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On McCoy condition and semicommutative rings

MOHAMED LOUZARI

Abstract. Let $R$ be a ring and $\sigma$ an endomorphism of $R$. We give a generalization of McCoy’s Theorem [Annihilators in polynomial rings, Amer. Math. Monthly 64 (1957), 28–29] to the setting of skew polynomial rings of the form $R[x;\sigma]$. As a consequence, we will show some results on semicommutative and $\sigma$-skew McCoy rings. Also, several relations among McCoyness, Nagata extensions and Armendariz rings and modules are studied.

Keywords: Armendariz rings; McCoy rings; Nagata extension; semicommutative rings; $\sigma$-skew McCoy

Classification: 16S36, 16U80

1. Introduction

Throughout the paper, $R$ will always denote an associative ring with identity and $M_R$ will stand for a right $R$-module. Given a ring $R$, the polynomial ring with an indeterminate $x$ over $R$ is denoted by $R[x]$. According to Nielsen [20] and Rege and Chhawchharia [22], a ring $R$ is called right McCoy (resp., left McCoy) if, for any polynomials $f(x), g(x) \in R[x] \setminus \{0\}$, $f(x)g(x) = 0$ implies $f(x)r = 0$ (resp., $sg(x) = 0$) for some $0 \neq r \in R$ (resp., $0 \neq s \in R$). A ring is called McCoy if it is both left and right McCoy. By McCoy [18], commutative rings are McCoy rings. Recall that a ring $R$ is reversible if $ab = 0$ implies $ba = 0$ for $a, b \in R$, and $R$ is semicommutative if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. It is obvious that commutative rings are reversible and reversible rings are semicommutative, but the converse does not hold, respectively. With the help of [8, Theorem 2.2], $R$ is a McCoy ring when $R[x]$ is semicommutative. Nielsen [20, Theorem 2] showed that reversible rings are McCoy and he gave an example of a semicommutative ring which is not right McCoy. Recall that a ring is reduced if it has no nonzero nilpotent elements. Rege and Chhawchharia called $R$ an Armendariz ring [22, Definition 1.1], if whenever any polynomials $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) = 0$, then $ab = 0$ for each coefficient $a$ of $f(x)$ and $b$ of $g(x)$. Any reduced ring is Armendariz by [2, Lemma 1] and Armendariz rings are clearly McCoy. We have the following diagram:

\[
\begin{align*}
R \text{ is reversible} & \quad \Rightarrow \quad R \text{ is Armendariz} \\
R[x] \text{ is semicommutative} & \quad \quad \Rightarrow \quad R \text{ is McCoy}
\end{align*}
\]
The Ore extension of a ring $R$ is denoted by $R[x; \sigma, \delta]$, where $\sigma$ is an endomorphism of $R$ and $\delta$ is a $\sigma$-derivation, i.e., $\delta: R \to R$ is an additive map such that $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$ for all $a, b \in R$. Recall that elements of $R[x; \sigma, \delta]$ are polynomials in $x$ with coefficients written on the left. Multiplication in $R[x; \sigma, \delta]$ is given by the multiplication in $R$ and the condition $xa = \sigma(a)x + \delta(a)$, for all $a \in R$. For $\delta = 0$, we put $R[x; \sigma, 0] = R[x; \sigma]$. Başer et al. [6], introduced a concept of $\sigma$-skew McCoy for an endomorphism $\sigma$ of $R$. A ring $R$ is called $\sigma$-skew McCoy, if for any nonzero polynomials $p(x) = \sum_{i=0}^{n} a_i x^i$ and $q(x) = \sum_{j=0}^{m} b_j x^j \in R[x; \sigma]$, $p(x)q(x) = 0$ implies $p(x)c = 0$ for some nonzero $c \in R$, and they have proved the following:

$$R[x; \sigma] \text{ is right McCoy } \frac{R[x; \sigma] \text{ is reversible}}{\Rightarrow R \text{ is } \sigma\text{-skew McCoy}}$$

Hong et al. [13, Theorem 1] proved that if $\sigma$ is an automorphism of $R$ and $I$ a right ideal of $S = R[x; \sigma, \delta]$ then $r_S(I) \neq 0$ implies $r_R(I) \neq 0$, which extends McCoy’s Theorem [17].

In this paper, we give another generalization of McCoy’s Theorem, by showing that for any right ideal $I$ of $S = R[x; \sigma]$, we have $r_S(I) \neq 0$ implies $r_R(I) \neq 0$ when $R$ is $\sigma$-compatible or $r_S(I)$ is $\sigma$-ideal. As a consequence, if $R[x; \sigma]$ is semicommutative then $R$ is $\sigma$-skew McCoy. Furthermore, we show some results on Nagata extensions. For a commutative ring $R$, we have

1) If $R$ is a domain, then
   
   (a) $M_R$ is Armendariz if and only if $R \oplus_\sigma M_R$ is Armendariz;
   
   (b) the ring $R \oplus_\sigma M_R$ is semicommutative and right McCoy.

A module $M_R$ is called Armendariz if whenever polynomials $m = \sum_{i=0}^{n} m_i x^i \in M[x]$ and $f = \sum_{j=0}^{m} a_j x^j \in R[x]$ satisfy $mf = 0$, then $m_i a_j = 0$ for each $i, j$.

2) If $R$ and $M_R$ are Armendariz such that $M_R$ satisfies the condition $(C^2_\sigma)$ (see Definition 2.7), then $R \oplus_\sigma M_R$ is Armendariz.

2. A generalization of McCoy’s Theorem

McCoy [17] proved that for any right ideal $I$ of $S = R[x_1, x_2, \ldots, x_n]$ over a ring $R$, if $r_S(I) \neq 0$ then $r_R(I) \neq 0$. This result was extended by Hong et al. [13] to the Ore extensions of several types, the skew monoid rings and the skew power series rings over noncommutative rings, where $\sigma$ is an automorphism of $R$. Herein, we will extend McCoy’s Theorem to skew polynomial rings of the form $R[x; \sigma]$ with $\sigma$ an endomorphism of $R$. According to Annin [3], a ring $R$ is $\sigma$-compatible, if for any $a, b \in R$, $ab = 0$ if and only if $\sigma(a)(b) = 0$. Let $\sigma$ be an endomorphism of $R$ and $I$ an ideal of $R$, we say that the ideal $I$ is $\sigma$-ideal, if $\sigma(I) \subseteq I$. Let $\sigma$ be an endomorphism of a ring $R$, then for any $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x; \sigma]$, we denote by $\sigma(f(x))$ the polynomial $\sum_{i=0}^{n} \sigma(a_i)x^i \in R[x; \sigma]$. 

Theorem 2.1. Let $R$ be a ring, $\sigma$ an endomorphism of $R$ and $I$ a right ideal in $S = R[x; \sigma]$. Suppose that $R$ is $\sigma$-compatible or $r_S(I)$ is $\sigma$-ideal. If $r_S(I) \neq 0$ then $r_R(I) \neq 0$.

Proof: Suppose that $r_S(I) \neq 0$. If $I = 0$, then it's trivial. Assume that $I \neq 0$. Let $g(x) = \sum_{j=0}^{m} b_j x^j \in r_S(I)$ with $b_m \neq 0$. If $m = 0$, then we are done, so we can suppose that $m \geq 1$. In this situation, if $I b_m = 0$, then we are done. Otherwise, there exists $0 \neq f(x) = \sum_{i=0}^{n} a_i x^i \in I$ such that $f(x)b_m \neq 0$. Suppose that $r_S(I) = 0$ and then $0 \neq f(x)g(x) = \sum_{i=0}^{n} a_i x^i \in I$.

If $R$ is $\sigma$-compatible, then $(\ast)$ implies $a_i \sigma^i(b_m) \neq 0$ for some $i \in \{0,1,\ldots,n\}$, so $a_i b_m = 0$ because $R$ is $\sigma$-compatible, therefore $a_i g(x) \neq 0$ for some $i \in \{0,1,\ldots,n\}$. Take $p = \max\{i | a_i g(x) \neq 0\}$, so $a_p g(x) \neq 0$ and $a_{p+1} g(x) = \cdots = a_n g(x) = 0$. On the other hand, we get $a_p b_m = 0$ from $f(x)g(x) = 0$. So that the degree of $a_p g(x)$ is less than $m$ such that $a_p g(x) \neq 0$. But $I(a_p g(x)) = (I a_p) g(x) = 0$ since $I$ is a right ideal of $S$, so $0 \neq a_p g(x) \in r_S(I)$. We can write $a_p g(x) = \sum_{k=0}^{\ell} a_p b_k x^k$ with $a_p b_\ell \neq 0$ and $\ell < m$. We have the two possibilities: If $\ell = 0$ then $a_p g(x)$ is a nonzero element in $r_R(I)$. Otherwise, $\ell \geq 1$. Then we will consider $a_p g(x)$ in place of $g(x)$. We have two cases $I(a_p b_\ell) = 0$ or $I(a_p b_\ell) \neq 0$. The first implies $0 \neq a_p b_\ell \in r_R(I)$, for the second, there exists $h(x) = \sum_{k=0}^{n} c_k x^k \in I$ such that $h(x) a_p b_\ell \neq 0$. Here, we can find $q$ as the largest integer such that $c_q a_p g(x) \neq 0$ and then $0 \neq c_q a_p g(x) \in r_S(I)$ such that the degree of $c_q a_p g(x)$ is smaller than one of $a_p g(x)$.

If $r_S(I)$ is $\sigma$-ideal, then $(\ast)$ implies $a_i x^i b_m = 0$ for some $i \in \{0,1,\ldots,n\}$, therefore $a_i x^i g(x) = 0$. Take $p = \max\{i | a_i x^i g(x) \neq 0\}$, then $a_p \sigma^p(g(x)) \neq 0$ and $a_{p+1} x^i g(x) = 0$ for $i \geq p + 1$. We obtain $a_p \sigma^p(b_m) = 0$ from $f(x)g(x) = 0$. Also, we have $I(a_p \sigma^p(g(x))) = (I a_p) \sigma^p(g(x)) = 0$ because $I$ is a right ideal of $S$ and $\sigma^p(g(x)) \in r_S(I)$. So $0 \neq a_p \sigma^p(g(x)) \in r_S(I)$. We can write $a_p \sigma^p(g(x)) = a_p \sigma^p(b_0) + a_p \sigma^p(b_1) x + \cdots + a_p \sigma^p(b_\ell) x^\ell$, where $a_p \sigma^p(b_\ell) \neq 0$ and $\ell < m$. If $\ell = 0$ then $a_p \sigma^p(b_\ell) = 0$, so $0 \neq a_p \sigma^p(b_\ell) \in r_R(I)$.

Corollary 2.2 ([8, Theorem 2.2]). Let $f(x) \in R[x]$. If $r_R[f(x)R[x]] \neq 0$ then $r_R[f(x)R[x]] \cap R \neq 0$.

Proof: Consider the right ideal $I = f(x)R[x]$.

Corollary 2.3. Let $R$ be a ring, $\sigma$ an endomorphism of $R$ and $I$ a right ideal of $S = R[x; \sigma]$. If $S$ is semicommutative, then $r_S(I) \neq 0$ implies $r_R(I) \neq 0$. 

\[\square\]
Proof: Let $I$ be a right ideal of $S = R[x; \sigma]$, $f(x) \in r_S(I)$ and $g(x) \in I$. Then $g(x)f(x) = 0$. Since $S$ is semicommutative we have $g(x)Sf(x) = 0$, in particular, $g(x)x f(x) = g(x)\sigma(f)(x) = 0$, so $\sigma(f)(x) \in r_S(I)$. Thus $r_S(I)$ is $\sigma$-ideal and we have the result by Theorem 2.1. \hfill $\Box$

Corollary 2.4. Let $\sigma$ be an endomorphism of a ring $R$. If $R[x; \sigma]$ is a semicommutative ring then $R$ is $\sigma$-skew McCoy.

Proof: It follows directly from Corollary 2.3, by letting $I = f(x)R[x; \sigma]$. \hfill $\Box$

From Corollary 2.4, we obtain immediately [6, Corollary 6] and [8, Corollary 2.3]. According to Clark [7], a ring $R$ is said to be quasi-Baer if the right annihilator of each right ideal of $R$ is generated (as a right ideal) by an idempotent. Following Başer et al. [4] and Zhang and Chen [24], a ring $R$ is said to be $\sigma$-semicommutative if, for any $a, b \in R$, $ab = 0$ implies $aR\sigma(b) = 0$. A ring $R$ is called right (left) $\sigma$-reversible [5, Definition 2.1] if whenever $ab = 0$ for $a, b \in R$, $b\sigma(a) = 0$ ($\sigma(b)a = 0$). A ring $R$ is called $\sigma$-reversible if it is both right and left $\sigma$-reversible. Hong et al. [9], proved that, if $R$ is $\sigma$-rigid then $R$ is quasi-Baer if and only if $R[x; \sigma]$ is quasi-Baer. Hong et al. [12] have proved the same result when $R$ is semi-prime and all ideals of $R$ are $\sigma$-ideals.

Proposition 2.5. Let $R$ be a $\sigma$-semicommutative ring. If $R[x; \sigma]$ is quasi-Baer then $R$ is so.

Proof: Let $I$ be a right ideal of $R$. We have $r_{R[x; \sigma]}(IR[x; \sigma]) = eIR[x; \sigma]$ for some idempotent $e = e_0 + e_1x + \cdots + e_mx^m \in R[x; \sigma]$. By [4, Proposition 3.9], $r_R(IR[x; \sigma]) = e_0R$. Clearly, $r_R(IR[x; \sigma]) \subseteq r_R(I)$. Conversely, let $b \in r_R(I)$ then $Ib = 0$. Since $R$ is $\sigma$-semicommutative, we have $IR[x; \sigma]b = 0$, so $b \in r_R(IR[x; \sigma])$. Therefore $r_R(I) = e_0R$. \hfill $\Box$

Example 2.6. Let $Z$ be the ring of integers and consider the ring

$$R = \{(a, b) \in Z \oplus Z \mid a \equiv b \pmod{2}\}$$

and $\sigma: R \rightarrow R$ defined by $\sigma(a, b) = (b, a)$.

1) $R[x; \sigma]$ is quasi-Baer and $R$ is not quasi-Baer, by [9, Example 9].

2) $R$ is not $\sigma$-semicommutative. Let $a = (2, 0), b = (0, 2)$. We have $ab = 0$, but $a\sigma(b) = (2, 0)(2, 0) = (4, 0) \neq 0$. Thus $R$ is not $\sigma$-semicommutative. Therefore the condition “$R$ is $\sigma$-semicommutative” is not a superfluous condition in Proposition 2.5.

Definition 2.7. Let $R$ be a ring, $M_R$ an $R$-module and $\sigma$ an endomorphism of $R$. For $m \in M_R$ and $a \in R$, we say that $M_R$ satisfies the condition ($C^1_\sigma$) (resp., ($C^2_\sigma$)) if $ma = 0$ (resp., $m\sigma(a)a = 0$) implies $m\sigma(a) = 0$. 

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Proposition 2.8. Let $\sigma$ be an endomorphism of a ring $R$.

1. If $R$ is semicommutative and satisfies the condition $(C_3')$ then it is $\sigma$-skew McCoy.

2. If $R$ is reduced and right $\sigma$-reversible then it is $\sigma$-skew McCoy.

Proof: (1) Immediately from [23, Proposition 3.4]. (2) Clearly from (1).

3. Nagata extensions and McCoyness

Let $R$ be a commutative ring, $M_R$ be an $R$-module and $\sigma$ an endomorphism of $R$. The $R$-module $R \oplus_{\sigma} M_R$ acquires a ring structure (possibly noncommutative), where the product is defined by $(a, m)(b, n) = (ab, n\sigma(a) + mb)$, for $a, b \in R$ and $m, n \in M_R$. We shall call this extension the Nagata extension of $R$ by $M_R$ and $\sigma$. If $\sigma = id_R$, then $R \oplus id_R M_R$ (denoted by $R \oplus M_R$) is a commutative ring. Anderson and Camillo [1] have proved that if $R$ is a commutative domain then $M_R$ is Armendariz if and only if $R \oplus M_R$ is Armendariz. We will see that this result holds for $R \oplus_{\sigma} M_R$ as well. Kim et al. [21] have proved that, if $R$ is a commutative domain and $\sigma$ is a monomorphism of $R$ then $R \oplus_{\sigma} R$ is reversible, and so it is McCoy. Recall that if $\sigma$ is an endomorphism of a ring $R$, then the map $R[x] \rightarrow R[x]$ defined by $\sum_{i=0}^{n} a_i x^i \mapsto \sum_{i=0}^{n} \sigma(a_i)x^i$ is an endomorphism of the polynomial ring $R[x]$ and clearly this map extends $\sigma$. We shall also denote the extended map $R[x] \rightarrow R[x]$ by $\sigma$ and the image of $f \in R[x]$ by $\sigma(f)$. In this section, we will discuss when the Nagata extension $R \oplus_{\sigma} M_R$ is McCoy.

Let $R$ be a commutative domain. The set $T(M) = \{m \in M | r_R(m) \neq 0\}$ is called the torsion submodule of $M_R$. If $T(M) = M$ (resp., $T(M) = 0$) then $M_R$ is torsion (resp., torsion-free).

Lemma 3.1. If $M_R$ is a torsion-free module then it is Armendariz.

Proof: Let $m(x) = m_0 + m_1 x + \cdots + m_p x^p \in M[x]$ and $f(x) = a_0 + a_1 x + \cdots + a_q x^q \in R[x]$ such that $m(x)f(x) = 0$. We may assume that $a_0 \neq 0$ (if not, set $f(x) = f'(x)x^k$ with a minimal $k$ such that $a_k \neq 0$). This implies the following system of equations:

\[
\begin{align*}
0 & : m_0a_0 = 0, \\
1 & : m_0a_1 + m_1a_0 = 0, \\
2 & : m_0a_2 + m_1a_1 + m_2a_0 = 0, \\
\vdots & ,
(p + q) & : m_pa_q = 0.
\end{align*}
\]

Since $M_R$ is a torsion-free module, then from these equations, we obtain $m_i = 0$ for all $i \in \{0, 1, \ldots, p\}$. Thus $M_R$ is an Armendariz module.

Proposition 3.2. Let $R$ be a commutative domain and $M_R$ an $R$-module. Then $R \oplus_{\sigma} M_R$ is Armendariz if and only if $M_R$ is Armendariz. In particular, if $M_R$ is torsion-free then $R \oplus_{\sigma} M_R$ is Armendariz.
PROOF: Let $R' = R \oplus_{\sigma} M_R$, then we have $R'[x] = R[x] \oplus_{\sigma} M[x]$. Suppose that $R'$ is Armendariz. Let $m = \sum_{i=0}^{p} m_i x^i \in M[x]$ and $f = \sum_{j=0}^{q} a_j x^j \in R[x]$ with $mf = 0$. We have $(0, m) = \sum_{i=0}^{p} (0, m_i) x^i \in R'[x]$ and $(f, 0) = \sum_{j=0}^{q} (a_j, 0) x^j \in R'[x]$, since $R'$ is Armendariz then $(0, m_i)(a_j, 0) = (0, m_i a_j) = (0, 0)$ for all $i, j$. Thus $m_i a_j = 0$ for all $i, j$. Conversely, suppose that $M_R$ is Armendariz. Let $f, g \in R[x]$ and $m, n \in M[x]$ such that $(f, m)(g, n) = (0, 0)$. Write $(f, m) = \sum (a_i, m_i) x^i \in R'[x]$ and $(g, n) = \sum (b_j, n_j) x^j \in R'[x]$. From $(f, m)(g, n) = (0, 0)$, we have $(fg, n\sigma(f) + mg) = (0, 0)$. Since $R[x]$ is a commutative domain, then $f = 0$ or $g = 0$. If $f = 0$, we get $mg = 0$. Then $m_i b_j = 0$ and $a_i = 0$ for all $i, j$. Thus $(a_i, m_i)(b_j, n_j) = (a_i b_j, n_j \sigma(a_i) + m_i b_j) = (0, 0)$. Otherwise, we get $n\sigma(f) = 0$. Then $b_j = 0$ and $n_j \sigma(a_i) = 0$ for all $i, j$. Thus $(a_i, m_i)(b_j, n_j) = (a_i b_j, n_j \sigma(a_i) + m_i b_j) = (0, 0)$. Therefore $R \oplus_{\sigma} M_R$ is Armendariz. In particular, if $M_R$ is torsion-free then $M_R$ is Armendariz by Lemma 3.1. Therefore $R \oplus_{\sigma} M_R$ is Armendariz.

**Corollary 3.3.** Let $R$ be a commutative domain and $M_R$ an $R$-module satisfying the condition $(C^2_{idR})$. Then $R \oplus_{\sigma} M_R$ is Armendariz.

PROOF: Since $M_R$ is semicommutative then it is Armendariz by [23, Lemma 3.3].

**Proposition 3.4.** Let $R$ be a commutative ring and $M_R$ an $R$-module such that $R$ satisfies $(C_1^1)$ and $M_R$ satisfies $(C_2^2)$. Then $R \oplus_{\sigma} M_R$ is a semicommutative ring.

PROOF: We will use freely the conditions $(C_1^1)$ and $(C_2^2)$. Let $(r, m), (s, n) \in R \oplus_{\sigma} M_R$ such that

\[ (1) \quad (r, m)(s, n) = (rs, n\sigma(r) + ms) = (0, 0). \]

We will show that for any $(t, u) \in R \oplus_{\sigma} M_R$

\[ (2) \quad (r, m)(t, u)(s, n) = (rts, n\sigma(r)t + u\sigma(r)s + mts) = (0, 0). \]

It suffices to show $n\sigma(rt) + u\sigma(r)s + mts = 0$. Multiplying $n\sigma(r) + ms = 0$ of equation (1) on the right hand by $r$, gives $n\sigma(r)r = 0$, so we get $n\sigma(r) = 0$ and hence $ms = 0$. Thus $n\sigma(rt) = mts = 0$. Clearly $rs = 0$ implies $\sigma(r)s = 0$ and so $u\sigma(r)s = 0$. Therefore $n\sigma(rt) + u\sigma(r)s + mts = 0$.

**Proposition 3.5.** Let $R$ be a commutative domain and $M_R$ an $R$-module. Then $R \oplus_{\sigma} M_R$ is a semicommutative right McCoy ring.

PROOF: Consider equations (1) and (2) of Proposition 3.4. From equation (1), we get $r = 0$ or $s = 0$ since $R$ is a domain. Say $r = 0$, then $rts = n\sigma(rt) = u\sigma(r)s = 0$, and $mats = 0$ from (1), hence we have (2). Next say $s = 0$, it follows $rts = u\sigma(r)s = mts = 0$ and $n\sigma(rt) = 0$ from (1), and so we have (2). Therefore $(r, m)(R \oplus_{\sigma} M)(s, n) = 0$. For McCoyness, let $(r, m), (s, n) \in R' = R \oplus_{\sigma} M_R$. Suppose that $(r, m)(s, n)^2 = (rs^2, n\sigma(r^2) + n\sigma(r) + ms^2) = 0$, then $r = 0$ or $s = 0$ which implies $(r, m)(s, n) = (rs, n\sigma(r) + ms) = 0$. Thus by Proposition 2.8(1), $R \oplus_{\sigma} M_R$ is right McCoy.
The next example shows that under the conditions of Proposition 3.5, \( R \oplus_\sigma M_R \) cannot be reversible.

**Example 3.6.** Let \( D \) be a commutative domain and \( R = D[x] \) be the polynomial ring over \( D \) with an indeterminate \( x \). Consider the endomorphism \( \sigma: R \to R \) defined by \( \sigma(f(x)) = f(0) \). Since \( (x,1)(0,1) = (0,0) \) and \( (0,1)(x,1) = (0,x) \neq (0,0) \), then \( R \oplus_\sigma R \) is not reversible. Thus \( R \oplus_\sigma M_R \) cannot be reversible under the conditions of Proposition 3.5.

**Lemma 3.7.** Let \( M_R \) be an Armendariz module, \( m(x) \in M[x] \) and \( f(x), g(x) \in R[x] \) such that \( m(x) = \sum_{i=0}^{n} m_i x^i \), \( f(x) = \sum_{j=0}^{p} a_j x^j \) and \( g(x) = \sum_{k=0}^{q} b_k x^k \). Then

\[
m(x)f(x)g(x) = 0 \iff m_i a_j b_k = 0 \quad \text{for all} \quad i, j, k.
\]

**Proof:** (\( \Leftarrow \)) Clear. (\( \Rightarrow \)) If \( m(x)f(x) = 0 \) then \( m(x)a_j = 0 \) for all \( j \). Now, if \( m(x)f(x)g(x) = 0 \) then \( m(x)[f(x)b_k] = 0 \) for all \( k \). Since \( M_R \) is Armendariz we have \( m_i(a_j b_k) = 0 \) for all \( i, j, k \). Thus \( m_i a_j b_k = 0 \) for all \( i, j, k \).

**Lemma 3.8.** If \( M_R \) is an Armendariz module satisfying the condition \((C^2_\sigma)\). Then \( M[x]_{R[x]} \) satisfies the condition \((C^2_\sigma)\).

**Proof:** Let \( m(x) = \sum_{i=0}^{n} m_i x^i \in M[x] \) and \( f(x) = \sum_{j=0}^{p} a_j x^j \in R[x] \). Suppose that \( m(x)f(x)g(x) = 0 \). By Lemma 3.7, \( m_i(a_j b_k) = 0 \) for all \( i, j, k \). In particular, \( m_i(a_j b_k) = 0 \) for all \( i, j \). Then \( m_i a_j b_k = 0 \) for all \( i, j \). Therefore \( m(x)f(x)g(x) = 0 \).

**Theorem 3.9.** Let \( R \) be a commutative Armendariz ring, \( \sigma \) an endomorphism of \( R \) and \( M_R \) a module satisfying the condition \((C^2_\sigma)\). Then \( M_R \) is Armendariz if and only if \( R \oplus_\sigma M_R \) is Armendariz.

**Proof:** Let \( f, g \in R[x] \) and \( m, n \in M[x] \) such that \((f,m)(g,n) = (0,0)\). Write \((f,m) = \sum (a_i, m_i) x^i \in R'[x] \) and \((g,n) = \sum (b_j, n_j) x^j \in R'[x] \). From \((f,m)(g,n) = (0,0)\), we have \((fg, n\sigma(f) + mg) = (0,0)\). Since \( R \) is Armendariz, then \( a_i b_j = 0 \) for all \( i, j \). Multiplying \( n\sigma(f) + mg = 0 \) on the right by \( f \). By Lemma 3.8, we have \( n\sigma(f)f = 0 \), then \( n\sigma(f) = 0 \) and so \( mg = 0 \). Since \( M_R \) is Armendariz we have \( m_i b_j = 0 \) and \( n_i \sigma(a_j) = 0 \) for all \( i, j \). Thus \((a_i, m_i)(b_j, n_j) = (a_i b_j, n_j \sigma(a_i) + m_i b_j) = (0,0)\). Therefore \( R' \) is Armendariz. The converse is clear.

**Corollary 3.10.** If \( R \) is a commutative reduced ring which satisfies the condition \((C^1_\sigma)\) then \( R \oplus_\sigma R \) is semicommutative and Armendariz.

**Proof:** Immediately by Proposition 3.4 and Theorem 3.9.

**Example 3.11.** Consider the ring \( R = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) with the usual addition and multiplication. Let \( \sigma: R \to R \) be defined by \( \sigma(a,b) = (b,a) \). Clearly \( R \) is a commutative reduced ring but not a domain. Let \( A = ((0,1),(0,1)) \), \( B = ((1,0),(0,1)) \) and \( C = ((1,0),(1,0)) \). We have

\[
AB = (((0,1),(0,1))((1,0),(0,1)) = ((0,0),((0,1)\sigma(0,1) + (0,1)(1,0))) = 0.
\]
But
\[ ACB = ((0, 1), (0, 1))((1, 0), (1, 0))((1, 0), (0, 1)) = ((0, 0), (1, 0))((1, 0), (0, 1)) = ((0, 0), (1, 0)) \neq 0. \]

Hence \( R \oplus \sigma R \) is not semicommutative. On other hand, we have \((1, 0)(0, 1) = 0, \) but \((1, 0)\sigma((0, 1)) = (1, 0)(1, 0) = (1, 0) \neq 0, \) so \( R \) does not satisfy the condition \((C^1_\sigma). \) Thus the condition \((C^1_\sigma) \) in Corollary 3.10 is not superfluous.

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References

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