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THE METHOD OF INFINITE ASCENT APPLIED ON $A^4 \pm nB^3 = C^2$

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Abstract. Each of the Diophantine equations $A^4 \pm nB^3 = C^2$ has an infinite number of integral solutions (A, B, C) for any positive integer n. In this paper, we will show how the method of infinite ascent could be applied to generate these solutions. We will investigate the conditions when A, B and C are pair-wise co-prime. As a side result of this investigation, we will show a method of generating an infinite number of co-prime integral solutions (A, B, C) of the Diophantine equation $aA^3 + cB^3 = C^2$ for any co-prime integer pair (a, c).

Keywords: method of infinite ascent, Diophantine equation $A^4 \pm nB^3 = C^2$

MSC 2010: 11D41, 11D72

1. Introduction

Sometimes we are interested in finding integral solutions of polynomial equations with integral coefficients. They are called the Diophantine equations, named after Diophantus of Alexandria, a third century mathematician who studied them extensively. The study of Diophantine equations is one of the most fascinating subjects of Number Theory, the queen of Mathematics. The paper [2] gives a technique of generating an infinite number of co-prime integral solutions (A, B, C) of the Diophantine equations $A^4 \pm nB^2 = C^3$ for any integer $n \geqslant 1$. The Diophantine equation (1.1) has been studied by Beukers [1] only for $n = \pm 1$. But the Diophantine equations (1.1) are yet to be studied for all integral values of $n \geqslant 2$. Hence, let us consider the following Diophantine equations:

$$A^4 \pm nB^3 = C^2 \tag{1.1}$$

We can apply the Method of Infinite Ascent (MIA) to these equations (1.1) to generate an infinite number of integral solutions (A, B, C) for any positive integer n. For applying MIA to a Diophantine equation, we have to look for a polynomial algebraic identity that resembles the Diophantine equation under study. Hence, we prove Lemma 1.1, that gives us the base to construct two polynomial identities, (2.5) and (2.6), to be used later for establishing the main results of this paper.

Lemma 1.1. *For any two non-zero integers k and t,*

$$k\{4(k-t)\}^3 + t(8k+t)^3 = \{\pm(8k^2 + 20kt - t^2)\}^2. \tag{1.2}$$

P r o o f.

$$\begin{aligned}
\text{L. H. S. of (1.2)} &= k\{4(k-t)\}^3 + t(8k+t)^3 \\
&= 64k\{(k)^3 + 3(k)^2(-t) + 3(k)(-t)^2 + (-t)^3\} \\
&\quad + t\{(8k)^3 + 3(8k)^2(t) + 3(8k)(t)^2 + (t)^3\} \\
&= 64k\{k^3 - 3k^2t + 3kt^2 - t^3\} + t\{512k^3 + 192k^2t + 24kt^2 + t^3\} \\
&= 64k^4 - 192k^3t + 192k^2t^2 - 64kt^3 + 512k^3t + 192k^2t^2 + 24kt^3 + t^4 \\
&= 64k^4 + (-192 + 512)k^3t + (192 + 192)k^2t^2 + (-64 + 24)kt^3 + t^4 \\
&= 64k^4 + 320k^3t + 384k^2t^2 - 40kt^3 + t^4 \\
&= 64k^4 + 320k^3t + (400 - 16)k^2t^2 - 40kt^3 + t^4 \\
&= 64k^4 + 400k^2t^2 + t^4 + 320k^3t - 40kt^3 - 16k^2t^2 \\
&= (8k^2)^2 + (20kt)^2 + (-t^2)^2 + 2(8k^2)(20kt) + 2(20kt)(-t^2) \\
&\quad + 2(8k^2)(-t^2) \\
&= \{\pm(8k^2 + 20kt - t^2)\}^2 = \text{R. H. S. of (1.2)}.
\end{aligned}$$

Hence, Lemma 1.1 is established. □

2. Constructing solutions for $A^4 \pm nB^3 = C^2$

Take

$$X = k\{4(k-t)\}^3, \quad Y = t(8k+t)^3 \quad \text{and} \quad Z = \{\pm(8k^2 + 20kt - t^2)\}^2. \tag{2.1}$$

Comparing (2.1) with (1.2) we get

$$X + Y = Z. \tag{2.2}$$

We know that

$$\{\pm(X+Y)\}^2-4XY=\{\pm(X-Y)\}^2. \tag{2.3}$$

From (2.2) and (2.3) we get

$$(\pm Z)^2-4XY=\{\pm(X-Y)\}^2. \tag{2.4}$$

From (2.4) and (2.1) we get

$$\begin{aligned}
&\{\pm(8k^2+20kt-t^2)\}^4-4k\{4(k-t)\}^3\cdot t(8k+t)^3\\
&\qquad=\{\pm(k(4(k-t))^3-t(8k+t)^3)\}^2\\
&\Longrightarrow \{\pm(8k^2+20kt-t^2)\}^4-2^8\cdot kt\{(k-t)(8k+t)\}^3\\
&\qquad=\{\pm(k(4(k-t))^3-t(8k+t)^3)\}^2\\
&\Longrightarrow \{\pm(8k^2+20kt-t^2)\}^4-2^8\cdot kt(8k^2+kt-8kt-t^2)^3\\
&\qquad=\{\pm(4^3\cdot k(k-t)^3-t(8k+t)^3)\}^2\\
&\Longrightarrow \{\pm(8k^2+20kt-t^2)\}^4-2^8\cdot kt(8k^2-7kt-t^2)^3\\
&\qquad=\{\pm(4^3\cdot k(k^3-3k^2t+3kt^2-t^3)-t(8^3k^3+3\cdot 8^2k^2t+3\cdot 8kt^2+t^3))\}^2\\
&\Longrightarrow \{\pm(8k^2+20kt-t^2)\}^4-2^8\cdot kt(8k^2-7kt-t^2)^3\\
&\qquad=\{\pm(64\cdot k(k^3-3k^2t+3kt^2-t^3)-t(512k^3+192k^2t+24kt^2+t^3))\}^2\\
&\Longrightarrow \{\pm(8k^2+20kt-t^2)\}^4-2^8\cdot kt(8k^2-7kt-t^2)^3\\
&\qquad=\{\pm(64k^4-192k^3t+192k^2t^2-64kt^3-512k^3t-192k^2t^2-24kt^3-t^4)\}^2\\
&\Longrightarrow \{\pm(8k^2+20kt-t^2)\}^4-2^8\cdot kt(8k^2-7kt-t^2)^3\\
&\qquad=\{\pm(64k^4+(-192-512)k^3t+(192-192)k^2t^2+(-64-24)kt^3-t^4)\}^2\\
&\Longrightarrow \{\pm(8k^2+20kt-t^2)\}^4-2^8\cdot kt(8k^2-7kt-t^2)^3\\
&\qquad=\{\pm(64k^4-704k^3t-88kt^3-t^4)\}^2.
\end{aligned}$$

Hence, we get an important polynomial algebraic identity (2.5):

$$\begin{aligned}\{\pm(8k^2+20kt-t^2)\}^4-2^8\cdot kt(8k^2-7kt-t^2)^3&\\=\{\pm(64k^4-704k^3t-88kt^3-t^4)\}^2.&\end{aligned} \tag{2.5}$$

Transforming t to $-t$, the identity of (2.5) becomes

$$\begin{aligned}\{\pm(8k^2-20kt-t^2)\}^4+2^8\cdot kt(8k^2+7kt-t^2)^3&\\=\{\pm(64k^4+704k^3t+88kt^3-t^4)\}^2.&\end{aligned} \tag{2.6}$$

Now, we state the following two theorems.

Theorem 2.1. *For any positive integer n, the Diophantine equation $A^4+nB^3=C^2$ has infinitely many co-prime integral solutions of the form*

$$(A,B,C)=\{\pm(32n^2p^6-40np^3q^3-q^6),8pq(32n^2p^6+14np^3q^3-q^6),\\ \pm(1024n^4p^{12}+5632n^3p^9q^3+176np^3q^9-q^{12})\},\tag{2.7}$$

where $npq\neq 0$, np and q are co-prime integers such that p takes any integral value having 3 as a factor and q is odd.

Proof. We will prove Theorem 2.1 in two steps.

Step I. According to the statement of Theorem 2.1, if

$$A^4+nB^3=C^2,\tag{2.8}$$

then, we have to show that

$$\{\pm(32n^2p^6-40np^3q^3-q^6)\}^4+n\{8pq(32n^2p^6+14np^3q^3-q^6)\}^3\\ =\{\pm(1024n^4p^{12}+5632n^3p^9q^3+176np^3q^9-q^{12})\}^2.\tag{2.9}$$

Put $k=2np^3$ and $t=q^3$ in (2.6). We get

$$\begin{aligned}&\{\pm(8(2np^3)^2-20(2np^3)(q^3)-(q^3)^2)\}^4\\ &\quad+2^8\cdot(2np^3)(q^3)\{8(2np^3)^2+7(2np^3)(q^3)-(q^3)^2\}^3\\ &\quad=\{\pm(64(2np^3)^4+704(2np^3)^3(q^3)+88(2np^3)(q^3)^3-(q^3)^4)\}^2\\ &\Longrightarrow\{\pm(32n^2p^6-40np^3q^3-q^6)\}^4+2^9np^3q^3(32n^2p^6+14np^3q^3-q^6)^3\\ &\quad=\{\pm(1024n^4p^{12}+5632n^3p^9q^3+176np^3q^9-q^{12})\}^2\\ &\Longrightarrow\{\pm(32n^2p^6-40np^3q^3-q^6)\}^4+n\{8pq(32n^2p^6+14np^3q^3-q^6)\}^3\\ &\quad=\{\pm(1024n^4p^{12}+5632n^3p^9q^3+176np^3q^9-q^{12})\}^2,\end{aligned}$$

which establishes (2.9).

Step II. We have to show that the given integral solutions of (2.8) are pair-wise co-prime.

$A=(32n^2p^6-40np^3q^3-q^6)$, which is odd, because q is odd. $B=8pq(32n^2p^6+14np^3q^3-q^6)$ is always even and $C=(1024n^4p^{12}+5632n^3p^9q^3+176np^3q^9-q^{12})$ is always odd. Since np and q are given to be co-prime, also p and q are co-prime.

Now, $A=(32n^2p^6-40np^3q^3-q^6)$ and $8pq$ do not have a common factor >1, because A does not share a factor with 8, p or q. Take $B=8pqB_1$, where $B_1=(32n^2p^6+14np^3q^3-q^6)$. Hence, to show that B and A are co-prime, we need to prove that $B_1=(32n^2p^6+14np^3q^3-q^6)$ and $A=(32n^2p^6-40np^3q^3-q^6)$ do not have a common factor > 1. Suppose that B_1 and A share a common factor $k > 1$. So, k must be a factor of $(B_1 - A) = \{(32n^2p^6 + 14np^3q^3 - q^6) - (32n^2p^6 - 40np^3q^3 - q^6)\} = (32n^2p^6 + 14np^3q^3 - q^6 - 32n^2p^6 + 40np^3q^3 + q^6) = 54np^3q^3 = 2 \cdot 3^3 \cdot p^3q^3$. But k, being a factor of A, is not divisible by 2. Also 3, being a factor of p, will not appear as a factor of A because p and q are co-prime. Hence, we see that k has to be a factor of both A and np^3q^3, which leads to a contradiction because A and np^3q^3 do not have a common factor > 1. So, under the conditions of Theorem 2.1, A and B are co-prime. Since A and n do not share a common factor > 1, A and nB are co-prime. Knowing that A and nB are co-prime, from equation (2.8) we see that (A, B, C) are pair-wise co-prime. Thus, combining Step I and Step II, Theorem 2.1 is proved. □

Theorem 2.2. *For any positive integer n, the Diophantine equation $A^4 - nB^3 = C^2$ has infinitely many co-prime integral solutions of the form*

$$(A, B, C) = \{\pm(32n^2p^6 + 40np^3q^3 - q^6), 8pq(32n^2p^6 - 14np^3q^3 - q^6), \\ \pm(1024n^4p^{12} - 5632n^3p^9q^3 - 176np^3q^9 - q^{12})\}, \tag{2.10}$$

where $npq \neq 0$, np and q are co-prime integers such that p takes any integral value having 3 as a factor and q is odd.

Proof. We will prove Theorem 2.2 in two steps.

Step I. According to the statement of Theorem 2.2, if

$$A^4 - nB^3 = C^2, \tag{2.11}$$

then we have to show that

$$\{\pm(32n^2p^6 + 40np^3q^3 - q^6)\}^4 - n\{8pq(32n^2p^6 - 14np^3q^3 - q^6)\}^3 \\ = \{\pm(1024n^4p^{12} - 5632n^3p^9q^3 - 176np^3q^9 - q^{12})\}^2. \tag{2.12}$$

Put $k = 2np^3$ and $t = q^3$ in (2.5) and proceed as we did for Theorem 2.1 to get the identity (2.12).

Step II. We can use the same logic as we did for Theorem 2.1 to prove that the solutions (A, B, C) of the Diophantine equation (2.11) are be pair-wise co-prime.

Hence, combining Step I and Step II, Theorem 2.2 is proved completely. □

3. Conclusion

The algebraic identity (1.2) is a Corollary of a Theorem given in [3]. It can be used to give a new result relating to the Diophantine equation $aA^3 + cB^3 = C^2$ as follows.

Theorem 3.1. *For any two non-zero co-prime integers a and c, the Diophantine equation $aA^3+cB^3=C^2$ has infinitely many co-prime integral solutions of the form*

$$(A,B,C)=\{4b(ab^3-cd^3), d(8ab^3+cd^3), (8a^2b^6+20acb^3d^3-c^2d^6)\}, \tag{3.1}$$

where b and d take non-zero integral values such that $3\mid b$, $ab^3\neq cd^3$, ab and cd are co-prime, with cd having an odd value.

P r o o f. We will prove Theorem 3.1 in two steps.

Step I. According to the statement of Theorem 3.1, if

$$aA^3+cB^3=C^2, \tag{3.2}$$

then we have to show that

$$a\{4b(ab^3-cd^3)\}^3+c\{d(8ab^3+cd^3)\}^3=(8a^2b^6+20acb^3d^3-c^2d^6)^2. \tag{3.3}$$

We can easily obtain the identity (3.3) if we substitute $k=ab^3$ and $t=cd^3$ in (1.2). Now, for any pair of integers (a,c), we change the values for b and d in accordance with the given conditions to generate an infinite number of solutions for (A,B,C). This proves one part of the Theorem 3.1.

Step II. To ensure that the Diophantine equation (3.2) has co-prime solutions (A,B,C) we note that $A=4b(ab^3-cd^3)$. Since $ab^3\neq cd^3$, A is always non-zero and even. Similarly, $B=d(8ab^3+cd^3)$ is odd because cd is odd, and hence, non-zero. Since ab and cd are co-prime, the factor $4b$ in A is co-prime to B. The other factor in A is (ab^3-cd^3), which is co-prime to d.

Since $3\mid b$, 3 is neither a factor of (ab^3-cd^3) nor $(8ab^3+cd^3)$ because ab and cd are co-prime. The sum of (ab^3-cd^3) and $(8ab^3+cd^3)$ is $9ab^3$, which contains factors different from (ab^3-cd^3) as well as $(8ab^3+cd^3)$. So, we observe that the constituent factors of A are co-prime to the constituent factors of B. Thus, A and B are proved to be co-prime. Noting that aA^3 and cB^3 are co-prime, from equation (3.2) we see that (A,B,C) are pair-wise co-prime.

Combining Step I and Step II, Theorem 3.1 is fully established. □

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