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TOTALLY REFLEXIVE MODULES WITH RESPECT TO A SEMIDUALIZING BIMODULE

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Abstract. Let S and R be two associative rings, let ${}_SC_R$ be a semidualizing (S,R)-bimodule. We introduce and investigate properties of the totally reflexive module with respect to ${}_SC_R$ and we give a characterization of the class of the totally C_R-reflexive modules over any ring R. Moreover, we show that the totally C_R-reflexive module with finite projective dimension is exactly the finitely generated projective right R-module. We then study the relations between the class of totally reflexive modules and the Bass class with respect to a semidualizing bimodule. The paper contains several results which are new in the commutative Noetherian setting.

Keywords: semidualizing bimodule, totally reflexive module, Bass class, precover, preenvelope

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Introduction

In 1967, Auslander [1] introduced the *Gorenstein dimension*, or G-dimension for finitely generated modules, and the finer details were developed in his joint paper [2] with Bridger. The G-dimension is a relative homological dimension and Christensen [4] studied the modules that serve as building blocks in the resolutions, which were called modules in the G-class by Auslander [1] and [2]. In 1995, Yassemi [22] studied Gorenstein dimensions for complexes and showed the possibility of defining the G-dimension with respect to a semidualizing complex C. The study of semidualizing modules goes back at least to Vasconcelos [19] who calls them spherical modules. This module is a PG-module, which was defined by Foxby in [7] as a generalization

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of a projective module and a Gorenstein module. A dualizing module is always a semidualizing module. Relative homological algebra with respect to a semidualizing module has caught many authors' attention. For this topic, we refer the reader to see Holm and White's work [12], but also to [10], [15], [16], [17]. In [8], Golod introduced the totally C-reflexive module with respect to a semidualizing module C over a commutative Noetherian ring, and the homological dimension which arises by resolving a given finitely generated module by totally C-reflexive modules is known as the G_C-dimension of a finitely generated module. In the case $C = R$, totally C-reflexive modules are exactly the modules in the G-class. Hence studying the totally C-reflexive modules is very useful; for this we refer the readers to [14].

On the other hand, Holm and White [12] extended the notion of semidualizing modules to the associative ring, where they defined the semidualizing (S, R) bimodule ${}_SC_R$ for any associative rings R and S (see Definition 1.3), and the Auslander class and Bass class with respect to ${}_SC_R$. Araya, Takahashi and Yoshino [3, Definition 2.1] defined totally C_R-reflexive modules with respect to a semidualizing (S, R)-bimodule ${}_SC_R$ over any associative rings S and R, which extends Golod's notion of totally C-reflexive modules with respect to a semidualizing module C to the non-commutative non-Noetherian setting and generalizes the modules in the G-class within this setting. In this paper, we denote the class of all totally C_R-reflexive modules by $\mathcal{T}_C(R)$ (see Definition 2.1), and we show that many conclusions over a commutative Noetherian ring also hold in an associative ring. Moreover, we show several results which are new in the commutative Noetherian setting.

Section 2 is devoted to the study of the totally reflexive modules with respect to a semidualizing bimodule ${}_SC_R$. We get the following result about the class $\mathcal{T}_C(R)$ over any ring R, see Theorem 2.3, and for the notation see Section 1:

$$\mathcal{T}_C(R) = \operatorname{gen}^*(R_R) \cap \operatorname{cog}^*(C_R) \cap {}^{\perp}(C_R).$$

Additionally, we show that when $M \cong \operatorname{Hom}_S(N, C)$ for some finitely generated left S-module N, then M is totally C_R-reflexive if and only if $\operatorname{Hom}_R(M, C)$ is totally ${}_SC$-reflexive, see Corollary 2.7. Moreover, we investigate the $\mathcal{T}_C$-dimension and the $\mathcal{T}_C(R)$-precover (and preenvelope) for a finitely generated right R-module M with degreewise finitely generated projective resolution, see Proposition 2.8.

On the other hand, recall that $\operatorname{Add}(X_R)$ ($\operatorname{add}(X_R)$) denotes the class of right R-modules M which is a direct summand of a (finite) direct sum of copies of X_R. Particularly, $\operatorname{Add}(R_R)$ is the class of all projective right R-modules and $\operatorname{add}(R_R)$ is the class of all finitely generated projective right R-modules. It is proved in Corollary 2.4 and Remark 2.2(1) that both $\operatorname{add}(C_R)$ and $\operatorname{add}(R_R)$ are all contained in the class $\mathcal{T}_C(R)$ (see Definition 2.1), and the totally C_R-reflexive modules with finite

$\mathrm{add}(C_R)$-projective dimensions must be contained in $\mathrm{add}(C_R)$, see Observation 2.10. It is natural to ask whether a totally C_R-reflexive module with finite projective dimension must be in $\mathrm{add}(R_R)$? The affirmative answer is shown in the following theorem (Theorem 2.11), and it answers a special case of the question put forward by D. White in [21, Question 2.15], i.e., when the semidualizing bimodule ${}_SC_R$ is faithful, White's conjecture is true for the right R-modules with degreewise finitely generated projective resolutions over any rings R and S.

Theorem 2.11. *Let ${}_SC_R$ be faithfully semidualizing (see Definition 1.3), and $M_R \in \mathcal{T}_C(R)$. If $\mathrm{pd}_R M < \infty$, then M is finitely generated projective.*

In Section 3, motivated by the work of Mantese and Reiten [13], we show that there exist some relations between the classes $\mathcal{T}_C(R)$ and $\mathcal{B}_C(R)$ (see Definition 1.4).

Theorem 3.2. *Let ${}_SC_R$ be faithfully semidualizing. Denote by $\mathcal{P}_R^{<\infty}$ the class of right R-modules which are in $\mathrm{gen}^*(R_R)$ (see Section 1) and have finite projective dimensions. Then*
(1) ${}^{\perp}\mathcal{B}_C(R) \cap \mathrm{gen}^*(R_R) \subseteq \mathcal{T}_C(R)$ *and* ${}^{\perp}\mathcal{B}_C(R) \cap \mathcal{P}_R^{<\infty} = \mathcal{T}_C(R) \cap \mathcal{P}_R^{<\infty}$;
(2) $\mathcal{T}_C(R)^{\perp} \subseteq \mathcal{B}_C(R)$.

Throughout this paper, R and S are always two associative rings and ${}_SC_R$ is always a semidualizing (S,R)-bimodule, see Definition 1.3. A subcategory or a class of right R-modules (left S-modules) is a full subcategory of the category of right R-modules (left S-modules), which is closed under isomorphisms. For unexplained concepts and notation, we refer the reader to [13], [20], [14].

1. Preliminaries

In this section, we recall a number of notions and results which will be used throughout this work. First, we employ some notions used by S. Sather-Wagstaff, T. Wakamatsu and D. White in [14], [20], [21].

Definition 1.1. Let $\mathcal{X}$ be a class of right R-modules and M_R a right R-module. A left $\mathcal{X}$-*resolution* of M_R is an exact sequence of right R-modules $\mathbf{X} = \ldots \to X_1 \to X_0 \to M \to 0$ with each $X_i \in \mathcal{X}$. The right $\mathcal{X}$-*resolution* of M_R is defined dually.

The $\mathcal{X}$-*projective dimension* of M_R is the quantity

$$\mathcal{X}\text{-}\mathrm{pd}_R(M) = \inf\{\sup\{n \geqslant 0\colon X_n \neq 0\}\colon \mathbf{X} \text{ is a left } \mathcal{X}\text{-resolution of } M_R\}.$$

Particularly, we denote by $\mathrm{pd}_R M$ the projective dimension of a right R-module M_R.

Denote by $\widehat{\mathcal{X}}$ the class of right R-modules with finite $\mathcal{X}$-projective dimension.

We denote by ${}^{\perp}\mathcal{X}$ the subcategory of right R-modules M such that $\operatorname{Ext}^i_R(M,X)=0$ for all $i\geqslant 1$ and all $X\in\mathcal{X}$ and similarly, $\mathcal{X}^{\perp}=\{M\colon\ \operatorname{Ext}^i_R(X,M)=0$ for all $i\geqslant 1$ and all $X\in\mathcal{X}\}$.

Definition 1.2 [15, Definition 1.6]**.** Let $\mathcal{X}$ be the class of right R-modules. For a right R-module M, an $\mathcal{X}$-*precover* of M is a right R-module homomorphism $\varphi\colon\ X\to M$ where $X\in\mathcal{X}$ is such that, for each $X'\in\mathcal{X}$, the homomorphism $\operatorname{Hom}_R(X',\varphi)\colon \operatorname{Hom}_R(X',X)\to\operatorname{Hom}_R(X',M)$ is surjective. The term *preenvelope* is defined dually.

Following [6, Definition 7.1.6], an $\mathcal{X}$-precover φ of M is called *special* provided that the sequence $0\to L\to A\xrightarrow{\varphi}M\to 0$ of right R-modules with $A\in\mathcal{X}$ is exact and $L\in\mathcal{X}^{\perp}$. The term *special preenvelope* is defined dually.

Holm and White [12, Definition 2.1] extended the definition of semidualizing modules to associative rings. They also defined faithfully semidualizing bimodules over non-commutative rings, i.e., a semidualizing bimodule ${}_SC_R$ is *faithfully semidualizing* if $\operatorname{Hom}_S(C,N)=0$ implies $N=0$ and $\operatorname{Hom}_{R^{op}}(C,M)=0$ implies $M=0$ for all modules ${}_SN$ and M_R, see [12, Definition 3.1], and they showed that if R is commutative, then a semidualizing module is always faithfully semidualizing, see [12, Proposition 3.1].

Definition 1.3 [12, Definition 2.1]**.** An (S,R)-bimodule $C={}_SC_R$ is called *semidualizing* if

(1) ${}_SC$ admits a degreewise finitely generated S-projective resolution;
(2) C_R admits a degreewise finitely generated R-projective resolution;
(3) the natural homothety map ${}_SS_S\to\operatorname{Hom}_R(C,C)$ is an isomorphism;
(4) the natural homothety map ${}_RR_R\to\operatorname{Hom}_S(C,C)$ is an isomorphism;
(5) $\operatorname{Ext}^{\geqslant 1}_R(C,C)=0=\operatorname{Ext}^{\geqslant 1}_S(C,C)$.

Holm and White [12] defined the Bass class $\mathcal{B}_C(S)$ with respect to the semidualizing module ${}_SC_R$ over any rings R and S.

Definition 1.4 [12]**.** The Bass class $\mathcal{B}_C(R)$ with respect to ${}_SC_R$ consists of all right R-modules N satisfying

(1) $\operatorname{Ext}^i_R(C,N)=0$ for all $i\geqslant 1$,
(2) $\operatorname{Tor}^S_i(\operatorname{Hom}_R(C,N),C)=0$ for all $i\geqslant 1$,
(3) the natural evaluation homomorphism $\nu_N\colon\ \operatorname{Hom}_R(C,N)\otimes_S C\to N$ is an isomorphism.

Remark 1.5. Recall that $\mathcal{B}_C(R)$ are closed under direct products and direct sums. By [12, Proposition 4.2] we know that $\mathcal{B}_C(R)$ is also closed under direct summands and direct limits. Moreover, by [12, Corollary 6.3], if ${}_SC_R$ is a faithfully semidualizing bimodule, $\mathcal{B}_C(R)$ has the property that if two modules in a short exact sequence are in $\mathcal{B}_C(R)$, so is the third.

The following lemma is used frequently in this paper, so we present it here and give the proof.

Lemma 1.6. *Let ${}_SC_R$ be a semidualizing bimodule. Then*
(1) $\mathrm{Add}(C_R) = \{P \otimes_S C\colon P_S \in \mathrm{Add}(S_S)\} = \mathcal{P}_C(R)$ *and* $\mathrm{add}(C_R) = \{Q \otimes_S C\colon Q_S \in \mathrm{add}(S_S)\}$;
(2) $\mathrm{Hom}_R(P, C) \in \mathrm{add}({}_SC)$ *for all* $P \in \mathrm{add}(R_R)$ *and* $\mathrm{Hom}_R(C_i, C) \in \mathrm{add}({}_SS)$ *for all* $C_i \in \mathrm{add}(C_R)$.

Proof. (1) Let P_S be a projective right S-module. Then there exists a projective right S-module P'_S such that $P \oplus P' = S^{(I)}$ for some index set I, and so $(P \otimes_S C) \oplus (P' \otimes_S C) \cong S^{(I)} \otimes_S C \cong C^{(I)} \in \mathrm{Add}(C_R)$.

Conversely, let $M_R \in \mathrm{Add}(C_R)$, then there exists a right R-module N such that $M \oplus N = C^{(J)}$ for some index set J. Since $C^{(J)} \in \mathcal{B}_C(R)$ and $\mathcal{B}_C(R)$ is closed under direct summands by Remark 1.5, we have that $M \in \mathcal{B}_C(R)$. Thus $M \cong \mathrm{Hom}_R(C, M) \otimes_S C$. On the other hand, $\mathrm{Hom}_R(C, M) \oplus \mathrm{Hom}_R(C, N) \cong \mathrm{Hom}_R(C, C^{(J)}) \cong S^{(J)}$, which implies that $\mathrm{Hom}_R(C, M)$ is S-projective. In the same way we can prove that $\mathrm{add}(C_R) = \{Q \otimes_S C\colon Q_S \in \mathrm{add}(S_S)\}$.

(2) For a semidualizing bimodule ${}_SC_R$, we have that $\mathrm{Hom}_R(C, C) \cong S$ and $\mathrm{Hom}_S(C, C) \cong R$. Thus the result is easy to prove. □

At last, we recall notation used in [20]. Let X_R be a right R-module. We denote by $\mathrm{cog}^*(X_R)$ the class of right R-modules M_R which admits an exact sequence: $0 \to M \to X_0 \to X_1 \to \ldots$ such that $X_i \in \mathrm{add}\, X_R$ and the sequence is $\mathrm{Hom}_R(-, X)$-exact. Dually, $\mathrm{gen}^*(X_R) = \{M_R\colon M$ admits a $\mathrm{Hom}_R(X, -)$ exact sequence: $\ldots \to X^1 \to X^0 \to M \to 0$, with $X^i \in \mathrm{add}\, X_R\}$. Particularly, $\mathrm{gen}^*(R_R)$ is exactly the class of all finitely generated right R-modules with degreewise finitely generated projective resolutions.

We will show some properties of these two classes.

Lemma 1.7. *Let X_R be a right R-module with $\mathrm{Ext}^1_R(X, X) = 0$ and let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of right R-modules. The following assertions hold.*
(1) *Both the two classes $\mathrm{cog}^*(X_R)$ and $\mathrm{gen}^*(X_R)$ are closed under finite direct sums and direct summands.*
(2) *If $\mathrm{Ext}^1_R(M'', X) = 0$ and any two of the three modules M', M and M'' are in $\mathrm{cog}^*(X_R)$, so is the third.*
(3) *If $\mathrm{Ext}^1_R(X, M') = 0$ and any two of the three modules M', M and M'' are in $\mathrm{gen}^*(X_R)$, so is the third.*

Proof. (1) It is easy to see that both the class $\mathrm{cog}^*(X_R)$ and the class $\mathrm{gen}^*(X_R)$

are closed under finite direct sums by their definition. And by [20, Lemma 2.2], both the two classes are closed under direct summands.

(2) Assume that $\mathrm{Ext}^1_R(M'', X) = 0$. If $M' \in \mathrm{cog}^*(X_R)$ and $M'' \in \mathrm{cog}^*(X_R)$, then $M \in \mathrm{cog}^*(X_R)$ follows from [20, Lemma 2.3(1)].

If $M \in \mathrm{cog}^*(X_R)$ and $M'' \in \mathrm{cog}^*(X_R)$, we will show that $M' \in \mathrm{cog}^*(X_R)$. In fact, since $M \in \mathrm{cog}^*(X_R)$, there exists a $\mathrm{Hom}_R(-, X)$ exact exact sequence: $0 \to M \to X_0 \to X_1 \to \ldots$ with $X_i \in \mathrm{add}\, X$ for $i \geqslant 0$. Let $K_1 = \ker(X_1 \to X_2)$, then clearly $K_1 \in \mathrm{cog}^*(X_R)$. Moreover, by [20, Remark 2.1(1)] we have that $\mathrm{Ext}^1_R(K_1, X) = 0$. We have the following pushout:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M' & \longrightarrow & X_0 & \longrightarrow & D & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K_1 & = & K_1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Consider the exact sequence $0 \to M'' \to D \to K_1 \to 0$. As $\mathrm{Ext}^1_R(M'', X) = 0$ and $\mathrm{Ext}^1_R(K_1, X) = 0$, we have that $\mathrm{Ext}^1_R(D, X) = 0$ and the exact sequence in the middle row of the above pushout is $\mathrm{Hom}_R(-, X)$-exact. Moreover, since $M'' \in \mathrm{cog}^*(X_R)$ and $K_1 \in \mathrm{cog}^*(X_R)$, we have $D \in \mathrm{cog}^*(X_R)$ by [20, Lemma 2.3(1)]. Hence $M' \in \mathrm{cog}^*(X_R)$.

If $M' \in \mathrm{cog}^*(X_R)$ and $M \in \mathrm{cog}^*(X_R)$, we will show that $M'' \in \mathrm{cog}^*(X_R)$. In fact, since $M' \in \mathrm{cog}^*(X_R)$, there exists an exact sequence $0 \to M' \to X'_0 \to K'_1 \to 0$ with $X'_0 \in \mathrm{add}\, X$ and $\mathrm{Ext}^1_R(K'_1, X) = 0$ which is $\mathrm{Hom}_R(-, X)$ exact and $K'_1 \in \mathrm{cog}^*(X_R)$. We have the following pushout:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & X'_0 & \longrightarrow & D & \longrightarrow & M'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & K'_1 & = & K'_1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Consider the exact sequence in the second row: $0 \to X_0' \to D \to M'' \to 0$. Since $\operatorname{Ext}_R^1(M'', X) = 0$ and $X_0' \in \operatorname{add} X$, we have $\operatorname{Ext}_R^1(M'', X_0') = 0$. Thus the exact sequence splits and M'' is a direct summand of D. On the other hand, we have the exact sequence in the second column: $0 \to M \to D \to K_1' \to 0$. By the above proof, we know that $K_1' \in \operatorname{cog}^*(X_R)$ and $\operatorname{Ext}_R^1(K_1', X) = 0$. Moreover, $M \in \operatorname{cog}^*(X_R)$, thus $D \in \operatorname{cog}^*(X_R)$ by [20, Lemma 2.3(1)]. Hence $M'' \in \operatorname{cog}^*(X_R)$ by (1).

(3) is dual to (2), so we omit the proof. □

2. Totally reflexive modules with respect to a semidualizing bimodule

In this section, we introduce and investigate properties of the totally reflexive module with respect to a semidualizing bimodule ${}_SC_R$ over any associative rings S and R. Over a commutative Noetherian ring the following definition can be found in [14, Definition 2.1.3]. And over any left Noetherian ring S and right Noetherian R, the notion of the totally C-reflexive module was also given by Araya, Takahashi and Yoshino [3, Theorem 2.1].

Definition 2.1. Let ${}_SC_R$ be a semidualizing bimodule. A finitely generated right R-module M_R is *totally C_R-reflexive* if it satisfies the following conditions:

(1) M_R admits a degreewise finitely generated R-projective resolution;

(2) the biduality map $\delta_M^C\colon M \to \operatorname{Hom}_S(\operatorname{Hom}_R(M, C), C)$ is an R-module isomorphism;

(3) $\operatorname{Hom}_R(M, C)$ admits a degreewise finitely generated S-projective resolution;

(4) $\operatorname{Ext}_R^i(M, C) = 0 = \operatorname{Ext}_S^i(\operatorname{Hom}_R(M, C), C)$ for all $i \geqslant 1$.

We denote the class of all totally C_R-reflexive right R-modules by $\mathcal{T}_C(R)$.

Similarly we can define the totally ${}_SC$-reflexive left S-modules, denoting them by $\mathcal{T}_C(S)$.

Remark 2.2.

(1) Clearly, finitely generated projective right R-modules and the semidualizing right R-module C are all totally C_R-reflexive.

(2) For each $G \in \mathcal{T}_C(R)$ and $i \geqslant 1$, we can get that $\operatorname{Ext}_R^i(G, L) = 0$ for any right R-module L with finite $\operatorname{add} C_R$-projective dimension by dimension shifting.

(3) It is easy to see that the functors $\operatorname{Hom}_R(-, C)$ and $\operatorname{Hom}_S(-, C)$ induce a duality between the class $\mathcal{T}_C(R)$ and the class $\mathcal{T}_C(S)$ by Definition 2.1, which is also proved by Araya, Takahashi and Yoshino [3, Theorem 2.1].

Wakamatsu [20] defined the Wakamatsu tilting module over any ring and proved that a semidualizing (S,R)-bimodule ${}_SC_R$ is always a Wakamatsu tilting module [20, Corollary 3.2]. Note that the Wakamatsu tilting module is called a tilting module in [20]. Hence the semidualizing bimodule shares the same properties with the Wakamatsu tilting modules. Particularly, using results from [20, Sec. 4] we have the following equality for the class of totally C_R-reflexive modules over any ring R.

Theorem 2.3. *Let ${}_SC_R$ be a semidualizing bimodule. Let us denote $(-)^C_R = \operatorname{Hom}_R(-,C)$. Then*

$$\mathcal{T}_C(R) = \operatorname{gen}^*(R_R) \cap \operatorname{cog}^*(C_R) \cap {}^{\perp}(C_R).$$

Proof. Let $M_R \in \mathcal{T}_C(R)$, then $M \in \operatorname{gen}^*(R_R) \cap {}^{\perp}C_R$ and $M \xrightarrow{\cong} \operatorname{Hom}_S(M^C_R, C)$ by Definition 2.1. So we only need to show $M \in \operatorname{cog}^*(C_R)$. In fact, we have that $M^C_R \in \mathcal{T}_C(S)$ by Remark 2.2(3). Thus $M^C_R \in \operatorname{gen}^*({}_SS) \cap {}^{\perp}_SC$ and $M^C_R \xrightarrow{\cong} \operatorname{Hom}_R(\operatorname{Hom}_S(M^C_R, C), C)$. Hence $\operatorname{Hom}_S(M^C_R, C) \in \operatorname{cog}^*({}_SC)$ by [20, Proposition 4.1]. Thus $M \in \operatorname{cog}^*({}_SC)$ as $M \xrightarrow{\cong} \operatorname{Hom}_S(M^C_R, C)$. Therefore, $M_R \in \operatorname{gen}^*(R_R) \cap \operatorname{cog}^*(C_R) \cap {}^{\perp}(C_R)$. For the reverse inclusion, since $M \in \operatorname{cog}^*(C_R)$, we have $M^C_R \in {}^{\perp}_SC \cap \operatorname{gen}^*({}_SS)$ by [20, Proposition 4.1]. So by Definition 2.1 we only need to show that the biduality map δ^C_M is an isomorphism. In fact, we have the following two commutative diagrams with exact rows by the definition of $\operatorname{cog}^*(C_R)$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{f_0} & C_0 & \longrightarrow & \operatorname{cok} f_0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \delta^C_M} & & \downarrow{\scriptstyle \delta^C_{C_0}} & & \downarrow{\scriptstyle \delta^C_{\operatorname{cok} f_0}} & & \\
0 & \longrightarrow & \operatorname{Hom}_S(M^C_R, C) & \longrightarrow & \operatorname{Hom}_S((C_0)^C_R, C) & \longrightarrow & \operatorname{Hom}_S((\operatorname{cok} f_0)^C_R, C) & &
\end{array}$$

and

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{cok} f_0 & \xrightarrow{f_1} & C_1 & \longrightarrow & \operatorname{cok} f_1 & \longrightarrow & 0. \\
 & & \downarrow{\scriptstyle \delta^C_{\operatorname{cok} f_0}} & & \downarrow{\scriptstyle \delta^C_{C_1}} & & \downarrow{\scriptstyle \delta^C_{\operatorname{cok} f_1}} & & \\
0 & \longrightarrow & \operatorname{Hom}_S((\operatorname{cok} f_0)^C_R, C) & \longrightarrow & \operatorname{Hom}_S((C_1)^C_R, C) & \longrightarrow & \operatorname{Hom}_S(\operatorname{cok}(f_1)^C_R, C) & &
\end{array}$$

Clearly, $\delta^C_{C_0}$ and $\delta^C_{C_1}$ are isomorphisms. Hence by the Snake Lemma, we get that δ^C_M is an isomorphism. Hence $M \in \mathcal{T}_C(R)$. □

From Theorem 2.3 we can get the following Corollary.

Corollary 2.4. *Let ${}_SC_R$ be a semidualizing (S,R)-bimodule and let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of right R-modules. Then the following assertions hold.*

(1) *The class $\mathcal{T}_C(R)$ is closed under finite direct sums and direct summands.*
(2) *If $M'' \in \mathcal{T}_C(R)$, then $M' \in \mathcal{T}_C(R)$ if and only if $M \in \mathcal{T}_C(R)$.*
(3) *If both $M' \in \mathcal{T}_C(R)$ and $M \in \mathcal{T}_C(R)$, then $M'' \in \mathcal{T}_C(R)$ if and only if $\mathrm{Ext}^1_R(M'', C) = 0$.*

Proof. (1) Clearly ${}^{\perp}C_R$ is closed under finite direct sums and direct summands. Moreover, by Lemma 1.7 we know that both $\mathrm{cog}^*(C_R)$ and $\mathrm{gen}^*(R_R)$ are closed under finite direct sums and direct summands. Hence the class $\mathcal{T}_C(R)$ is closed under finite direct sums and direct summands by Theorem 2.3.

(2) Since $M'' \in \mathcal{T}_C(R)$, we have $M'' \in {}^{\perp}(C_R)$ by Definition 2.1. Moreover, ${}^{\perp}(C_R)$ is closed under extensions and kernels of epimorphisms. Hence (2) follows from Theorem 2.3 and Lemma 1.7.

(3) $(\Rightarrow)$ follows from Definition 2.1. Next we will show $(\Leftarrow)$. In fact, since $M' \in \mathcal{T}_C(R)$ and $M \in \mathcal{T}_C(R)$, we have $M' \in {}^{\perp}(C_R)$ and $M \in {}^{\perp}(C_R)$. Applying $\mathrm{Hom}_R(-, C)$ to the exact sequence $0 \to M' \to M \to M'' \to 0$, we get that $\mathrm{Ext}^{i+1}_R(M'', C) = 0$ for $i \geqslant 1$. Hence $M'' \in {}^{\perp}C_R$. Moreover, $M' \in \mathcal{T}_C(R)$ and $M \in \mathcal{T}_C(R)$, so $M'' \in \mathrm{gen}^*(R_R) \cap \mathrm{cog}^*(C_R)$ by Lemma 1.7. Hence $M'' \in \mathcal{T}_C(R)$ by Theorem 2.3. □

When $R = S$ is a commutative ring and $C = R$, the following proposition is [4, Proposition 1.1.9]. Since the proof is similar, we omit it.

Proposition 2.5. *Let ${}_SC_R$ be a semidualizing bimodule and M a right R-module. If $M \cong \mathrm{Hom}_S(N, C)$ for some finitely generated left S-module N, then M is a direct summand of $\mathrm{Hom}_S(\mathrm{Hom}_R(M, C), C)$.*

Remark 2.6. From Remark 2.2(3) we know that if a right R-module M is totally C_R-reflexive, then $\mathrm{Hom}_R(M, C)$ is totally ${}_SC$-reflexive. However, the reverse implication does not hold true in general, see [4, Observation 1.1.7]. But when $M \cong \mathrm{Hom}_S(N, C)$ for some finitely generated left S-module N, we have the following corollary.

Corollary 2.7. *Let M be a right R-module. Assume that $M \cong \mathrm{Hom}_S(N, C)$ for some finitely generated left S-module N. Then M is a totally C_R-reflexive module if and only if $\mathrm{Hom}_R(M, C)$ is a totally ${}_SC$-reflexive module.*

Proof. The forward implication follows from Remark 2.2(3). For the converse, since $\mathrm{Hom}_R(M, C)$ is totally ${}_SC$-reflexive, $\mathrm{Hom}_S(\mathrm{Hom}_R(M, C), C)$ is totally C_R-reflexive also by Remark 2.2(3). As M is a direct summand of $\mathrm{Hom}_S(\mathrm{Hom}_R(M, C), C)$

by Proposition 2.5, we have that M is a totally C_R-reflexive module by Corollary 2.4(1). $\square$

By Remark 2.2(1) we know that finitely generated projective right R-modules are totally C_R-reflexive, thus we can define $\mathcal{T}_C$-dimension for every finitely generated right R-module M which admits a degreewise finitely generated projective resolution (e.g., the finitely generated right R-module over the right Noetherian ring R), denoted by $\mathcal{T}_C\text{-}\dim_R(M)$, see [20, Sec. 3]. For a non-negative integer n, we write $\mathcal{T}_C\text{-}\dim_R(M) \leqslant n$ if there exists an exact sequence $0 \to G_n \to \ldots \to G_0 \to M \to 0$ with each $G_i \in \mathcal{T}_C(R)$. In the next proposition, we investigates the $\mathcal{T}_C$-dimension and the $\mathcal{T}_C(R)$-precover (preenvelope) for $M \in \operatorname{gen}^*(R_R)$.

Proposition 2.8. *Let ${}_SC_R$ be a semidualizing bimodule and n a non-negative integer. The following conditions are equivalent for $M \in \operatorname{gen}^*(R_R)$ with finite $\mathcal{T}_C$ dimension:*

(1) $\mathcal{T}_C\text{-}\dim_R(M) \leqslant n$.

(2) *For any degreewise finitely generated projective resolution of M, $\ldots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$, we have that the* $\ker(f_i)$ *is totally C_R-reflexive for $i \geqslant n-1$, and when $n = 0$, then* $\ker(f_{-1}) = M$.

(3) *For any exact sequence $\ldots \to G_i \xrightarrow{g_i} G_{i-1} \ldots \to G_1 \xrightarrow{g_1} G_0 \xrightarrow{g_0} M \to 0$ with $G_j \in \mathcal{T}_C(R)$ for $j \geqslant 0$, we have that* $\ker(g_i)$ *for $i \geqslant n-1$ is totally C_R-reflexive, and when $n = 0$, then* $\ker(f_{-1}) = M$.

(4) $\operatorname{Ext}^i_R(M, C) = 0$ *for $i \geqslant n+1$.*

(5) *M_R has a special $\mathcal{T}_C(R)$-precover $0 \to K \to G \to M \to 0$ such that $G \in \mathcal{T}_C(R)$ and* $\operatorname{add}(C_R)\text{-}\operatorname{pd}_R K \leqslant n-1$ *if $n \geqslant 1$ and $K = 0$ if $n = 0$.*

(6) *M_R has a special $\widehat{\operatorname{add}(C_R)}$-preenvelope $0 \to M \to L \to G' \to 0$ such that* $\operatorname{add}(C_R)\text{-}\operatorname{pd}_R L \leqslant n$ *and $G' \in \mathcal{T}_C(R)$.*

Proof. Using a proof similar to [3, Lemma 2.1 and Theorem 2.2], we can prove that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4).

(5) $\Rightarrow$ (1) It is straightforward to prove.

(1) $\Rightarrow$ (5) Since $\mathcal{T}_C\text{-}\dim_R(M) \leqslant n$, using a proof similar to [9, Theorem 2.10] and Lemmas 1.6, 1.7 and Theorem 2.3 we can find an exact sequence of right R-modules, $0 \to K \to G \xrightarrow{\varphi} M \to 0$ such that $G \in \mathcal{T}_C(R)$ and $\operatorname{add}(C_R)\text{-}\operatorname{pd}_R K = \mathcal{T}_C\text{-}\dim_R(M) - 1$. So $\operatorname{add}(C_R)\text{-}\operatorname{pd}_R K \leqslant n-1$. Moreover, by Remark 2.2(2), we have that $\operatorname{Ext}^i_R(N, K) = 0$ for any $N \in \mathcal{T}_C(R)$ and $i \geqslant 1$. Hence φ is a special $\mathcal{T}_C(R)$-precover of M by Definition 1.2.

At last we will show that (5) $\Leftrightarrow$ (6). In fact, assume that (5) holds, then $\mathcal{T}_C\text{-}\dim_R(M) \leqslant n < \infty$. Thus using a proof similar to [5, Lemma 2.17] and Lemmas 1.6

and 1.7, we can find an exact sequence of right R-modules

$$0 \to M \xrightarrow{\varphi} L \to G' \to 0$$

such that $G' \in \mathcal{T}_C(R)$ and $\mathrm{add}(C_R)$-$\mathrm{pd}_R L = \mathcal{T}_C$-$\dim_R(M) \leqslant n$. Thus $L \in \widehat{\mathrm{add}(C_R)}$, see Definition 1.1. Moreover, we have that $\mathrm{Ext}^i_R(G', L') = 0$ for any $L' \in \widehat{\mathrm{add}(C_R)}$ and $i \geqslant 1$ by Remark 2.2(2). Hence φ is a special $\widehat{\mathrm{add}(C_R)}$-preenvelope of M by Definition 1.2.

Conversely, assume that (6) holds. Then there is an exact sequence $0 \to M \to L \to G' \to 0$ such that $\mathrm{add}(C_R)$-$\mathrm{pd}_R L \leqslant n$ and $G' \in \mathcal{T}_C(R)$. If $n = 0$, then $L \in \mathrm{add}(C_R)$. By Remark 2.2(1) and Corollary 2.4(2), we know that $M \in \mathcal{T}_C(R)$. Hence the exact sequence $0 \to M \xrightarrow{\cong} M \to 0$ satisfies the condition of (5). Next we assume that $n \geqslant 1$, then we can find an exact sequence of right R-modules, $0 \to L' \to C_0 \to L \to 0$ with $C_0 \in \mathrm{add}(C_R)$ and $\mathrm{add}(C_R)$-$\mathrm{pd}_R L' \leqslant n-1$. Thus we have the following pullback diagram.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & L' & = & L' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & G'' & \to & C_0 & \to & G' & \to & 0 \\
 & & \downarrow f & & \downarrow & & \| & & \\
0 & \to & M & \to & L & \to & G' & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

From the second row we know that $G'' \in \mathcal{T}_C(R)$ by Corollary 2.4(2). Since $L' \in \widehat{\mathrm{add}(C_R)}$, f is a special $\mathcal{T}_C(R)$-precover of M by Remark 2.2(2). Thus the first column $0 \to L' \to G'' \xrightarrow{f} M \to 0$ is the desired exact sequence and (5) holds true.

□

Because semidualizing modules are Wakamatsu tilting modules, see the argument above Proposition 2.8, so by [20, Proposition 5.6, Theorem 6.6] and the Baer Criterion, we can also obtain the result over the non-commutative Noetherian ring, which gives a necessary and sufficient condition for a semidualizing module to be a dualizing module. Note that we can define a dualizing bimodule ${}_SD_R$ over any rings R and S. We call a bimodule ${}_SD_R$ dualizing if it is a semidualizing bimodule with finite left S- and right R-injective dimension.

Proposition 2.9. *Let S be left Noetherian, R right Noetherian and let m, n be nonnegative integers. Then $\mathcal{T}_C(R)$-$\dim_R M \leqslant m$ for every finitely generated right R-module M and $\mathcal{T}_C(R)$-$\dim_R N \leqslant n$ for every finitely generated left S-module N if and only if $\mathrm{id}_R(C) \leqslant m$ and $\mathrm{id}_S(C) \leqslant n$.*

P r o o f. ($\Rightarrow$) For any ideal I of R, R/I is a finitely generated right R-module. Thus $\mathcal{T}_C(R)$-$\dim_R R/I \leqslant m$. Consider the injective resolution of C_R:

$$0 \to C \to E_0 \to E_1 \to \ldots \to E_{m-1} \to C_m \to 0.$$

Applying $\mathrm{Hom}_R(R/I, -)$, we get that $\mathrm{Ext}_R^1(R/I, C_m) \cong \mathrm{Ext}_R^{m+1}(R/I, C)$. Hence $\mathrm{Ext}_R^1(R/I, C_m) = 0$ by Proposition 2.8. Thus C_m is injective by the Baer Criterion and $\mathrm{id}_R(C) \leqslant m$. Using the same method we can prove that $\mathrm{id}_S(C) \leqslant n$.

($\Leftarrow$) Since $\mathrm{id}_R(C) \leqslant m$, we have $\mathrm{Ext}_R^{m+i}(M, C) = 0$ for each right R-module M and $i \geqslant 1$. Consider the projective resolution of M:

$$0 \to \Omega^m M \to P_{m-1} \to \ldots \to P_1 \to P_0 \to M \to 0,$$

then we have that $0 = \mathrm{Ext}_R^{m+i}(M, C) \cong \mathrm{Ext}_R^i(\Omega^m M, C)$. Thus $\Omega^m M \in {}^{\perp}C_R$. Since R is right Noetherian, $\Omega^m M \in \mathrm{gen}^*(R_R)$. Moreover, as S is left Noetherian and $\mathrm{id}_S(C) \leqslant n < \infty$, we have that $\mathcal{T}_C(R) = \mathrm{gen}^*(R_R) \cap \mathrm{cog}^* C_R \cap {}^{\perp}C_R = \mathrm{gen}^*(R_R) \cap {}^{\perp}C_R$ by Theorem 2.3 and [20, Proposition 5.6]. So $\Omega^m M \in \mathcal{T}_C(R)$ and $\mathcal{T}_C(R)$-$\dim_R M \leqslant m$. Similarly, we have that $\mathcal{T}_C(R)$-$\dim_R N \leqslant n$ for every finitely generated left S-module N. □

Observation 2.10. For every totally C_R-reflexive module M, from Theorem 2.3 we know that there exists a $\mathrm{Hom}_R(-, C)$-exact exact sequence of right R-modules $\ldots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} C_0 \xrightarrow{g_0} C_1 \xrightarrow{g_1} \ldots$ with P_i finitely generated projective and $C_j \in \mathrm{add}(C_R)$ and $M \cong \ker(g_0)$. As $\mathrm{Ext}_R^1(C, C) = 0$, it is easy to see that $\mathrm{Ext}_R^1(\ker(g_j), C) = 0$ for each $j \geqslant 0$. Moreover, by Remark 2.2(1) we know that P_i and C_j are all totally C_R-reflexive, hence every kernel in this exact sequence is totally C_R-reflexive by Corollary 2.4. Hence we can get an exact sequence: $0 \to M \to C_0 \to \ker(g_1) \to 0$ with $\ker(g_1)$ totally C_R-reflexive. If $M \in \widehat{\mathrm{add}(C_R)}$, then the sequence splits by Remark 2.2(2). Thus $M \in \mathrm{add}(C_R)$.

It is natural to ask whether a totally C_R-reflexive module with finite projective dimension is finitely generated projective. When ${}_SC_R$ is a faithfully semidualizing module, the next theorem gives an affirmative answer to this question. Moreover, by [21, Theorem 4.4] we know that a right R-module M with $M \in \mathrm{gen}^*(R_R)$ is G_C-projective if and only if M is totally C_R-reflexive. Note that the conclusion holds true in any ring and the condition $\mathrm{Hom}_R(M, C) \in \mathrm{gen}^*(R_R)$ is not needed in the proof of [21, Theorem 4.4]. Hence the theorem is also the answer the special case of the question put forward by D. White in [21, Question 2.15], i.e., over a non-commutative non-local ring R, her conjecture is true for the right R-module M with $M \in \operatorname{gen}^*(R_R)$.

Theorem 2.11. *Let ${}_SC_R$ be faithfully semidualizing and $M_R \in \mathcal{T}_C(R)$. If $\operatorname{pd}_R M = n < \infty$, then M is finitely generated projective.*

Proof. By Theorem 2.3 we have that $\mathcal{T}_C(R) = \operatorname{gen}^*(R_R) \cap \operatorname{cog}^*(C_R) \cap {}^{\perp}(C_R)$. Since $M_R \in \mathcal{T}_C(R)$ and $\operatorname{pd}_R M = n$, there exists an exact sequence of right R-modules

$$(*) \qquad 0 \to P_n \to \ldots \to P_0 \to M \to 0$$

with P_i finitely generated projective. Applying $\operatorname{Hom}_R(-, C)$ to $(*)$, we get a sequence

$$0 \to \operatorname{Hom}_R(M, C) \to \operatorname{Hom}_R(P_0, C) \to \ldots \to \operatorname{Hom}_R(P_n, C) \to 0.$$

Since $M \in {}^{\perp}(C_R)$, the sequence is exact. By Lemma 1.6, $\operatorname{Hom}_R(P_i, C) \in \operatorname{add}({}_SC)$. Assume that $\operatorname{Hom}_R(P_i, C) = C_i$, $K_0 = \operatorname{Hom}_R(M, C)$, $K_n = C_n$ and $K_i = \ker(C_i \to C_{i+1})$ for $(n-1) \geqslant i \geqslant 1$. Then we can get several short exact sequences:

$$\begin{gathered} 0 \to K_{n-1} \to C_{n-1} \to C_n \to 0, \\ \vdots \\ 0 \to K_i \to C_i \to K_{i+1} \to 0, \\ \vdots \\ 0 \to \operatorname{Hom}_R(M, C) \to C_0 \to K_1 \to 0. \end{gathered}$$

Since $\operatorname{add}({}_SC) \subseteq \mathcal{B}_C(S)$, we have $K_i \in \mathcal{B}_C(S)$ for $n \geqslant i \geqslant 0$ by Remark 1.5. Thus $\operatorname{Ext}^1_S(C, K_i) = 0$. So we get that $\operatorname{Ext}^1_S(C_n, K_{n-1}) = 0$ and the first short exact sequence splits, thus $K_{n-1} \in \operatorname{add}({}_SC)$. Repeating this process we get that $\operatorname{Hom}_R(M, C) \in \operatorname{add}({}_SC)$. As ${}_SC_R$ is a semidualizing bimodule, so $\operatorname{Hom}_S(C, C) \cong R$. Thus $M \overset{\cong}{\to} \operatorname{Hom}_S(\operatorname{Hom}_R(M, C), C) \in \operatorname{add}(R_R)$ and M is finitely generated projective. $\square$

Corollary 2.12. *Let ${}_SC_R$ be faithfully semidualizing and let M_R be a right R-module such that $M \in \operatorname{gen}^*(R_R)$. Then $\mathcal{T}_C\text{-}\dim_R M = \operatorname{pd}_R M$ when $\operatorname{pd}_R M < \infty$.*

Proof. By Remark 2.2(1), we know that finitely generated projective right R-modules are totally C_R-reflexive, so $\mathcal{T}_C\text{-}\dim_R M \leqslant \operatorname{pd}_R M$. On the other hand, assume that $\mathcal{T}_C$-$\dim_R M = n < \infty$. Then there exists an exact sequence of right R-modules

$$0 \to G_n \to P_{n-1} \to \ldots \to P_1 \to P_0 \to M \to 0$$

such that P_i is finitely generated projective for $0 \leqslant i \leqslant n-1$ and $G_n \in \mathcal{T}_C(R)$ by Proposition 2.8. Since $\mathrm{pd}_R M < \infty$, we have $\mathrm{pd}_R G_n < \infty$. Hence G_n is finitely generated projective by Theorem 2.11. It follows that $\mathrm{pd}_R M \leqslant n$. Therefore $\mathcal{T}_C$-$\dim_R M = \mathrm{pd}_R M$. □

3. Connections with Bass class

In this section, we will show that there exist some relations between the class $\mathcal{T}_C(R)$ and the class $\mathcal{B}_C(R)$. First, we employ the notions of Mantese and Reiten in [13]. For a Wakamatsu tilting right R-module T_R, denote by $\mathrm{Gen}^*(T_R)$ the subcategory of all right R-modules M such that there exists an exact sequence $\ldots \to T^1 \overset{g_1}{\to} T^0 \overset{g_0}{\to} M \to 0$ where $T^i \in \mathrm{Add}(T_R)$ and $\mathrm{Ext}^1_R(T, \ker g_i) = 0$ for $i \geqslant 0$. When T_Λ is a Wakamatsu tilting module over an Artin algebra Λ, there is an exact sequence $0 \to \Lambda \overset{f_0}{\to} T_0 \overset{f_1}{\to} T_1 \to \ldots$ with $T_i \in \mathrm{add}(T_R)$ and $\mathrm{cok} f_i \in {}^{\perp}(C_R)$ for $i \geqslant 0$. Denote $K_i = \mathrm{cok} f_i$, Mantese and Reiten [13, Proposition 3.6] showed the following equality:

$$T^{\perp} \cap \mathrm{Gen}^*(T) = \Bigl(\bigoplus_{i \geqslant 0} K_i \oplus T\Bigr)^{\perp}.$$

Moreover, it is not hard to see from the proof of [13, Proposition 3.6] that the equality holds over any ring R. On the other hand, by [20, Corollary 3.2] we know that a semidualizing bimodule ${}_S C_R$ is a Wakamatsu tilting, so there exists an exact sequence of right R-modules $0 \to R \overset{f_0}{\to} C^{n_0} \overset{f_1}{\to} C^{n_1} \to \ldots$ where n_i are positive integers and $\mathrm{cok} f_i \in {}^{\perp}C$. Denote the modules $\mathrm{cok} f_i$ by K_i for $i \geqslant 0$, then we have a similar equality for a semidualizing bimodule ${}_S C_R$, that is, $(C_R)^{\perp} \cap \mathrm{Gen}^*(C_R) = \Bigl(\bigoplus_{i \geqslant 0} K_i \oplus C\Bigr)^{\perp}$. It is easy to see that $K_i \in \mathrm{cog}^*(C_R) \cap \mathrm{gen}^*(R_R)$ by Lemma 1.7. Thus $K_i \in \mathrm{gen}^*(R_R) \cap \mathrm{cog}^*(C_R) \cap {}^{\perp}(C_R) = \mathcal{T}_C(R)$ for $i \geqslant 0$ by Theorem 2.3.

Now, we show the following proposition.

Proposition 3.1. *Let ${}_S C_R$ be an (R, S) semidualizing bimodule. Then $\mathcal{B}_C(R) = \Bigl(\bigoplus_{i \geqslant 0} K_i \oplus C_R\Bigr)^{\perp}$.*

P r o o f. By Definition 1.4, we know that for a right R-module M, $M_R \in \mathcal{B}_C(R)$ if and only if $M \in (C_R)^{\perp}$, $\mathrm{Tor}^S_{i \geqslant 1}(\mathrm{Hom}_R(C, M), C) = 0$ and $\mathrm{Hom}_R(C, M) \otimes_S C \overset{\cong}{\to} M$.

On the other hand, Takahashi and White [18, Proposition 2.2] proved the following result: over a commutative ring R, for any R-module M, M admits an exact proper $\mathcal{P}_C$-resolution if and only if $\operatorname{Tor}^R_{i\geqslant 1}(\operatorname{Hom}_R(C,M),C)=0$ and $\operatorname{Hom}_R(C,M)\otimes_R C\xrightarrow{\cong} M$. Note that the result holds true over any associtative ring R from the proof of Takahashi and White [18, Proposition 2.2]. By Lemma 1.6 and the definition of the proper $\mathcal{P}_C$-resolution, see [18, 1.5], we have that M admits an exact proper $\mathcal{P}_C$-resolution if and only if $M\in\operatorname{Gen}^*(C_R)$. Hence we have that $\mathcal{B}_C(R)=(C_R)^{\perp}\cap\operatorname{Gen}^*(C_R)$. So by the above argument, we have that $\mathcal{B}_C(R)=\Big(\bigoplus\limits_{i\geqslant 0}K_i\oplus C_R\Big)^{\perp}$. □

Theorem 3.2. *Let ${}_SC_R$ be faithfully semidualizing. Denote by $\mathcal{P}_R^{<\infty}$ the class of right R-modules which are in $\operatorname{gen}^*(R_R)$ and have finite projective dimensions. Then*

(1) ${}^{\perp}\mathcal{B}_C(R)\cap\operatorname{gen}^*(R_R)\subseteq\mathcal{T}_C(R)$ *and* ${}^{\perp}\mathcal{B}_C(R)\cap\mathcal{P}_R^{<\infty}=\mathcal{T}_C(R)\cap\mathcal{P}_R^{<\infty}$,

(2) $\mathcal{T}_C^{\perp}(R)\subseteq\mathcal{B}_C(R)$.

Proof. (1) Assume that $M\in{}^{\perp}\mathcal{B}_C(R)\cap\operatorname{gen}^*(R_R)$. Then $M\in{}^{\perp}(C_R)$ because $C_R\in\mathcal{B}_C(R)$. The Bass class $\mathcal{B}_C(R)$ is preenveloping by [11, Theorem 3.2(b)] and contains all the injective right R-modules, so there exists an exact sequence for any right R-module M, $0\to M\xrightarrow{\varphi}B\to M'\to 0$ with $B\in\mathcal{B}_C(R)$, where φ is a $\mathcal{B}_C(R)$-preenvelope. By [18, Corollary 2.4] and Lemma 1.6, there is an exact sequence $0\to B'\to C^{(I)}\to B\to 0$ for some index set I. Hence we have a pullback

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & B' & = & B' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & P & \to & C^{(I)} & \to & M' & \to & 0\\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & M & \xrightarrow{\varphi} & B & \to & M' & \to & 0\\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

By Remark 1.5, $B'\in\mathcal{B}_C(R)$, so the first column splits and we have an exact sequence $0\to M\to C^{(I)}\to M''\to 0$. Since $M\in\operatorname{gen}^*(R_R)$, M is finitely generated. So M is contained in a finite direct sum of copies C. That is, the image of M is contained in a finitely generated submodule C^n of $C^{(I)}$. Thus we have the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & C^n & \longrightarrow & M_1 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & C^{(I)} & \longrightarrow & M'' & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \xrightarrow{\varphi} & B & \longrightarrow & M' & \longrightarrow & 0.
\end{array}$$

Applying $\mathrm{Hom}_R(-, B'')$ with $B'' \in \mathcal{B}_C(R)$ to the first row and the last row of the commutative diagram, we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_R(M', B'') & \longrightarrow & \mathrm{Hom}_R(B, B'') & \longrightarrow & \mathrm{Hom}_R(M, B'') & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathrm{Hom}_R(M_1, B'') & \longrightarrow & \mathrm{Hom}_R(C^n, B'') & \longrightarrow & \mathrm{Hom}_R(M, B'') & &
\end{array}$$

Note that the first row is exact because φ is a $\mathcal{B}_C(R)$-preenvelope. It is easy to see from the last commutative square of the commutative diagram that $\mathrm{Hom}_R(C^n, B'') \to \mathrm{Hom}_R(M, B'')$ is surjective. By Definition 1.4, we know that $\mathrm{add}(C_R) \subseteq {}^{\perp}\mathcal{B}_C(R)$, so $\mathrm{Ext}^1_R(C^n, B'') = 0$. Thus we have the long exact sequence induced by $\mathrm{Hom}_R(-, B'')$,

$$\mathrm{Hom}_R(C^n, B'') \to \mathrm{Hom}_R(M, B'') \to \mathrm{Ext}^1_R(M_1, B'') \to 0$$

and

$$0 \to \mathrm{Ext}^i_R(M, B'') \to \mathrm{Ext}^{i+1}_R(M_1, B'') \to 0 \quad \text{for } i \geqslant 1.$$

So we get that $\mathrm{Ext}^1_R(M_1, B'') = 0$ and $\mathrm{Ext}^{i+1}_R(M_1, B'') \cong \mathrm{Ext}^i_R(M, B'')$ for $i \geqslant 1$. Hence $M_1 \in {}^{\perp}\mathcal{B}_C(R)$. As $\mathrm{add}(C_R) \subseteq \mathcal{B}_C(R)$, repeating this process, we get that $M \in \mathrm{cog}^*(C_R)$. Hence $M \in \mathrm{gen}^*(R_R) \cap \mathrm{cog}^*(C_R) \cap {}^{\perp}(C_R) = \mathcal{T}_C(R)$ and ${}^{\perp}\mathcal{B}_C(R) \cap \mathrm{gen}^*(R_R) \subseteq \mathcal{T}_C(R)$.

By [18, Proposition 2.2] and Lemma 1.6, we know that for any right R-module $B \in \mathcal{B}_C(R)$ there exists an exact sequence of right R-modules

$$(*) \qquad \ldots \to C_1 \xrightarrow{f_1} C_0 \xrightarrow{f_0} B \to 0$$

with $C_i \in \mathrm{Add}(C_R)$ and the sequence is $\mathrm{Hom}_R(C, -)$-exact. Let $M_R \in {}^{\perp}(C_R) \cap \mathcal{P}^{<\infty}_R$, then $M \in \mathrm{gen}^*(R_R)$, so M has degree-wise finitely generated projective resolution. Hence $\mathrm{Ext}^j_R(M, \bigoplus C) \cong \bigoplus \mathrm{Ext}^j_R(M, C)$ for $j \geqslant 0$ by [6, Lemma 3.1.16]. Thus $\mathrm{Ext}^j_R(M, C_i) = 0$ for $j \geqslant 1$ and $i \geqslant 0$. Applying $\mathrm{Hom}_R(M, -)$ to $(*)$, we get that $\mathrm{Ext}^j_R(M, B) \cong \mathrm{Ext}^{j+n}_R(M, \ker(f_n))$ for $j \geqslant 1$ and $n \geqslant 1$. Since $M \in \mathcal{P}^{<\infty}_R$, we have $\mathrm{pd}_R M < \infty$. So $\mathrm{Ext}^j_R(M,B) = 0$ for all $j \geqslant 1$. Hence ${}^{\perp}(C_R) \cap \mathcal{P}_R^{<\infty} \subseteq {}^{\perp}\mathcal{B}_C(R) \cap \mathcal{P}_R^{<\infty}$. But $\mathcal{T}_C(R) \subseteq {}^{\perp}(C_R)$ by Definition 2.1. So $\mathcal{T}_C(R) \cap \mathcal{P}_R^{<\infty} \subseteq {}^{\perp}(C_R) \cap \mathcal{P}_R^{<\infty} \subseteq {}^{\perp}\mathcal{B}_C(R) \cap \mathcal{P}_R^{<\infty}$. On the other hand, we have that ${}^{\perp}\mathcal{B}_C(R) \cap \mathcal{P}_R^{<\infty} \subseteq \mathcal{T}_C(R) \cap \mathcal{P}_R^{<\infty}$ by the above argument, as $\mathcal{P}_R^{<\infty} \subseteq \mathrm{gen}^*(R_R)$. Therefore, ${}^{\perp}\mathcal{B}_C(R) \cap \mathcal{P}_R^{<\infty} = \mathcal{T}_C(R) \cap \mathcal{P}_R^{<\infty}$.

(2) By Definition 2.1, we know that $C_R \in \mathcal{T}_C(R)$, and the argument above Proposition 3.1 indicates that $K_i \in \mathcal{T}_C(R)$ for $i \geqslant 1$. Let $M_R \in \mathcal{T}_C^{\perp}(R)$, then $\mathrm{Ext}^i_R(C,M) = 0$ for $i \geqslant 1$. So $\mathrm{Ext}^i_R\bigl(\bigoplus K_i, M\bigr) \cong \prod \mathrm{Ext}^i_R(K_i,M) = 0$. Hence $M_R \in \bigl(\bigoplus K_i \oplus C\bigr)^{\perp} = \mathcal{B}_C(R)$ by Proposition 3.1. It follows that $\mathcal{T}_C^{\perp}(R) \subseteq \mathcal{B}_C(R)$. □

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