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THE \( b \)-WEAK COMPACTNESS OF WEAK BANACH-SAKS OPERATORS

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Abstract. We characterize Banach lattices on which every weak Banach-Saks operator is \( b \)-weakly compact.

Keywords: \( b \)-weakly compact operator, weak Banach-Saks operator, Banach lattice, \( (b) \)-property, KB-space

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1. Introduction and notation

Many authors have studied the Banach-Saks property, the weak Banach-Saks property and operators with these properties (see [8], [10], [11], [12], [13]). The notions of Banach-Saks property and weak Banach-Saks property are introduced in [12]. Note that the later property was introduced for the first time in [8] with the name Banach-Saks-Rosenthal property.

A Banach space \( X \) is said to have the Banach-Saks property if every bounded sequence \( (x_n) \) in \( X \) has a subsequence \( (x_{n_k}) \) which is Cesàro convergent. The origin of the Banach-Saks property can be traced back to a result of S. Mazur. If a sequence \( (x_n) \) in a Banach space is weakly convergent to some point \( x \), then there is a sequence formed by convex combinations of \( (x_n) \) that converges in norm to \( x \). It is proved that a space with the Banach-Saks property must be reflexive but not all reflexive spaces have the Banach-Saks property.

Also, a Banach space \( X \) is said to have the weak Banach-Saks property if every weakly null sequence \( (x_n) \) in \( X \) has a Cesàro convergent subsequence. Note that a Banach space with the Banach-Saks property satisfies the weak Banach-Saks property, and that not all spaces have the weak Banach-Saks property.
We say that an operator \( T : X \rightarrow Y \) is \textit{weak Banach-Saks} if every weakly null sequence \( (x_n) \) in \( X \) has a subsequence such that \( (Tx_n_k) \) is Cesàro convergent. As examples, the identity operator of the Banach lattice \( l^1 \) is weak Banach-Saks but the identity operator of the Banach lattice \( l^\infty \) is not.

On the other hand, let us recall that an operator \( T \) from a Banach lattice \( E \) into a Banach space \( X \) is said to be b-weakly compact whenever \( T \) carries each b-order bounded subset of \( E \) into a relatively weakly compact subset of \( X \). For information on this class of operators see [3], [4], [5].

Note that a b-weakly compact operator is not necessarily weak Banach-Saks. In fact, the identity operator of the Banach lattice \( L^2(c_0) \) is b-weakly compact (because \( L^2(c_0) \) is a KB-space), but it is not weak Banach-Saks (because \( L^2(c_0) \) does not have the weak Banach-Saks property). Conversely, there exists a weak Banach-Saks operator which is not b-weakly compact. In fact, the identity operator of \( c_0 \) is weak Banach-Saks (because \( c_0 \) has the weak Banach-Saks property) but it is not b-weakly compact (because \( c_0 \) is not a KB-space).

The goal of this paper is to characterize Banach lattices on which each \textit{weak Banach-Saks} operator is b-weakly compact. In another paper, we will look at the reciprocal problem. In fact, in this paper we will prove that if \( E \) and \( F \) are two Banach lattices such that the norm of \( E \) is order continuous, then each \textit{weak Banach-Saks} operator \( T : E \rightarrow F \) is b-weakly compact if and only if \( E \) or \( F \) is a KB-space. As consequences, we will obtain some characterizations for KB-spaces. Also, we will characterize Banach lattices under which the second power of each \textit{weak Banach-Saks} operator is b-weakly compact.

To state our results, we need to fix some notation and recall some definitions. Let \( E \) be a vector lattice. For each \( x, y \in E \) with \( x \leq y \), the set \( [x, y] = \{ z \in E : x \leq z \leq y \} \) is called an order interval. A subset of \( E \) is said to be order bounded if it is included in some order interval. Recall that a nonzero element \( x \) of a vector lattice \( E \) is discrete if the order ideal generated by \( x \) equals the subspace generated by \( x \). The vector lattice \( E \) is discrete, if it admits a complete disjoint system of discrete elements.

A Banach lattice is a Banach space \( (E, \| \cdot \|) \) such that \( E \) is a vector lattice and its norm satisfies the following property: for each \( x, y \in E \) such that \( |x| \leq |y| \), we have \( \|x\| \leq \|y\| \). If \( E \) is a Banach lattice, its topological dual \( E' \), endowed with the dual norm, is also a Banach lattice. A norm \( \| \cdot \| \) of a Banach lattice \( E \) is order continuous if for each generalized sequence \( (x_\alpha) \) such that \( x_\alpha \downarrow 0 \) in \( E \), the sequence \( (x_\alpha) \) converges to 0 for the norm \( \| \cdot \| \) where the notation \( x_\alpha \downarrow 0 \) means that the sequence \( (x_\alpha) \) is decreasing, its infimum exists and \( \inf(x_\alpha) = 0 \).

Let us recall that a Banach lattice \( E \) is said to be a KB-space whenever every increasing norm bounded sequence of \( E^+ \) is norm convergent. As an example, each reflexive Banach lattice is a KB-space.
We refer the reader to [1] for unexplained terminology on Banach lattice theory.

2. Main results

We will use the term operator $T: E \to F$ between two Banach lattices to mean a bounded linear mapping. It is positive if $T(x) \geq 0$ in $F$ whenever $x \geq 0$ in $E$. An operator $T: E \to F$ is regular if $T = T_1 - T_2$ where $T_1$ and $T_2$ are positive operators from $E$ into $F$. It is well known that each positive linear mapping on a Banach lattice is continuous. For terminology concerning positive operators, we refer the reader to the excellent book of Aliprantis-Burkinshaw [1].

Recall that the definition of $b$-weakly compact operators is based on the notion of $b$-order bounded subsets. A subset $A$ of a Banach lattice $E$ is called $b$-order bounded if it is order bounded in the topological bidual $E''$. It is clear that every order bounded subset of $E$ is $b$-order bounded. However, the converse is not true in general. A Banach lattice $E$ is said to have the $(b)$-property if $A \subset E$ is order bounded in $E$ whenever it is order bounded in its topological bidual $E''$.

Let $E$ be a Banach lattice and let $X$ be a Banach space. An operator $T: E \to X$ is said to be $b$-weakly compact whenever $T$ carries each $b$-order bounded subset of $E$ into a relatively weakly compact subset of $X$.

It follows from Aliprantis-Burkinshaw ([1], p. 222) that a Banach lattice $E$ is lattice embeddable into another Banach lattice $F$ whenever there exists a lattice homomorphism $T: E \to F$ and there exist two positive constants $K$ and $M$ satisfying

$$K\|x\| \leq \|T(x)\| \leq M\|x\| \quad \text{for all } x \in E.$$ 

$T$ is called a lattice embedding from $E$ into $F$. In this case $T(E)$ is a closed sublattice of $F$ which can be identified with $E$.

Note that each KB-space has the $(b)$-property but a Banach lattice with the $(b)$-property is not necessarily a KB-space. However, by Proposition 2.1 of [3], a Banach lattice $E$ is a KB-space if and only if it has the $(b)$-property and its norm is order continuous.

We note that there exists a Banach lattice with an order continuous norm without the $(b)$-property. In fact, the norm of $c_0$ is order continuous but $c_0$ does not have the $(b)$-property.

On the other hand, the norm of $l^\infty$ is not order continuous and $l^\infty$ has the $(b)$-property, but does not contain a complemented copy of $c_0$.

Before stating our main results, we would like to recall that “$E$ has an order continuous norm” does not imply “$E$ has the weak Banach-Saks property”. In fact, it follows from [12] that $E$ has the Banach-Saks property if, and only if, $E$ has the
weak Banach-Saks property and is reflexive. By way of contradiction, suppose that $E$ is a KB-space implies $E$ has the weak Banach-Saks property. Then every reflexive Banach lattice would have the Banach-Saks property, and this is impossible (because Baernstein’s space is a reflexive Banach lattice without the Banach-Saks property). So, there is an operator which is not weak Banach-Saks, however, $E$ has an order continuous norm. In fact, $L^2(c_0)$ has an order continuous norm, but its identity operator is not weak Banach-Saks.

Also, the class of weak Banach-Saks operators is a two sided ideal of the space of all operators on a Banach lattice.

**Theorem 2.1.** Let $E$ be a Banach lattice with an order continuous norm, and $F$ a Banach lattice. Then the following assertions are equivalent:

(1) Each operator $T: E \to F$ is $b$-weakly compact.

(2) Each weak Banach-Saks operator $T: E \to F$ is $b$-weakly compact.

(3) Each positive weak Banach-Saks operator $T: E \to F$ is $b$-weakly compact.

(4) One of the following assertions holds:

   (a) $E$ is a KB-space.

   (b) $F$ is a KB-space.

**Proof.** (1) $\implies$ (2) Obvious.

(2) $\implies$ (3) Obvious.

(3) $\implies$ (4) By way of contradiction, we suppose that neither $E$ nor $F$ is a KB-space and we construct a positive weak Banach-Saks operator which is not $b$-weakly compact. In fact, since $E$ has an order continuous norm, Proposition 2.1 of [3] implies that $E$ does not have the (b)-property. So it follows from Lemma 2.1 of [7] that the Banach lattice $E$ contains a complemented copy of $c_0$. Denote by $P: E \to c_0$ the positive projection of $E$ in $c_0$ and by $i: c_0 \to E$ the canonical injection of $c_0$ into $E$.

As $F$ is not a KB-space, Theorem 4.61 of [1] implies that $c_0$ is lattice embeddable in $F$, so there exists a lattice embedding $T$ from $c_0$ into $F$. Hence, there exists a constant $K > 0$ such that $\|T((\gamma_n))\| \geq K\|(\gamma_n)\|_{\infty}$ for all $(\gamma_n) \in c_0$. Note that the embedding $T: c_0 \to F$ is not $b$-weakly compact. Otherwise, as the canonical basic $(e_n)$ of $c_0$ is a disjoint $b$-order bounded sequence, it would follow from Proposition 2.8 of [3] that $\lim_n \|T((e_n))\| = 0$, but this is false because $\|T((e_n))\| \geq K\|(e_n)\|_{\infty} = K$ for each $n$.

Now, we consider the composed operator $T \circ P: E \to c_0 \to F$. Since $T \circ P = T \circ \text{Id}_{c_0} \circ P$ and the identity operator $\text{Id}_{c_0}: c_0 \to c_0$ is weak Banach-Saks, hence $T \circ P$ is also weak Banach-Saks. But it is not a $b$-weakly compact operator. Otherwise, the composed operator $T \circ P \circ i$, which is exactly the embedding $T: c_0 \to F$, would be $b$-weakly compact, but this is a contradiction.
Remark. The assumption “E with an order continuous norm” is essential in Theorem 2.1. In fact, each positive operator $T$ from $l^\infty$ into $c_0$ is $b$-weakly compact, but neither $l^\infty$ nor $c_0$ is a KB-space.

As consequences, we obtain the following characterizations of KB-spaces.

**Corollary 2.2.** Let $E$ be a Banach lattice with an order continuous norm. Then the following assertions are equivalent:
1. Each operator $T: E \to E$ is $b$-weakly compact.
2. Each weak Banach-Saks operator $T: E \to E$ is $b$-weakly compact.
3. Each positive weak Banach-Saks operator $T: E \to E$ is $b$-weakly compact.
4. $E$ is a KB-space.

**Corollary 2.3.** Let $E$ be a Banach lattice with an order continuous norm. Then the following assertions are equivalent:
1. Each operator $T: E \to c_0$ is $b$-weakly compact.
2. Each weak Banach-Saks operator $T: E \to c_0$ is $b$-weakly compact.
3. Each positive weak Banach-Saks operator $T: E \to c_0$ is $b$-weakly compact.
4. $E$ is a KB-space.

**Corollary 2.4.** Let $F$ be a Banach lattice. Then the following assertions are equivalent:
1. Each operator $T: c_0 \to F$ is $b$-weakly compact.
2. Each weak Banach-Saks operator $T: c_0 \to F$ is $b$-weakly compact.
3. Each positive weak Banach-Saks operator $T: c_0 \to F$ is $b$-weakly compact.
4. $F$ is a KB-space.

Now, we note that there exists an operator which is weak Banach-Saks but its second power is not $b$-weakly compact. In fact, the identity operator of the Banach lattice $c_0$ is weak Banach-Saks, but its second power which is also the identity operator of $c_0$ is not $b$-weakly compact.

In the next result we give necessary and sufficient conditions under which the second power of each weak Banach-Saks operator is $b$-weakly compact.

**Theorem 2.5.** Let $E$ be a Banach lattice with an order continuous norm. Then the following assertions are equivalent:
(1) For all positive operators $S$ and $T$ from $E$ into $E$ with $0 \leq S \leq T$ and $T$ weak Banach-Saks, $S$ is b-weakly compact.

(2) Each positive weak Banach-Saks operator $T: E \to E$ is b-weakly compact.

(3) For each positive weak Banach-Saks operator $T: E \to E$, the second power $T^2$ is b-weakly compact.

(4) $E$ is a KB-space.

Proof. (1) $\implies$ (2) Let $T: E \to E$ be a positive weak Banach-Saks operator. Since $0 \leq T \leq T$, by our hypothesis $T$ is b-weakly compact.

(2) $\implies$ (3) By our hypothesis $T$ is b-weakly compact and hence $T^2$ is b-weakly compact.

(3) $\implies$ (4) By way of contradiction, suppose that $E$ is not a KB-space. As the norm of $E$ is order continuous, it follows from Proposition 2.4 of [4] and Lemma 2.1 of [7] that $E$ contains a complemented copy of $c_0$, and there exists a positive projection $P: E \to c_0$. Denote by $i: c_0 \to E$ the canonical injection.

Consider the operator $T = i \circ P: E \to c_0 \to E$. Clearly the operator $T$ is weak Banach-Saks (because $T = i \circ \text{Id}_{c_0} \circ P$) but it is not b-weakly compact. Otherwise, the operator $P \circ T \circ i = \text{Id}_{c_0}$ would be b-weakly compact, and this is a contradiction.

Hence, the operator $T^2 = T$ is not b-weakly compact.

(4) $\implies$ (1) Follows from Corollary 2.2 and [3], Corollary 2.9.

Recall from [6] that an operator $T$ from a Banach lattice $E$ into a Banach space $X$ is said to be b-AM-compact if it carries each b-order bounded subset of $E$ into a relatively compact subset of $X$.

Note that each b-AM-compact operator is b-wealy compact but the converse is false in general. In fact, the identity operator of the Banach lattice $L^1[0,1]$ is b-weakly compact (because $L^1[0,1]$ is a KB-space), but it is not b-AM-compact (because $L^1[0,1]$ is not a discrete KB-space). Also, there exists a weak Banach-Saks operator which is not b-AM-compact. In fact, the identity operator of the Banach lattice $c_0$ is weak Banach-Saks but it is not b-AM-compact (because $c_0$ is not a discrete KB-space).

However, we have the following necessary conditions.

Theorem 2.6. Let $E$ be a Banach lattice with an order continuous norm and let $F$ be a Banach lattice. If each positive weak Banach-Saks operator $T: E \to F$ is b-AM-compact, then one of the following assertions holds:

(1) $E$ is a KB-space.

(2) $F$ is a KB-space.
Proof. Suppose that neither $E$ nor $F$ is a KB-space. Consider the same operator $T \circ P$ as that used in the proof of Theorem 2.1. This operator is positive and weak Banach-Saks but it is not b-AM-compact (because it is not b-weakly compact).

Remark. The assumption “$E$ with an order continuous norm” is essential in Theorem 2.6. In fact, each positive operator $T: l^\infty \to c_0$ is b-AM-compact, but neither $l^\infty$ nor $c_0$ is a KB-space.

Remark. The converse of Theorem 2.6 is false, i.e. there exist KB-spaces $E$ and $F$ such that a positive weak Banach-Saks operator $T: E \to F$ is not necessarily b-AM-compact. In fact, it follows from Theorem 5 of [10] that there exists a positive operator $T: L^1[0,1] \to l^\infty$ which is not b-AM-compact. However, the operator $T: L^1[0,1] \to l^\infty$ is weak Banach-Saks and $L^1[0,1]$ is a KB-space. As another example, put $E = L^1[0,1] \oplus l^2$; the identity operator of the Banach lattice $E$ is weak Banach-Saks, but it is not b-AM-compact. However, $E$ is KB-space.

References


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