x) = -\mu x$. Analogous regularity results hold for the adjoint problem.

Now we can prove existence. Let

$$A_\mu(u,v) = A(u,v) + \frac{1}{2}\mu(u,v).$$

By the Lax-Milgram lemma we know that the shifted problem

$$A_\mu(u,\varphi) = (g,\varphi) \quad \forall \varphi \in H_0^1(a,b)$$

has a unique solution $u \in H_0^1(a,b)$ for each $g \in L_2(a,b)$. This defines the bounded linear operator $\mathcal{A}_\mu^{-1}\colon L_2(a,b) \to H_0^1(a,b)$ by $u = \mathcal{A}_\mu^{-1} g$. The equation (3.10) is now equivalent to

$$v = \mathcal{A}_\mu^{-1} f + \frac{1}{2}\mu \mathcal{A}_\mu^{-1} v,$$

or $v - Kv = h$, where $h = \mathcal{A}_\mu^{-1} f$ and where $K = \frac{1}{2}\mu\mathcal{A}_\mu^{-1}\colon\ L_2(a,b) \to L_2(a,b)$ is a compact operator, because $H_0^1(a,b)$ is compactly embedded in $L_2(a,b)$.

By the Fredholm alternative we know that the latter equation is uniquely solvable for every $h \in L_2(a,b)$ if and only if the corresponding homogeneous equation has no non-trivial solution. But a non-trivial solution of $v - Kv = 0$ would be a weak solution, and hence a classical solution, of (3.2) with $f = 0$.

Then we can apply the maximum principle for classical solutions of (3.2), see [5, Theorem 3.1.3]. It says that if a classical function satisfies $(\mathcal{L}-\mathcal{I})u \leqslant 0$ in (a,b), then $\max\limits_{[a,b]} u = \max\limits_{\mathbb{R}\setminus(a,b)} u$. (The maximum principle for the integro-differential equation is proved in the same way as for the differential equation after noting that $-\mathcal{I}u(x_0) \geqslant 0$ if u has a maximum at x_0.) We conclude that the homogeneous equation has no non-trivial solution and therefore (3.2) has a unique weak solution for every $f \in L_2(a,b)$. By the Fredholm theory the adjoint problem (3.12) is then also uniquely solvable for all $g \in L_2(a,b)$.

Finally, we prove the bounds (3.13) and (3.14). Let $v = \mathcal{A}^{-1} f$ and $w = (\mathcal{A}^*)^{-1} g$ denote the solution operators of (3.2) and (3.12), respectively.

Let $f \in H_0^1(a,b)$. Then $v = \mathcal{A}^{-1} f$ is classical and the maximum principle gives

$$\|v\|_{L_\infty(a,b)} \leqslant c_4 \|f\|_{L_\infty(a,b)}. \tag{3.15}$$

In order to compute the explicit constant we briefly recall the proof. Let

$$\varphi(x) = \begin{cases} \mathrm{e}^{\gamma(b-a)} - \mathrm{e}^{\gamma(x-a)}, & x \leqslant b, \\ 0, & x \geqslant b, \end{cases}$$

where $\gamma > 0$ is chosen so that $\mathcal{A}\varphi \geqslant 1$ in (a,b). Then $u(x) = \|f\|_{L_\infty(a,b)}\varphi(x)$ satisfies $\mathcal{A}u \geqslant \|f\|_{L_\infty(a,b)} \geqslant f = \mathcal{A}v$ in (a,b) and $u \geqslant 0 = v$ outside (a,b), so that the maximum principle gives $\max\limits_{[a,b]}(v-u) = \max\limits_{\mathbb{R}\setminus(a,b)}(v-u) = 0$, that is, $u \geqslant v$ in $[a,b]$. Hence $v \leqslant \|\varphi\|_{L_\infty(a,b)}\|f\|_{L_\infty(a,b)}$ in $[a,b]$. The lower bound $v \geqslant -\|\varphi\|_{L_\infty(a,b)}\|f\|_{L_\infty(a,b)}$ is obtained in a similar way and so we get

$$\|v\|_{L_\infty(a,b)} \leqslant \|\varphi\|_{L_\infty(a,b)}\|f\|_{L_\infty(a,b)} \leqslant \mathrm{e}^{\gamma(b-a)}\|f\|_{L_\infty(a,b)}.$$

To determine γ, let $x \in (a,b)$ and compute

$$\begin{aligned} -\mathcal{I}\varphi(x) &= \lambda \mathrm{e}^{\gamma(x-a)} \int_{-\infty}^{b-x} (\mathrm{e}^{\gamma y} - 1)\varphi(y)\,\mathrm{d}y + \lambda(\mathrm{e}^{\gamma(b-a)} - \mathrm{e}^{\gamma(x-a)}) \int_{b-x}^{\infty} \varphi(y)\,\mathrm{d}y \\ &\geqslant -\lambda \mathrm{e}^{\gamma(x-a)} \int_{-\infty}^{\infty} \varphi(y)\,\mathrm{d}y = -\lambda \mathrm{e}^{\gamma(x-a)}. \end{aligned}$$

Hence,

$$\mathcal{A}\varphi(x) \geqslant \Big(\frac{1}{2}\sigma^2\gamma^2 - \mu b\gamma - \lambda\Big)\mathrm{e}^{\gamma(x-a)} \geqslant 1, \quad x \in (a,b),$$

if $\frac{1}{2}\sigma^2\gamma^2 - \mu b\gamma - \lambda \geqslant 1$, that is, if

$$\gamma = \hat{\gamma} = \frac{\mu b}{\sigma^2} + \sqrt{\frac{2(\lambda+1)}{\sigma^2}}.$$

Then we conclude that (3.15) holds with $c_4 = \mathrm{e}^{\hat{\gamma}(b-a)}$.

Hence, since $\|v\| \leqslant (b-a)^{1/2}\|v\|_{L_\infty(a,b)}$ and $\|f\|_{L_\infty(a,b)} \leqslant (b-a)^{1/2}\|Df\|$, we obtain the bound

$$\|v\| = \|\mathcal{A}^{-1}f\| \leqslant c_5\|Df\| \quad \forall f \in H_0^1(a,b),\ c_5 = (b-a)c_4.$$

By duality we conclude

$$\|(\mathcal{A}^{-1})^*\|_{B(L_2,H^{-1})} = \|\mathcal{A}^{-1}\|_{B(H_0^1,L_2)} \leqslant c_5.$$

Hence

$$\|w\|_{H^{-1}} = \|(\mathcal{A}^*)^{-1}g\|_{H^{-1}} = \|(\mathcal{A}^{-1})^*g\|_{H^{-1}} \leqslant c_5\|g\| \quad \forall g \in L_2(a,b), \tag{3.16}$$

where $H^{-1}(a,b) = (H_0^1(a,b))^*$ and

$$\|w\|_{H^{-1}} = \sup_{\varphi \in H_0^1} \frac{(\varphi, w)}{\|D\varphi\|}.$$

Recall that $v \mapsto \|Dv\|$ is the chosen norm in $H_0^1(a,b)$. By using $\varphi = w \in H_0^1(a,b)$ here we obtain

$$\|w\|^2 \leqslant \|w\|_{H^{-1}}\|Dw\|. \tag{3.17}$$

We take $\varphi = w$ in the adjoint equation (3.11) and use coercivity (3.9), the inequality $2ab \leqslant \varepsilon a^2 + \varepsilon^{-1}b^2$, and (3.17) to get

$$\begin{aligned}\frac{1}{2}\sigma^2\|Dw\|^2 &\leqslant A(w,w) + \frac{1}{2}\mu\|w\|^2 \leqslant \|g\|\|w\| + \frac{1}{2}\mu\|w\|^2 \\ &\leqslant \frac{1}{2}\mu^{-1}\|g\|^2 + \mu\|w\|^2 \leqslant \frac{1}{2}\mu^{-1}\|g\|^2 + \mu\|w\|_{H^{-1}}\|Dw\| \\ &\leqslant \frac{1}{2}\mu^{-1}\|g\|^2 + \mu^2\sigma^{-2}\|w\|_{H^{-1}}^2 + \frac{1}{4}\sigma^2\|Dw\|^2.\end{aligned}$$

With (3.16) this leads to

$$\begin{aligned}\|Dw\|^2 &\leqslant 2\sigma^{-2}\mu^{-1}\|g\|^2 + 4\sigma^{-4}\mu^{-2}\|w\|_{H^{-1}}^2 \\ &\leqslant (2\sigma^{-2}\mu^{-1} + 4\sigma^{-4}\mu^{-2}c_5^2)\|g\|^2\end{aligned}$$

and with Poincaré's inequality (3.7),

$$\|w\| \leqslant c_2\|Dw\| \leqslant c_2(2\sigma^{-2}\mu^{-1} + 4\sigma^{-4}\mu^{-2}c_5^2)^{1/2}\|g\|.$$

Hence

$$\begin{aligned}(3.18) \qquad &\|(\mathcal{A}^*)^{-1}g\| = \|w\| \leqslant c_6\|g\| \quad \forall g \in L_2(a,b), \\ &c_6 = c_2(2\sigma^{-2}\mu^{-1} + 4\sigma^{-4}\mu^{-2}c_5^2)^{1/2}.\end{aligned}$$

By duality in L_2 we also have

$$(3.19) \qquad \|v\| = \|\mathcal{A}^{-1}f\| \leqslant c_6\|f\| \quad \forall f \in L_2(a,b).$$

In order to bound D^2v we recall that $v \in H^2(a,b)$. Hence it satisfies (3.2) strongly, so that with (3.5) we obtain

$$\begin{aligned}\frac{1}{2}\sigma^2\|D^2v\| &\leqslant \mu\|xDv\| + \|\mathcal{I}v\| + \|f\| \\ &\leqslant \mu\max(|a|,|b|)\|Dv\| + 2\lambda\|v\| + \|f\| \\ &\leqslant \mu\max(|a|,|b|)\|D^2v\|^{1/2}\|v\|^{1/2} + 2\lambda\|v\| + \|f\| \\ &\leqslant \frac{1}{4}\sigma^2\|D^2v\| + (2\lambda + \sigma^{-2}\mu^2\max(|a|,|b|)^2)\|v\| + \|f\|.\end{aligned}$$

Hence,

$$\begin{aligned}&\|D^2v\| \leqslant c_7\|f\| + c_8\|v\|, \\ &c_7 = 4\sigma^{-2}, \quad c_8 = 4\sigma^{-2}(2\lambda + \mu + \sigma^{-2}\mu^2\max(|a|,|b|)^2).\end{aligned}$$

In the last step we replaced 2λ by $2\lambda + \mu$ in c_8, so that the same result holds also for the adjoint equation (3.12). Using also (3.18) and (3.19) we finally conclude

$$\begin{aligned}&\|D^2v\| \leqslant c_3\|f\|, \quad \|D^2w\| \leqslant c_3\|g\|, \\ &c_3 = c_7 + c_6c_8.\end{aligned}$$

This completes the proof. □

3.2. The finite element method. The finite element method is based on a family of subdivisions $\mathcal{T}_h$ of the interval $[a,b]$ parametrized by the maximal mesh size h. Each mesh is of the form

$$\mathcal{T}_h\colon\ a = x_0 < x_1 < \ldots < x_{j-1} < x_j < \ldots < x_N = b, \quad h = \max_{j=1,\ldots,N}(x_j - x_{j-1}).$$

We introduce the space $V_h \subset H_0^1(a,b)$ consisting of all continuous functions that reduce to piecewise polynomials of degree at most 1 with respect to $\mathcal{T}_h$. See [8, Ch. 5] or [1, Ch. 1]. Then there is an interpolator $I_h\colon\ C([a,b]) \to V_h$ such that $I_h u(x_j) = u(x_j)$, $j = 1,\ldots,N$, and

$$\|D(u - I_h u)\|_{L_p(a,b)} \leqslant h^{1/2+\frac{1}{p}}\|D^2u\|, \quad u \in H^2(a,b) \cap H_0^1(a,b),\ p = 2,\infty. \tag{3.20}$$

To prove this we use the identity

$$D(u - I_h u)(x) = h_j^{-1}\int_{x_{j-1}}^{x_j}(u'(x) - u'(y))\,\mathrm{d}y = h_j^{-1}\int_{x_{j-1}}^{x_j}\int_y^x u''(z)\,\mathrm{d}z\,\mathrm{d}y$$

for $x \in (x_{j-1}, x_j)$ and with $h_j = x_j - x_{j-1}$, which yields

$$|D(u - I_h u)(x)| \leqslant h_j^{1/2}\|D^2u\|_{L_2(x_{j-1},x_j)} \leqslant h^{1/2}\|D^2u\|, \quad x \in (x_{j-1}, x_j).$$

This proves the case $p = \infty$ and for $p = 2$ we have

$$\|D(u - I_h u)\|^2 \leqslant \sum_{j=1}^N h_j^2\|D^2u\|^2_{L_2(x_{j-1},x_j)} \leqslant h^2\|D^2u\|^2.$$

The finite element problem is based on the weak formulation in (3.10): find $v_h \in V_h$ such that

$$A(v_h, \varphi_h) = (f, \varphi_h) \quad \forall \varphi_h \in V_h, \tag{3.21}$$

where $A(\cdot,\cdot)$ is defined in (3.3) with the integral operator computed as in (3.4). In the following theorem we prove convergence estimates with explicit constants.

Theorem 3.2. *Let v be the solution of* (3.2) *as in Theorem* 3.1. *There is $h_0 = \sigma/(2^{1/2}\mu^{1/2}c_1c_3)$ such that, for $h \leqslant h_0$,* (3.21) *has a unique solution $v_h \in V_h$ and*

$$\|v - v_h\| \leqslant 4c_1^2c_3^2\sigma^{-2}h^2\|f\|, \quad \|D(v - v_h)\| \leqslant 4c_1c_3\sigma^{-2}h\|f\|. \tag{3.22}$$

Proof. We adapt an argument from [11]. Let $e = v - v_h$ denote the error. By subtraction of (3.21) and (3.10) with $\varphi = \varphi_h \in V_h \subset H_0^1(a,b)$ we get

$$A(e, \varphi_h) = 0 \quad \forall \varphi_h \in V_h. \tag{3.23}$$

Consider the adjoint problem (3.11) with $g = e$ and a solution $w = (\mathcal{A}^*)^{-1}e$. With $\varphi = e$ this yields

$$\begin{aligned}\|e\|^2 = A(e,w) = A(e, w - I_hw) &\leqslant c_1\|De\|\|D(w - I_hw)\| \\ &\leqslant c_1\|De\|h\|D^2w\| \leqslant c_1c_3h\|De\|\|e\|.\end{aligned}$$

Here we used (3.23), (3.8), (3.20), and (3.14). We conclude

$$\|e\| \leqslant c_1c_3h\|De\|. \tag{3.24}$$

In view of (3.23) we have $A(e,e) = A(e, v - v_h) = A(e,v)$, so that by (3.9) and (3.24),

$$\begin{aligned}\frac{1}{2}\sigma^2\|De\|^2 \leqslant A(e,e) + \frac{1}{2}\mu\|e\|^2 &= A(e,v) + \frac{1}{2}\mu\|e\|^2 \\ &\leqslant c_1\|De\|\|Dv\| + \frac{1}{2}\mu c_1^2c_3^2h^2\|De\|^2.\end{aligned} \tag{3.25}$$

Hence, for $h \leqslant h_0$ sufficiently small ($h_0^2 = \sigma^2/(2\mu c_1^2c_3^2)$), we have

$$\|De\| \leqslant c_9\|Dv\|, \quad c_9 = 4c_1\sigma^{-2}.$$

Now if $f = 0$ in (3.10) and (3.21), then $v = 0$ by uniqueness, and hence $e = 0$, so that $v_h = 0$. This means that we have uniqueness for the finite element problem (3.21). But this is an equation in a finite dimensional space so existence also follows. Therefore, (3.21) has a unique solution for all $f \in L_2(a,b)$ if $h \leqslant h_0$.

In order to prove the error estimate (3.22) we return to (3.25) but use $A(e,e) = A(e, v - v_h) = A(e, v - I_hv)$ instead:

$$\begin{aligned}\frac{1}{2}\sigma^2\|De\|^2 \leqslant A(e,e) + \frac{1}{2}\mu\|e\|^2 &= A(e, v - I_hv) + \frac{1}{2}\mu\|e\|^2 \\ &\leqslant c_1\|De\|\|D(v - I_hv)\| + \frac{1}{2}\mu c_1^2c_3^2h^2\|De\|^2,\end{aligned}$$

and we conclude, for $h \leqslant h_0$,

$$\|De\| \leqslant c_9 \|D(v - I_h v)\|, \quad c_9 = 4c_1\sigma^{-2}.$$

Hence, by (3.20), (3.13), and (3.24),

$$\begin{aligned} \|De\| &\leqslant c_9 h \|D^2 v\| \leqslant c_9 c_3 h \|f\| = 4c_1 c_3 \sigma^{-2} h \|f\|, \\ \|e\| &\leqslant c_1 c_3 h \|De\| \leqslant 4c_1^2 c_3^2 \sigma^{-2} h^2 \|f\|, \end{aligned}$$

which is (3.22). □

We finish by proving the pointwise convergence of the derivative.

Corollary 3.3. *Assume that each finite element mesh $\mathcal{T}_h$ is uniform, that is, $x_j - x_{j-1} = h$ for $j = 1, \ldots, N$. Then, for $h \leqslant h_0$ as in Theorem 3.2, we have*

$$|v'(b) - v_h'(b)| \leqslant c_{10} h^{1/2} \|f\|, \quad c_{10} = 2 + 4c_1 c_3 \sigma^{-2}.$$

P r o o f. We use the inverse inequality

$$\|D\varphi_h\|_{L_\infty(a,b)} \leqslant h^{-1/2} \|D\varphi_h\|, \quad \varphi_h \in V_h.$$

To prove this we note that

$$D\varphi_h(x) = h^{-1} \int_{x_{j-1}}^{x_j} D\varphi_h(y)\,\mathrm{d}y, \quad x \in (x_{j-1}, x_j),\ h = x_j - x_{j-1},$$

which yields

$$|D\varphi_h(x)| \leqslant h^{-1/2} \|D\varphi_h\|_{L_2(x_{j-1},x_j)} \leqslant h^{-1/2} \|D\varphi_h\|, \quad x \in (x_{j-1}, x_j).$$

Hence, by (3.20) and (3.22),

$$\begin{aligned} \|De\|_{L_\infty(a,b)} &\leqslant \|D(v - I_h v)\|_{L_\infty(a,b)} + \|D(I_h v - v_h)\|_{L_\infty(a,b)} \\ &\leqslant \|D(v - I_h v)\|_{L_\infty(a,b)} + h^{-1/2} \|D(I_h v - v_h)\| \\ &\leqslant \|D(v - I_h v)\|_{L_\infty(a,b)} + h^{-1/2} \|D(I_h v - v)\| + h^{-1/2} \|D(v - v_h)\| \\ &\leqslant 2h^{1/2} \|D^2 v\| + h^{-1/2} \|D(v - v_h)\| \leqslant (2 + 4c_1 c_3 \sigma^{-2}) h^{1/2} \|f\|. \end{aligned}$$

Therefore

$$|v'(b) - v_h'(b)| \leqslant (2 + 4c_1 c_3 \sigma^{-2}) h^{1/2} \|f\|.$$

□

In particular, with $f(x) = -\mu x$, Corollary 3.3 gives

$$|v'(b) - v_h'(b)| \leqslant c_{11} h^{1/2}, \quad c_{11} = c_{10}\mu\sqrt{\frac{b^3 - a^3}{3}}. \tag{3.26}$$

Given numerical values for the parameters $a, b, \sigma, \mu, \lambda$ we may now compute numerical values for h_0 and c_{11}. Alternatively, we may conclude that there are uniform bounds $h_0 \geqslant \hat{h}_0$, $c_{11} \leqslant \hat{c}_{11}$ for $b \in [b_1, b_2]$ and with the other parameters fixed.

3.3. The free boundary value problem. We use uniform meshes $\mathcal{T}_h$ with

$$x_j - x_{j-1} = h = \frac{b-a}{N}, \quad j = 1, \ldots, N.$$

Since we want to vary b, we parametrize by N instead of h. Let $f(x) = -\mu x$, fix $a < 0$ and let v, v_N denote the solutions of (3.10) and (3.21) for $b > a$. Define functions

$$F(b) = v'(b), \quad F_N(b) = v_N'(b).$$

From (3.26) we get for $a < b_1 < b_2$

$$\begin{aligned} \|F - F_N\|_{L_\infty(b_1,b_2)} \leqslant \hat{c}_{12} N^{-1/2}, \quad N \geqslant \widehat{N}_0, \\ \hat{c}_{12} = \hat{c}_{11}(b_2 - a)^{1/2}, \quad \widehat{N}_0 = \frac{b_2 - a}{\hat{h}_0}. \end{aligned} \tag{3.27}$$

By writing down the matrix equation for solving the finite element problem (3.21), it is easy to see that, for fixed N, the function $b \mapsto F_N(b)$ is continuous on (a, ∞). From (3.27) we conclude that $b \mapsto F(b)$ is also continuous on (a, ∞). Moreover, by a direct consequence of the strong maximum principle and the Hopf boundary point principle for our equation (see [5, Theorem 3.1.4–3.1.5]), we get the following:

Lemma 3.4. *If $a < b \leqslant 0$, then $F(b) < 0$. In particular, if (u, b) is a solution to the free boundary problem* (1.3), *then $b > 0$.*

We believe that there exists a unique $b > 0$ such that $F(b) = 0$. We are not able to provide a rigorous proof of this, but numerical simulations present strong evidence in the following way. Assign numerical values to the parameters a, σ, μ, λ and fix a jump density φ. In all our computations, we took φ to be the truncated normal distribution with mean zero, variance $\gamma > 0$ and support $[-J, J]$, that is,

$$\varphi(y) = \begin{cases} \dfrac{\mathrm{e}^{-y^2/2\gamma^2}}{\gamma\sqrt{2\pi}(2\Phi J/\gamma - 1)} & \text{if } -J < y < J, \\ 0 & \text{otherwise,} \end{cases}$$

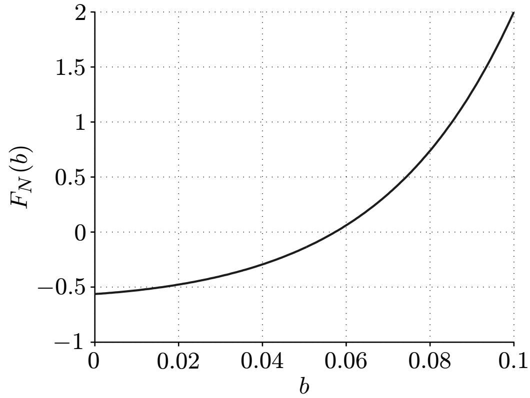

Figure 1. The function F_N when $a = -0.1$, $\lambda = 10$, $\sigma = 0.2$, $\mu = \sigma^2/0.005$, $\gamma = 0.02$ and $J = 0.05$.

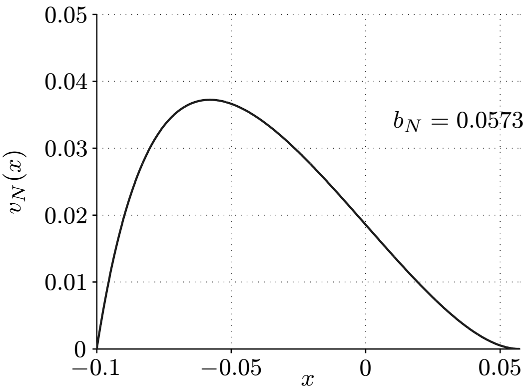

Figure 2. The solution (v_N, b_N) when $a = -0.1$, $\lambda = 10$, $\sigma = 0.2$, $\mu = \sigma^2/0.005$, $\gamma = 0.02$ and $J = 0.05$.

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-y^2/2}\,\mathrm{d}y, \quad x \in \mathbb{R}.$$

From computations of the boundary value problem (3.21) (see Figures 1 and 2) we can find $0 \leqslant b_1 < b_2$ and $\widetilde{N} \geqslant \widehat{N}_0$ such that

$$F_{\widetilde{N}}(b_1) \leqslant -\frac{1}{2}, \quad F_{\widetilde{N}}(b_2) \geqslant \frac{1}{2}, \quad \text{and} \quad \hat{c}_{12}\widetilde{N}^{-1/2} < \frac{1}{4}.$$

(The numbers 1/2 and 1/4 may vary if we change the parameters.) From (3.27) we can then conclude that

$$\begin{aligned} F(b_1) < 0, \quad & F(b_2) > 0, \\ F_N(b_1) < 0, \quad & F_N(b_2) > 0 \quad \text{for all } N \geqslant \widetilde{N}. \end{aligned}$$

Hence, there exists $b \in (b_1, b_2)$ such that $F(b) = 0$ and for each $N \geqslant \widetilde{N}$ there exists $b_N \in (b_1, b_2)$ such that $F_N(b_N) = 0$. Moreover, (3.27) gives us that

$$\lim_{N\to\infty} F(b_N) = 0.$$

Of course, we cannot conclude that b is unique and $b_N \to b$ as $N \to \infty$. However, Figure 1 suggests that b is unique and from computations with increasing N, it seems like b_N converges, see Table 3.1.

N	b_N
2000	0.0572939
4000	0.0572743
6000	0.0572678
8000	0.0572653

Table 3.1. $a = -0.1$, $\lambda = 10$, $\sigma = 0.2$, $\mu = \sigma^2/0.005$, $\gamma = 0.02$ and $J = 0.05$.

We now discuss whether the properties (a) and (b) in the statement of Theorem 1.1 hold for a solution (u, b) of (1.3). We have no rigorous proof, but computational evidence. The properties (a) and (b) boil down to

$$\lambda \int_a^b v(y)\varphi(y-x)\,\mathrm{d}y \leqslant \mu x \quad \text{for } x > b \tag{3.28}$$

and $v \geqslant 0$ respectively, where (v, b) solves (3.1). We believe that $v \geqslant 0$ holds for all values of the parameters, but computations suggest that (3.28) may fail for certain parameter values, typically when σ is small and λ is three or four times larger than μ. See Figures 3 and 4, where we check (3.28) for (v_N, b_N) instead of (v, b).

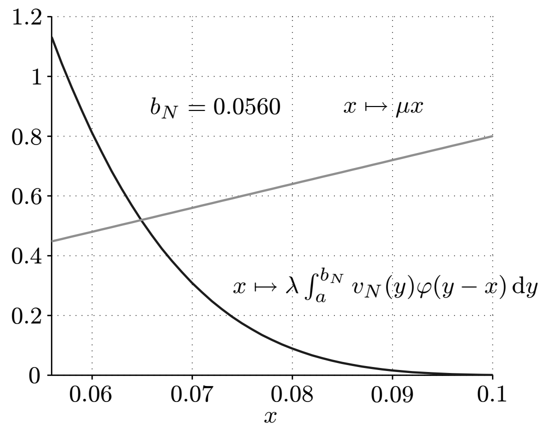

Figure 3. A simulation of (3.28) when $a = -0.1$, $\lambda = 30$, $\sigma = 0.2$, $\mu = \sigma^2/0.005$, $\gamma = 0.02$ and $J = 0.05$. The condition fails.

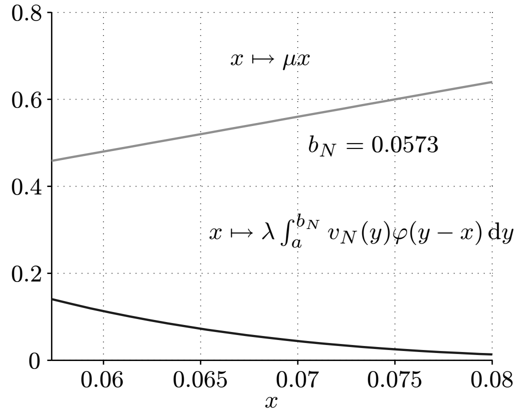

Figure 4. A simulation of (3.28) when $a = -0.1$, $\lambda = 10$, $\sigma = 0.2$, $\mu = \sigma^2/0.005$, $\gamma = 0.02$ and $J = 0.05$. The condition holds.

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