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Czechoslovak Mathematical Journal, Vol. 63 (2013), No. 2, 441–449

Persistent URL: http://dml.cz/dmlcz/143323

Czechoslovak Mathematical Journal, 63 (138) (2013), 441–449

STRONGLY $\mathcal{W}$-GORENSTEIN MODULES

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(Received February 7, 2012)

Abstract. Let $\mathcal{W}$ be a self-orthogonal class of left R-modules. We introduce a class of modules, which is called strongly $\mathcal{W}$-Gorenstein modules, and give some equivalent characterizations of them. Many important classes of modules are included in these modules. It is proved that the class of strongly $\mathcal{W}$-Gorenstein modules is closed under finite direct sums. We also give some sufficient conditions under which the property of strongly $\mathcal{W}$-Gorenstein module can be inherited by its submodules and quotient modules. As applications, many known results are generalized.

Keywords: self-orthogonal class, strongly $\mathcal{W}$-Gorenstein module, $\mathcal{C}$-resolution

MSC 2010: 16E05, 18G20, 18G25

1. INTRODUCTION

Throughout this article, R is an associative ring with identity and all modules are unitary. Auslander and Bridger [2] introduced the G-dimension for finitely generated modules. Enochs and Jenda [4] defined Gorenstein projective modules whether the modules are finitely generated or not. Also, they defined the Gorenstein projective dimension for arbitrary (non-finitely generated) modules. It is well-known that for finitely generated modules over a commutative Noetherian ring, the Gorenstein projective dimension agrees with the G-dimension. Along the same lines, Gorenstein injective modules were introduced in [4]. Since then, various generalizations of these modules have been given over specific rings [6], [7], [8], [11].

Let $\mathcal{W}$ be a class of left R-modules. We recall that a left R-module M is said to be $\mathcal{W}$-Gorenstein [9] if there exists an exact sequence $W_\bullet = \ldots \to W_1 \to W_0 \to$

This research was supported by National Natural Science Foundation of China (10901129, 11261050, 11101197, 11001222), SRFDP (20096203120001), Research Supervisor Program of Education Department of Gansu Province (0801-03), and NWNU-KJCXGC-03-51.

$W^0 \to W^1 \to \dots$ of modules in $\mathcal{W}$ such that $M = \operatorname{Ker}(W^0 \to W^1)$ and $W_\bullet$ is both $\operatorname{Hom}_R(\mathcal{W}, -)$ and $\operatorname{Hom}_R(-, \mathcal{W})$ exact. For different choices of $\mathcal{W}$, the class of $\mathcal{W}$-Gorenstein modules encompasses all of the modules aforementioned in the first paragraph, and some results for the modules above can be obtained as particular instances of the results on $\mathcal{W}$-Gorenstein modules.

In 2007 Driss Bennis and Najib Mahdou introduced strongly Gorenstein projective, injective, and flat modules, which are defined as follows [3]:

Definition 1.1. (1) An R-module M is said to be strongly Gorenstein projective, if there exists an exact sequence of projective modules

$$\mathbf{P} = \dots \longrightarrow P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \longrightarrow \cdots$$

such that $M \cong \operatorname{Ker} f$ and $\operatorname{Hom}_R(-, Q)$ leaves the sequence $\mathbf{P}$ exact whenever Q is a projective module. The exact sequence $\mathbf{P}$ is called a strongly complete projective resolution.

(2) The strongly Gorenstein injective modules are defined dually.

(3) An R-module M is said to be strongly Gorenstein flat, if there exists an exact sequence of flat modules

$$\mathbf{F} = \dots \longrightarrow F \xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} F \longrightarrow \cdots$$

such that $M \cong \operatorname{Ker} f$ and $-\otimes I$ leaves the sequence $\mathbf{F}$ exact whenever I is an injective module. The exact sequence $\mathbf{F}$ is called a strongly complete flat resolution.

Almost by definition one has the inclusion

$$\begin{aligned}\{\textit{projective modules}\} &\subsetneq \{\textit{strongly Gorenstein projective modules}\}\\ &\subsetneq \{\textit{Gorenstein projective modules}\}.\end{aligned}$$

In this paper, we define and study strongly $\mathcal{W}$-Gorenstein modules for a self-orthogonal class $\mathcal{W}$ of left R-modules, which is motivated by the definitions of the strongly Gorenstein projective, injective modules. Some results for the strongly Gorenstein projective and the strongly Gorenstein injective modules can be also obtained as particular instances of the results on strongly $\mathcal{W}$-Gorenstein modules.

Next we shall recall some notions and definitions which we need in the later sections.

Let $\mathcal{C}$ be a class of left R-modules. We define

$$^{\perp}\mathcal{C} = \bigcap_{i=1}^{\infty} {}^{\perp i}\mathcal{C}, \quad \text{where } {}^{\perp i}\mathcal{C} = \{X\colon \operatorname{Ext}^i_R(X, C) = 0 \text{ for all } C \in \mathcal{C}\},\ i \geqslant 1,$$

$$\mathcal{C}^{\perp} = \bigcap_{i=1}^{\infty} \mathcal{C}^{\perp i}, \quad \text{where } \mathcal{C}^{\perp i} = \{X\colon \operatorname{Ext}^i_R(C, X) = 0 \text{ for all } C \in \mathcal{C}\},\ i \geqslant 1.$$

A $\mathcal{C}$-resolution of a left R-module M is an exact sequence $\mathcal{C}_\bullet = \ldots \to C_1 \to C_0 \to M \to 0$ with $C_i \in \mathcal{C}$ for all $i \geqslant 0$; moreover, if the sequence $\mathrm{Hom}_R(C, \mathcal{C}_\bullet)$ is exact for every $C \in \mathcal{C}$, then we say that $\mathcal{C}_\bullet$ is proper. The $\mathcal{C}$ resolution dimension $\mathrm{resdim}_{\mathcal{C}} M$ of M is the minimal nonnegative n such that M has a $\mathcal{C}$ resolution of length n. Dually we have the definition of a (coproper) $\mathcal{C}$-coresolution and the $\mathcal{C}$ coresolution dimension $\mathrm{coresdim}_{\mathcal{C}} M$ of M.

We say that the class $\mathcal{C}$ is closed under extensions, if for every short exact sequence of left R-modules $0 \to A \to B \to C \to 0$, the condition A and C are in $\mathcal{C}$ implies that B is in $\mathcal{C}$.

We say that the class $\mathcal{C}$ is closed under kernels of epimorphisms, if for every short exact sequence of left R-modules $0 \to A \to B \to C \to 0$, the condition B and C are in $\mathcal{C}$ implies that A is in $\mathcal{C}$.

The class $\mathcal{C}$ is said to be projectively resolving, if it contains all projective left R-modules, and it is closed under both extensions and kernels of epimorphisms; that is, for every short exact sequence of left R-modules $0 \to A \to B \to C \to 0$ with $C \in \mathcal{C}$, the conditions $A \in \mathcal{C}$ and $B \in \mathcal{C}$ are equivalent.

As usual, $\mathrm{Add}_R M$ stands for the category consisting of all modules isomorphic to direct summands of direct sums copies of M and $\mathrm{Prod}_R M$ for the category consisting of all modules isomorphic to direct summands of direct products of copies of M.

2. Strongly $\mathcal{W}$-Gorenstein modules

Definition 2.1 ([9], Definition 2.1)**.** Let $\mathcal{W}$ be a class of left R-modules. $\mathcal{W}$ is called self-orthogonal if it satisfies the following condition: $\mathrm{Ext}^i_R(W, W') = 0$ for all $W, W' \in \mathcal{W}$ and all $i \geqslant 1$.

In what follows, $\mathcal{W}$ always denotes a self-orthogonal class of left R-modules which is closed under finite direct sums and direct summands.

Definition 2.2. A left R-module M is said to be strongly $\mathcal{W}$-Gorenstein if there exists an exact sequence

$$W_\bullet = \ldots \longrightarrow W \xrightarrow{f} W \xrightarrow{f} W \xrightarrow{f} W \longrightarrow \cdots$$

of modules in $\mathcal{W}$ such that $M = \mathrm{Ker} f$ and $W_\bullet$ is both $\mathrm{Hom}_R(\mathcal{W}, -)$ and $\mathrm{Hom}_R(-, \mathcal{W})$ exact. We call the exact sequence $W_\bullet$ a strongly complete $\mathcal{W}$-resolution.

In the following, we denote by $\mathcal{SG}(\mathcal{W})$ the class of strongly $\mathcal{W}$-Gorenstein left R-modules.

Remark 2.3. (1) By the definition, we know that every strongly $\mathcal{W}$-Gorenstein module is a $\mathcal{W}$-Gorenstein module.

(2) If the sequence $W_\bullet = \ldots \longrightarrow W \xrightarrow{f} W \xrightarrow{f} W \longrightarrow \cdots$ is a $\mathrm{Hom}_R(\mathcal{W}, -)$ and $\mathrm{Hom}_R(-, \mathcal{W})$ exact exact sequence of modules in $\mathcal{W}$, then by symmetry, all the images, the kernels and the cokernels of $W_\bullet$ are strongly $\mathcal{W}$-Gorenstein.

(3) If $\mathcal{W} = \mathrm{Add}_R R$ ($\mathrm{Prod}_R E$ with E an injective cogenerator), then strongly $\mathcal{W}$-Gorenstein modules are exactly strongly Gorenstein projective (injective) modules.

Proposition 2.4. *The class of strongly $\mathcal{W}$-Gorenstein modules is closed under finite direct sums.*

Proof. Simply note that a finite sum of strongly complete $\mathcal{W}$-resolutions is also a strongly complete $\mathcal{W}$-resolution (using the isomorphisms in ([1], Proposition 20.2 (1)) and ([5], Exercises 1.2.2 (a))). □

We know that every module in $\mathcal{W}$ is $\mathcal{W}$-Gorenstein ([9], Remark 2.3 (1)). The next result shows that the class of all strongly $\mathcal{W}$-Gorenstein modules is between the class $\mathcal{W}$ and the class of all $\mathcal{W}$-Gorenstein modules.

Proposition 2.5. *Every left R-module in $\mathcal{W}$ is strongly $\mathcal{W}$-Gorenstein.*

Proof. Let W be a left R-module in $\mathcal{W}$, and consider the exact sequence

$$W_\bullet = \ldots \longrightarrow W \oplus W \xrightarrow{f} W \oplus W \xrightarrow{f} W \oplus W \xrightarrow{f} \cdots$$

where f sends (x, y) to $(x, 0)$ for any $x, y \in W$. So we have $0 \oplus W = \mathrm{Ker} f \cong \mathrm{Im} f \cong W$. Applying the functors $\mathrm{Hom}_R(W', -)$ and $\mathrm{Hom}_R(-, W')$ to the above sequence $W_\bullet$ where $W' \in \mathcal{W}$, we get two commutative diagrams

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \mathrm{Hom}(W', W \oplus W) & \xrightarrow{\mathrm{Hom}(W', f)} & \mathrm{Hom}(W', W \oplus W) & \longrightarrow \cdots \\
& \cong \downarrow & & \cong \downarrow & \\
\cdots \longrightarrow & \mathrm{Hom}(W', W) \oplus \mathrm{Hom}(W', W) & \longrightarrow & \mathrm{Hom}(W', W) \oplus \mathrm{Hom}(W', W) & \longrightarrow \cdots
\end{array}$$

and

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \mathrm{Hom}(W \oplus W, W') & \xrightarrow{\mathrm{Hom}(f, W')} & \mathrm{Hom}(W \oplus W, W') & \longrightarrow \cdots \\
& \cong \downarrow & & \cong \downarrow & \\
\cdots \longrightarrow & \mathrm{Hom}(W, W') \oplus \mathrm{Hom}(W, W') & \longrightarrow & \mathrm{Hom}(W, W') \oplus \mathrm{Hom}(W, W') & \longrightarrow \cdots
\end{array}$$

Since the lower sequence in each diagram is exact, the proposition follows. □

Corollary 2.6 ([3], Proposition 2.3). *Every projective (injective) module is strongly Gorenstein projective (injective).*

The following proposition is immediate by definition.

Proposition 2.7. *If a left R-module M is strongly $\mathcal{W}$-Gorenstein, then $M \in {}^{\perp}\mathcal{W} \cap \mathcal{W}^{\perp}$ and M has a proper $\mathcal{W}$-resolution and a coproper $\mathcal{W}$-coresolution.*

The next result gives a simple characterization of the strongly $\mathcal{W}$-Gorenstein modules.

Proposition 2.8. *For any left R-module M, the following are equivalent:*

(1) *M is strongly $\mathcal{W}$-Gorenstein.*
(2) *There exists a short exact sequence $0 \to M \to W \to M \to 0$, where $W \in \mathcal{W}$, $\mathrm{Ext}^i_R(M, W') = 0$ and $\mathrm{Ext}^i_R(W', M) = 0$ for all $i \geqslant 1$ and any $W' \in \mathcal{W}$.*
(3) *There exists a short exact sequence $0 \to M \to W \to M \to 0$, where $W \in \mathcal{W}$, $\mathrm{Ext}^i_R(M, W'') = 0$ for all $i \geqslant 1$ and any W'' with finite $\mathcal{W}$-resolution and $\mathrm{Ext}^i_R(W'', M) = 0$ for all $i \geqslant 1$ and any W'' with finite $\mathcal{W}$-coresolution.*
(4) *There exists a short exact sequence $0 \to M \to W \to M \to 0$, where $W \in \mathcal{W}$, such that for any $W' \in \mathcal{W}$, the short sequences*

$$0 \longrightarrow \mathrm{Hom}_R(M, W') \longrightarrow \mathrm{Hom}_R(W, W') \longrightarrow \mathrm{Hom}_R(M, W') \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Hom}_R(W', M) \longrightarrow \mathrm{Hom}_R(W', W) \longrightarrow \mathrm{Hom}_R(W', M) \longrightarrow 0$$

are exact.
(5) *There exists a short exact sequence $0 \to M \to W \to M \to 0$, where $W \in \mathcal{W}$, such that for any W'' with finite $\mathcal{W}$-resolution, the short sequence*

$$0 \longrightarrow \mathrm{Hom}_R(M, W'') \longrightarrow \mathrm{Hom}_R(W, W'') \longrightarrow \mathrm{Hom}_R(M, W'') \longrightarrow 0$$

is exact and for any W'' with finite $\mathcal{W}$-coresolution, the short sequence

$$0 \to \mathrm{Hom}_R(W'', M) \longrightarrow \mathrm{Hom}_R(W'', W) \longrightarrow \mathrm{Hom}_R(W'', M) \longrightarrow 0$$

is exact.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4) and (3) $\Leftrightarrow$ (5) are obvious.

(2) $\Rightarrow$ (3) If W'' has finite W-resolution, then there exists an exact sequence $0 \to W_n \to \ldots \to W_1 \to W_0 \to W'' \to 0$. Applying the functors $\mathrm{Hom}_R(-, W'')$ to the sequence

$$0 \to M \to W \to M \to 0,$$

we have

$$\mathrm{Ext}^1_R(M, W'') \cong \mathrm{Ext}^2_R(M, W'') \cong \ldots.$$

Let $K_i = \mathrm{Ker}(W_{i-1} \to W_{i-2})$, $1 \leqslant i \leqslant n$, $W_{-1} = M$ and $K_n = W_n$. By dimension shifting, we have

$$\mathrm{Ext}^1_R(M, W'') \cong \mathrm{Ext}^{n+1}_R(M, W'') \cong \mathrm{Ext}^n_R(M, K_1) \cong \ldots \cong \mathrm{Ext}^1_R(M, W_n) = 0$$

by (2). If W'' has finite $\mathcal{W}$-coresolution, we have $\mathrm{Ext}^i_R(W'', M) = 0$ by similar argument.

(3) $\Rightarrow$ (2) is trivial. $\square$

Corollary 2.9 ([3], Proposition 2.9)**.** *For any left R-module M, the following are equivalent:*

(1) *M is strongly Gorenstein projective.*
(2) *There exists a short exact sequence $0 \to M \to P \to M \to 0$, where P is a projective module, and $\mathrm{Ext}^i_R(M, P') = 0$ for all $i \geqslant 1$ and any projective module P'.*
(3) *There exists a short exact sequence $0 \to M \to P \to M \to 0$, where P is a projective module, and $\mathrm{Ext}^i_R(M, P'') = 0$ for all $i \geqslant 1$ and any module P'' with finite projective dimension.*
(4) *There exists a short exact sequence $0 \to M \to P \to M \to 0$, where P is a projective module, such that for any projective module P', the short sequence $0 \to \mathrm{Hom}_R(M, P') \to \mathrm{Hom}_R(P, P') \to \mathrm{Hom}_R(M, P') \to 0$ is exact.*
(5) *There exists a short exact sequence $0 \to M \to P \to M \to 0$, where P is a projective module, such that for any module P'' with finite projective dimension, the short sequence $0 \to \mathrm{Hom}_R(M, P'') \to \mathrm{Hom}_R(P, P'') \to \mathrm{Hom}_R(M, P'') \to 0$ is exact.*

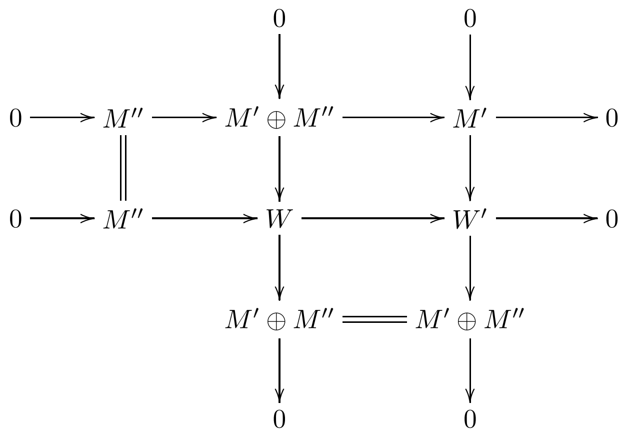
It was shown in [12] that direct summands of a strongly Gorenstein projective module need not be strongly Gorenstein projective and the class of all strongly Gorenstein projective R-modules is not projectively resolving. But the property of a strongly Gorenstein projective module can be inherited by its direct summands under certain condition. So when does the property of a strongly $\mathcal{W}$-Gorenstein module can be inherited by its direct summands? We have the following statement.

Theorem 2.10. *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of left R-modules. Then*

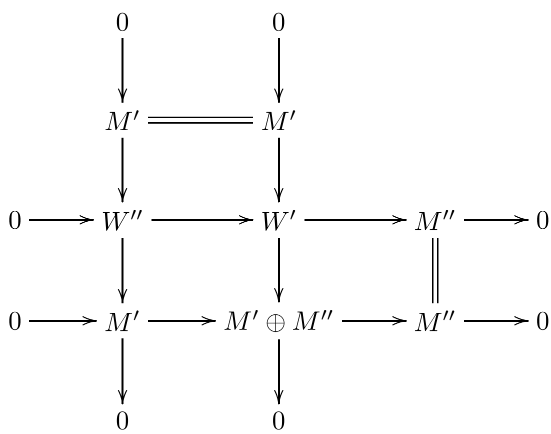
(1) *if $M'' \in \mathcal{W}$ and $M' \in \mathcal{W}^{\perp_1}$, then $M' \in \mathcal{SG}(\mathcal{W})$ if and only if $M \in \mathcal{SG}(\mathcal{W})$;*

(2) *if $M' \in \mathcal{W}$ and $M'' \in {}^{\perp_1}\mathcal{W}$, then $M'' \in \mathcal{SG}(\mathcal{W})$ if and only if $M \in \mathcal{SG}(\mathcal{W})$.*

P r o o f. (1) Since $M'' \in \mathcal{W}$ and $M' \in \mathcal{W}^{\perp_1}$, we always have $\mathrm{Ext}_R^1(M'', M') = 0$. So the sequence $0 \to M' \to M \to M'' \to 0$ is split and hence $M \cong M' \oplus M''$. If $M' \in \mathcal{SG}(\mathcal{W})$, by $M \cong M' \oplus M''$ it is clear that M is a strongly $\mathcal{W}$-Gorenstein module by Proposition 2.4. Conversely, suppose $M \in \mathcal{SG}(\mathcal{W})$. Then there exists an exact sequence $0 \to M' \oplus M'' \to W \to M' \oplus M'' \to 0$, $W \in \mathcal{W}$. Consider the pushout diagram



Since both $M' \oplus M''$ and M' are $\mathcal{W}$-Gorenstein modules by ([10], Corollary 4.11), W' is $\mathcal{W}$-Gorenstein by ([10], Corollary 4.5). Then $\mathrm{Ext}_R^1(W', M'') = 0$, the middle row splits. So $W' \in \mathcal{W}$. Consider the pullback diagram



Then the sequence $0 \to M' \to W'' \to M' \to 0$ is exact and W'' is $\mathcal{W}$-Gorenstein. So $W'' \in \mathcal{W}^{\perp}$ and then the middle row splits, and we have $W'' \in \mathcal{W}$. Since M' is $\mathcal{W}$-Gorenstein, so $\operatorname{Ext}^i_R(M', W_0) = 0$ and $\operatorname{Ext}^i_R(W_0, M') = 0$ for all $i \geqslant 1$ and any module $W_0 \in \mathcal{W}$, hence $M' \in \mathcal{SG}(\mathcal{W})$ by Proposition 2.8.

(2) The proof is similar to that of (1). □

Corollary 2.11 ([12], Theorem 2.1). *Let $0 \to N \to M \to Q \to 0$ be an exact sequence of left R-modules, where Q is a projective module. Then N is strongly Gorenstein projective if and only if M is strongly Gorenstein projective.*

Corollary 2.12 ([12], Theorem 2.2). *Let $0 \to E \to M \to N \to 0$ be an exact sequence of left R-modules, where E is an injective module. Then N is strongly Gorenstein injective if and only if M is strongly Gorenstein injective.*

Proposition 2.13. *Let M be a left R-module. Then M has a $\mathcal{W}$-resolution if and only if M has a $\mathcal{SG}(\mathcal{W})$-resolution.*

Proof. Note that if M has a $\mathcal{SG}(\mathcal{W})$-resolution, then M has a $\mathcal{G}(\mathcal{W})$-resolution, and so M has a $\mathcal{W}$-resolution by ([9], Proposition 2.10). The "only if" part is obvious. □

Remark 2.14. In fact, the result in the above statement on resolution has the dual version on coresolution.

Remark 2.15. (1) Note that using the characterizations of strongly $\mathcal{W}$-Gorenstein modules and strongly Gorenstein projective modules, all the images, the kernels and the cokernels of $W_\bullet$ investigated in Remark 2.3 are the same and Proposition 2.5 becomes straightforward. Indeed, we have the short exact sequence $0 \to W \to W \oplus W \to W \to 0$, and $\operatorname{Ext}^i_R(W, W') = 0$ and $\operatorname{Ext}^i_R(W', W) = 0$ for all $i \geqslant 1$ with any W' in $\mathcal{W}$.

(2) We can also obtain a characterization of the strongly Gorenstein injective modules by Proposition 2.9 in a way similar to the description of strongly Gorenstein projective modules.

Acknowledgement. The authors would like to appreciate the anonymous referee for many helpful comments and suggestions.

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