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ON THE MEAN VALUE OF THE MIXED EXPONENTIAL SUMS WITH DIRICHLET CHARACTERS AND GENERAL GAUSS SUM

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Abstract. The main purpose of the paper is to study, using the analytic method and the property of the Ramanujan's sum, the computational problem of the mean value of the mixed exponential sums with Dirichlet characters and general Gauss sum. For integers m, n, k, q, with $k \geqslant 1$ and $q \geqslant 3$, and Dirichlet characters χ, $\bar{\chi}$ modulo q we define a mixed exponential sum

$$C(m,n;k;\chi;\bar{\chi};q) = {\sum_{a=1}^{q}}' \chi(a) G_k(a,\bar{\chi}) e\Big(\frac{ma^k + n\overline{a^k}}{q}\Big),$$

with Dirichlet character χ and general Gauss sum $G_k(a,\bar{\chi})$ as coefficient, where $\sum'$ denotes the summation over all a such that $(a,q)=1$, $a\bar{a} \equiv 1 \bmod q$ and $e(y) = \mathrm{e}^{2\pi \mathrm{i} y}$. We mean value of

$$\sum_m \sum_\chi \sum_{\bar{\chi}} |C(m,n;k;\chi;\bar{\chi};q)|^4,$$

and give an exact computational formula for it.

Keywords: mixed exponential sum, mean value, Dirichlet character, general Gauss sum, computational formula

MSC 2010: 11L03, 11L05

1. Introduction

Let $q \geqslant 3$ be a positive integer. For any integers m, n and k, and Dirichlet character χ modulo q, we define exponential sums as follows:

$$S_1(m,n;q) = {\sum_{a=1}^{q}}' e\Big(\frac{ma + n\bar{a}}{q}\Big), \tag{1.1}$$

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$$S_2(m,n;\chi;q)=\sum_{a=1}^{q}{}'\chi(a)e\Big(\frac{ma+n\bar{a}}{q}\Big), \tag{1.2}$$

$$S_3(m,n;k;q)=\sum_{a=1}^{q}{}'e\Big(\frac{ma^k+na}{q}\Big), \tag{1.3}$$

$$S_4(m,n;k;\chi;q)=\sum_{a=1}^{q}{}'\chi(a)e\Big(\frac{ma^k+na}{q}\Big), \tag{1.4}$$

$$S_5(m,n;k;q)=\sum_{a=1}^{q}{}'e\Big(\frac{ma^k+n\overline{a^k}}{q}\Big), \tag{1.5}$$

$$S_6(m,n;k;\chi;q)=\sum_{a=1}^{q}{}'\chi(a)e\Big(\frac{ma^k+n\overline{a^k}}{q}\Big), \tag{1.6}$$

where $\sum'$ denotes the summation over all a such that $(a,q)=1$, $a\bar{a}\equiv 1 \bmod q$ and $e(y)=\mathrm{e}^{2\pi\mathrm{i}y}$.

These summations are very important, because they are generalizations of the classical Kloosterman sums (that is formula (1.1)) which were introduced in 1926 by H. D. Kloosterman [9], and we have

$$\begin{aligned}S_1(m,n;q)&=S_2(m,n;\chi_0;q)=S_3(m,n;-1;q)\\&=S_4(m,n;-1;\chi_0;q)=S_5(m,n;1;q)=S_6(m,n;1;\chi_0;q),\end{aligned}$$

where χ_0 is the principal character mod q. Many authors have obtained a lot of well-known results. For example, some authors obtained the most famous estimate (see [3], [5]),

$$|S_1(m,n;q)|\ll d(q)q^{1/2}(m,n,q)^{1/2}, \tag{1.7}$$

where $d(q)$ is the divisor function, and (m,n,q) denotes the greatest common divisor of m, n and q. W. P. Zhang [16], R. Evans [6] and K. Gong and D. Q. Wan [7] study the power mean of (1.1) and give some exact computation formulas for them. Many authors have investigated the summations (1.2)–(1.6) by various methods (see [4], [8], [10], [11], [12], [13], [15], and [18]).

Recently, C. Calderon, M. J. De Velasco and M. J. Zarate [2] have defined a new generalized Kloosterman sum

$$S(m,n;\chi;\bar{\chi};q)=\sum_{a=1}^{q}{}'\chi(a)G(a,\bar{\chi})e\Big(\frac{ma^k+na}{q}\Big), \tag{1.8}$$

where χ, $\bar{\chi}$ are Dirichlet characters modulo q at the same time, the sum $G(a,\bar{\chi})$ is a Gauss sum which is defined by $G(a,\bar{\chi})=\sum_{u=1}^{q}{}'\bar{\chi}(u)e(ua/q)$. Now we change the summation of (1.8), then we define the mixed exponential sum by general k-Kloosterman sums as follows:

$$C(m,n;k;\chi;\bar{\chi};q)=\sum_{a=1}^{q}{}'\chi(a)G_k(a,\bar{\chi})e\Big(\frac{ma^k+n\overline{a^k}}{q}\Big), \tag{1.9}$$

with a Dirichlet character and a general Gauss sum $G_k(a,\bar{\chi})$ as coefficient, where

$$G_k(a,\bar{\chi})=\sum_{u=1}^{q}{}'\bar{\chi}(u)e\Big(\frac{ua^k}{q}\Big).$$

In particular, if $k=1$, q is an odd prime and $q\nmid n$ and χ_0 is the principal character modulo q, then we have

$$C(m,n;1;\chi_0;\bar{\chi};q)=\sum_{a=1}^{q}{}'G_1(a,\bar{\chi})e\Big(\frac{ma+n\bar{a}}{q}\Big),$$

where $G_1(a,\bar{\chi})=\sum_{u=1}^{q}{}'\bar{\chi}(u)e(ua/q)$. Since

$$G_1(a,\bar{\chi})=\sum_{u=1}^{q}{}'\bar{\chi}(u)e\Big(\frac{ua}{q}\Big)=\sum_{r=1}^{q}{}'e\Big(\frac{nr^2}{q}\Big)=G(a,q)=(a|q)G(1,q)$$

where $(a|q)$ denotes Legendre's symbol, from [1] we know that

$$G(1,q)=\begin{cases} q^{1/2}, & q\equiv 1 \pmod 4,\\ \mathrm{i}q^{1/2}, & q\equiv 3 \pmod 4.\end{cases}$$

Thus we derive

$$\begin{aligned} C(m,n;1;\chi_0;\bar{\chi};q) &= \begin{cases} q^{1/2}\sum_{a=1}^{q}{}'(a|q)e\Big(\frac{ma+n\bar{a}}{q}\Big), & q\equiv 1 \pmod 4,\\ \mathrm{i}q^{1/2}\sum_{a=1}^{q}{}'(a|q)e\Big(\frac{ma+n\bar{a}}{q}\Big), & q\equiv 3 \pmod 4,\end{cases}\\ &= \begin{cases} q^{1/2}S_2(m,n;\chi;q), & q\equiv 1 \pmod 4,\\ \mathrm{i}q^{1/2}S_2(m,n;\chi;q), & q\equiv 3 \pmod 4.\end{cases}\end{aligned}$$

Finally, we obtain

$$|C(m,n;1;\chi_0;\bar{\chi};q)|=q^{1/2}|S_2(m,n;\chi;q)|. \tag{1.10}$$

By the formula (1.10) we can obtain many results about the summations (1.1)–(1.6) (see [14] and [17]).

In this paper,we use the analytic method and the property of Ramanujan's sum to study the fourth power mean value of the mixed exponential sum $\sum_m \sum_\chi \sum_{\bar\chi} |C(m,n;k;\chi;\bar\chi;q)|^4$ and give some explicit formulas. That is, we shall prove the following theorem and corollary.

Theorem 1.1. *Let m, $q \geqslant 3$ and k be positive integers with $(k,q)=1$. Then for any integer n with $(n,q)=1$ we have*

$$\begin{aligned}\sum_m \sum_\chi \sum_{\bar\chi} &|C(m,n;k;\chi;\bar\chi;q)|^4 \\ &= \varphi^3(q)q\prod_{p\|q}(k,p-1)(2p-(k,p-1)-1)(p^2-p-1) \\ &\quad\times \prod_{\substack{p^\alpha\|q\\ \alpha\geqslant 2}}(k,p-1)p^{3\alpha-2}(p-1)((\alpha p-p-\alpha)(k,p-1)+2p),\end{aligned}$$

where $\varphi(q)$ is the Euler function, and $\prod_{p^\alpha\|q}$ denotes the product over all prime divisors p of q with $p^\alpha \mid q$ and $p^{\alpha+1}\nmid q$.

We immediately get the following corollaries.

Corollary 1.1. *Let q be a square-free number and k a positive integer with $(k,q)=1$. Then for any integer n with $(n,q)=1$ we have*

$$\begin{aligned}\sum_m \sum_\chi \sum_{\bar\chi} &|C(m,n;k;\chi;\bar\chi;q)|^4 \\ &= \varphi^3(q)q\prod_{p\|q}(k,p-1)(2p-(k,p-1)-1)(p^2-p-1).\end{aligned}$$

Corollary 1.2. *Let q be a square-full number and k a positive integer with $(k,q)=1$. Then for any integer n with $(n,q)=1$ we get*

$$\begin{aligned}\sum_m \sum_\chi \sum_{\bar\chi} &|C(m,n;k;\chi;\bar\chi;q)|^4 \\ &= \varphi^3(q)q\prod_{\substack{p^\alpha\|q\\ \alpha\geqslant 2}}(k,p-1)p^{3\alpha-2}(p-1)((\alpha p-p-\alpha)(k,p-1)+2p).\end{aligned}$$

2. Several lemmas

Before starting our proof of the theorem, the following lemmas will be useful.

Lemma 2.1. *Let k, n, m, q_1 and q_2 be positive integers with $(n, q_1q_2) = (q_1, q_2) = 1$. Then for any characters χ modulo q_1q_2 and $\bar{\chi}$ modulo q_1q_2 there exist integers n_1 and n_2 with $(n_1, q_1) = (n_2, q_2) = 1$ such that $n \equiv n_1q_2^{k+1} + n_2q_1^{k+1} \pmod{q_1q_2}$, and for these integers we have*

$$|C(m, n; k; \chi; \bar{\chi}; q_1q_2)| \\ = |C(m(q_2)^{k-1}, n_1; k; \chi_1; \bar{\chi}_1; q_1)|\,|C(m(q_1)^{k-1}, n_2; k; \chi_2; \bar{\chi}_2; q_2)|$$

where $\chi = \chi_1\chi_2$ with χ_1 modulo q_1 and χ_2 modulo q_2, and also $\bar{\chi} = \bar{\chi}_1\bar{\chi}_2$ with $\bar{\chi}_1$ modulo q_1 and $\bar{\chi}_2$ modulo q_2.

P r o o f. From the property of reduced residue systems we get

$$\begin{aligned} C(m, n; k; \chi; \bar{\chi}; q_1q_2) &= \sideset{}{'}\sum_{a=1}^{q_1q_2} \chi(a)G_k(a, \bar{\chi})e\Big(\frac{ma^k + n\overline{a^k}}{q_1q_2}\Big) \\ &= \sideset{}{'}\sum_{a=1}^{q_1q_2} \chi(a) \sideset{}{'}\sum_{u=1}^{q_1q_2} \bar{\chi}(u)e\Big(\frac{ua^k}{q_1q_2}\Big)e\Big(\frac{ma^k + n\overline{a^k}}{q_1q_2}\Big). \end{aligned}$$

Since $(q_1, q_2) = 1$, we have $(q_1^{k+1}, q_2^{k+1}) = 1$. Therefore there exist integers n_1 such that

$$n_1q_2^{k+1} \equiv n \pmod{q_1^{k+1}},$$

and n_2 such that

$$n_2q_1^{k+1} \equiv n \pmod{q_2^{k+1}}.$$

This implies

$$q_1^{k+1} \mid n_1q_2^{k+1} + n_2q_1^{k+1} - n$$

and

$$q_2^{k+1} \mid n_2q_1^{k+1} + n_1q_2^{k+1} - n.$$

That is

$$n \equiv n_1{q_2}^{k+1} + n_2{q_1}^{k+1} \pmod{q_1q_2}.$$

It is clear that $(q_1, n_1) = (q_1, n_1 q_2^{k+1}) = (q_1, n) = 1$ and $(q_2, n_2) = 1$. For these integers q_1, q_2, n_1, n_2 we derive

$$\begin{aligned}
&C(m,n;k;\chi;\bar{\chi};q_1q_2)\\
&\quad= \sum_{b=1}^{q_1}{}'\sum_{c=1}^{q_2}{}'\chi_1\chi_2(bq_2+cq_1)\\
&\qquad\times \sum_{u=1}^{q_1q_2}{}'\bar{\chi}(u)e\Big(\frac{u(bq_2+cq_1)^k+m(bq_2+cq_1)^k+n\overline{(bq_2+cq_1)^k}}{q_1q_2}\Big)\\
&\quad= \sum_{b=1}^{q_1}{}'\sum_{c=1}^{q_2}{}'\chi_1\chi_2(bq_2+cq_1)\sum_{u=1}^{q_1}{}'\sum_{v=1}^{q_2}{}'\bar{\chi}_1\bar{\chi}_2(uq_2+vq_1)\\
&\qquad\times e\Big(\frac{(uq_2+vq_1)(aq_2+cq_1)^k+m(bq_2+cq_1)^k+n\overline{(bq_2+cq_1)^k}}{q_1q_2}\Big)\\
&\quad= \chi_1(q_2)\chi_2(q_1)\sum_{b=1}^{q_1}{}'\chi_1(b)\bar{\chi}_1(q_2)\bar{\chi}_2(q_1)\\
&\qquad\times \sum_{u=1}^{q_1}{}'\bar{\chi}_1(u)e\Big(\frac{u(bq_2)^k}{q_1}\Big)e\Big(\frac{mb^k(q_2)^{k-1}+n_1\overline{b^k}}{q_1}\Big)\\
&\qquad\times \sum_{c=1}^{q_2}{}'\chi_2(c)\sum_{v=1}^{q_2}\bar{\chi}_2(v)e\Big(\frac{v(cq_1)^k}{q_2}\Big)e\Big(\frac{mc^k(q_1)^{k-1}+n_2\overline{c^k}}{q_2}\Big)\\
&\quad= \chi_1(q_2)\chi_2(q_1)\bar{\chi}_1(q_2)\bar{\chi}_2(q_1)\\
&\qquad\times \sum_{b=1}^{q_1}{}'\chi_1(b)\sum_{u=1}^{q_1}{}'\bar{\chi}_1(u)e\Big(\frac{(u{q_2}^k)b^k}{q_1}\Big)e\Big(\frac{mb^k(q_2)^{k-1}+n_1\overline{b^k}}{q_1}\Big)\\
&\qquad\times \sum_{c=1}^{q_2}{}'\chi_2(c)\sum_{v=1}^{q_2}{}'\bar{\chi}_2(v)e\Big(\frac{(v{q_1}^k)c^k}{q_2}\Big)e\Big(\frac{mc^k(q_1)^{k-1}+n_2\overline{c^k}}{q_2}\Big)\\
&\quad= \chi_1(q_2)\chi_2(q_1)\bar{\chi}_1(q_2)\bar{\chi}_2(q_1)\overline{\bar{\chi}_1(q_2^k)\bar{\chi}_2(q_1^k)}\\
&\qquad\times \sum_{b=1}^{q_1}{}'\chi_1(b)\sum_{u=1}^{q_1}{}'\bar{\chi}_1(uq_2^k)e\Big(\frac{(u{q_2}^k)b^k}{q_1}\Big)e\Big(\frac{mb^k(q_2)^{k-1}+n_1\overline{b^k}}{q_1}\Big)\\
&\qquad\times \sum_{c=1}^{q_2}{}'\chi_2(c)\sum_{v=1}^{q_2}{}'\bar{\chi}_2(vq_1^k)e\Big(\frac{(v{q_1}^k)c^k}{q_2}\Big)e\Big(\frac{mc^k(q_1)^{k-1}+n_2\overline{c^k}}{q_2}\Big).
\end{aligned}$$

Since $(u, q_1) = (q_2, q_1) = 1$ and u runs through a reduced residue system modulo q_1, we have $(q_2^k, q_1) = 1$, $(uq_2^k, q_1) = 1$ and uq_2^k runs through a reduced residue system modulo q_1 (analogously vq_1^k runs through a reduced residue system modulo q_2). Thus the above formula simplifies to

$$\begin{aligned}
&C(m,n;k;\chi;\bar\chi;q_1q_2)\\
&\quad=\chi_1(q_2)\chi_2(q_1)\bar\chi_1(q_2)\bar\chi_2(q_1)\overline{\bar\chi_1(q_2^k)\bar\chi_2(q_1^k)}\\
&\qquad\times\sideset{}{'}\sum_{b=1}^{q_1}\chi_1(b)\sideset{}{'}\sum_{u=1}^{q_1}\bar\chi_1(u)e\Big(\frac{ub^k}{q_1}\Big)e\Big(\frac{mb^k(q_2)^{k-1}+n_1\overline{b^k}}{q_1}\Big)\\
&\qquad\times\sideset{}{'}\sum_{c=1}^{q_2}\chi_2(c)\sideset{}{'}\sum_{v=1}^{q_2}\bar\chi_2(v)e\Big(\frac{vc^k}{q_2}\Big)e\Big(\frac{mc^k(q_1)^{k-1}+n_2\overline{c^k}}{q_2}\Big)\\
&\quad=\chi_1(q_2)\chi_2(q_1)\overline{\bar\chi_1(q_2^{k-1})\bar\chi_2(q_1^{k-1})}\\
&\qquad\times C(m(q_2)^{k-1},n_1;k;\chi_1;\bar\chi_1;q_1)C(m(q_1)^{k-1},n_2;k;\chi_2;\bar\chi_2;q_2)\\
&\quad=\chi_1(q_2)\chi_2(q_1)\overline{(\bar\chi_1(q_2)\bar\chi_2(q_1))^{k-1}}\\
&\qquad\times C(m(q_2)^{k-1},n_1;k;\chi_1;\bar\chi_1;q_1)C(m(q_1)^{k-1},n_2;k;\chi_2;\bar\chi_2;q_2).
\end{aligned}$$

Then Lemma 2.1 follows from $|\chi_1(q_2)\chi_2(q_1)|=1$ and $|\bar\chi_1(q_2)\bar\chi_2(q_1)|=1$. □

Lemma 2.2. *Let k, n, m, p^α be positive integers. Let $C(m,n;k;\chi;\bar\chi;p^\alpha)$ be the first sum defined in* (1.9) *with Dirichlet character χ, $\bar\chi$ modulo p^α. Then we have the identity*

$$\sum_\chi\sum_{\bar\chi}|C(m,n;k;\chi;\bar\chi;p^\alpha)|^4=\varphi^2(p^\alpha)\sideset{}{'}\sum_{u=1}^{p^\alpha}\sideset{}{'}\sum_{a=1}^{p^\alpha}|S_1|^2,$$

where

$$S_1=\sideset{}{'}\sum_{v=1}^{p^\alpha}\sideset{}{'}\sum_{b=1}^{p^\alpha}e\Big(\frac{mb^k(a^k-1)+vb^k(ua^k-1)+n\overline{b^k}(\overline{a^k}-1)}{p^\alpha}\Big).$$

P r o o f. From the property of reduced residue systems we have

$$\begin{aligned}
|C(m,n;k;\chi;\bar\chi;p^\alpha)|^2&=\Big|\sideset{}{'}\sum_{a=1}^{q}\chi(a)G_k(a,\bar\chi)e\Big(\frac{ma^k+n\overline{a^k}}{q}\Big)\Big|^2\\
&=\sideset{}{'}\sum_{u=1}^{p^\alpha}\bar\chi(u)\sideset{}{'}\sum_{v=1}^{p^\alpha}\sideset{}{'}\sum_{a=1}^{p^\alpha}\sideset{}{'}\sum_{b=1}^{p^\alpha}\chi(a)e\Big(\frac{mb^k(a^k-1)+vb^k(ua^k-1)+n\overline{b^k}(\overline{a^k}-1)}{p^\alpha}\Big).
\end{aligned}$$

Therefore by the orthogonality relation for characters [1]

$$\sum_{\chi \bmod p}\chi(m)\overline{\chi(n)}=\begin{cases}\varphi(p), & m\equiv n \pmod p,\\ 0, & m\not\equiv n \pmod p.\end{cases}$$

We obtain

$$\sum_{\bar\chi}|C(m,n;k;\chi;\bar\chi;p^\alpha)|^4=\sum_{\bar\chi}\sideset{}{'}\sum_{u=1}^{p^\alpha}\sideset{}{'}\sum_{w=1}^{p^\alpha}\bar\chi(u)\overline{\bar\chi(w)}$$
$$\times\biggl|\sideset{}{'}\sum_{v=1}^{p^\alpha}\sideset{}{'}\sum_{a=1}^{p^\alpha}\sideset{}{'}\sum_{b=1}^{p^\alpha}\chi(a)e\Bigl(\frac{mb^k(a^k-1)+vb^k(ua^k-1)+n\overline{b^k}(\overline{a^k}-1)}{p^\alpha}\Bigr)\biggr|^2$$
$$=\sideset{}{'}\sum_{u=1}^{p^\alpha}\sideset{}{'}\sum_{w=1}^{p^\alpha}\sum_{\bar\chi}\bar\chi(u)\overline{\bar\chi(w)}$$
$$\times\biggl|\sideset{}{'}\sum_{v=1}^{p^\alpha}\sideset{}{'}\sum_{a=1}^{p^\alpha}\sideset{}{'}\sum_{b=1}^{p^\alpha}\chi(a)e\Bigl(\frac{mb^k(a^k-1)+vb^k(ua^k-1)+n\overline{b^k}(\overline{a^k}-1)}{p^\alpha}\Bigr)\biggr|^2$$
$$=\varphi(p^\alpha)\sideset{}{'}\sum_{u=1}^{p^\alpha}\biggl|\sideset{}{'}\sum_{v=1}^{p^\alpha}\sideset{}{'}\sum_{a=1}^{p^\alpha}\sideset{}{'}\sum_{b=1}^{p^\alpha}\chi(a)e\Bigl(\frac{mb^k(a^k-1)+vb^k(ua^k-1)+n\overline{b^k}(\overline{a^k}-1)}{p^\alpha}\Bigr)\biggr|^2.$$

Now by the same method we can easily obtain Lemma 2.2. □

Lemma 2.3. *Let p be an odd prime number. For any positive integer α we consider Ramanujan's sum $C_{p^\alpha}(n)=\sideset{}{'}\sum_{v=1}^{p^\alpha}e\bigl(nv/p^\alpha\bigr)$. Then we have the following identities:*

(i) *For $\alpha=1$,*

$$C_p(n)=\sideset{}{'}\sum_{v=1}^{p^\alpha}e\Bigl(\frac{nv}{p}\Bigr)=\begin{cases}\varphi(p) & \text{if } p\mid n,\\ -1 & \text{if } p\nmid n.\end{cases}$$

(ii) *For $\alpha\geqslant 2$,*

$$C_{p^\alpha}(n)=\sideset{}{'}\sum_{v=1}^{p^\alpha}e\Bigl(\frac{nv}{p^\alpha}\Bigr)=\begin{cases}\varphi(p^\alpha) & \text{if } p^\alpha\mid n,\\ -p^{\alpha-1} & \text{if } p^{\alpha-1}\parallel n,\\ 0 & \text{if } p^{\alpha-1}\nmid n\end{cases}$$

where $p^\alpha\parallel q$ denotes the divisor p of q with $p^\alpha\mid q$ and $p^{\alpha+1}\nmid q$.

P r o o f. See Lemma 2.2 of [7]. □

Lemma 2.4. *Let p be an odd prime number and α any positive integer. If k is a positive integer and $(k,p)=1$, then we have the identity*

$$\sideset{}{'}\sum_{\substack{a=1\\ p^\alpha|(a^k-1)}}^{p^\alpha} \sideset{}{'}\sum_{\substack{b=1\\ (b^k-1)}}^{p^\alpha} 1 = \varphi(p^\alpha)((\alpha-1)d^2+2d) - d^2p^{\alpha-1},$$

where $d=(k,p-1)$.

P r o o f. Using the property of the k-th power congruences and primitive roots, we obtain

$$\sideset{}{'}\sum_{\substack{a=1\\ p^\alpha|(a^k-1)}}^{p^\alpha} \sideset{}{'}\sum_{\substack{b=1\\ (b^k-1)}}^{p^\alpha} 1 = \sideset{}{'}\sum_{\substack{a=1\\ p^\alpha|(a^k-1)\\ p\nmid(a^k-1)}}^{p^\alpha} \sideset{}{'}\sum_{\substack{b=1\\ (b^k-1)}}^{p^\alpha} 1 + \sum_{i=1}^{\alpha-1} \sideset{}{'}\sum_{\substack{a=1\\ p^\alpha|(a^k-1)\\ p^{\alpha-i}\|(a^k-1)}}^{p^\alpha} \sideset{}{'}\sum_{\substack{b=1\\ (b^k-1)}}^{p^\alpha} 1 + \sideset{}{'}\sum_{\substack{a=1\\ p^\alpha|(a^k-1)\\ p^\alpha|(a^k-1)}}^{p^\alpha} \sideset{}{'}\sum_{\substack{b=1\\ (b^k-1)}}^{p^\alpha} 1$$
$$= E_1 + E_2 + E_3.$$

Now we compute E_1, E_2 and E_3 in the above formula. By Theorem 5.33 of [1] we have

$$E_1 = \sideset{}{'}\sum_{\substack{a=1\\ p^\alpha|(a^k-1)\\ p\nmid a^k-1)}}^{p^\alpha} \sideset{}{'}\sum_{\substack{b=1\\ (b^k-1)}}^{p^\alpha} 1 = (k,\varphi(p^\alpha))\Big(\varphi(p^\alpha) - \frac{\varphi(p^\alpha)}{\varphi(p)}(k,\varphi(p))\Big) = d\varphi(p^\alpha) - d^2p^{\alpha-1}.$$

From the same property we can also deduce that

$$\begin{aligned} E_2 &= \sum_{i=1}^{\alpha-1} \sideset{}{'}\sum_{\substack{a=1\\ p^\alpha|(a^k-1)\\ p^{\alpha-i}\|(a^k-1)}}^{p^\alpha} \sideset{}{'}\sum_{\substack{b=1\\ (b^k-1)}}^{p^\alpha} 1 \\ &= \sum_{i=1}^{\alpha-1} \frac{\varphi(p^\alpha)}{\varphi(p^i)}(k,\varphi(p^i))\Big(\frac{\varphi(p^\alpha)}{\varphi(p^{\alpha-i})}(k,\varphi(p^{\alpha-i})) - \frac{\varphi(p^\alpha)}{\varphi(p^{\alpha-i+1})}(k,\varphi(p^{\alpha-i+1}))\Big) \\ &= d\sum_{i=1}^{\alpha-1} p^{\alpha-i}d(p^id - p^{i-1}d) = (\alpha-1)d^2\varphi(p^\alpha). \end{aligned}$$

Finally, we can easily get

$$E_3 = \varphi(p^\alpha)(k,\varphi(p^\alpha)) = d\varphi(P^\alpha).$$

Combining E_1, E_2 and E_3 we immediately deduce Lemma 2.4. □

Lemma 2.5. *Let k, n, m, p^α be positive integers. Let $C(m,n;k;\chi;\bar\chi;p^\alpha)$ be the first sum defined in* (1.9) *with Dirichlet character χ, $\bar\chi$ modulo p^α, then we have the identity*

$$\sum_m\sum_\chi\sum_{\bar\chi}|C(m,n;k;\chi;\bar\chi;p^\alpha)|^4 = \begin{cases}\varphi^3(p)pd(2\varphi(p)-d)(\varphi(p^2)-1) & \text{if } \alpha=1,\\ \varphi^3(p^\alpha)p^\alpha d(\varphi(p^\alpha)((\alpha-1)d+2)-dp^{\alpha-1})\varphi^2(p^{2\alpha}) & \text{if } \alpha\geqslant 2,\end{cases}$$

where $d=(k,p-1)$.

P r o o f. By Lemma 2.3 we can obtain

$$\begin{aligned}
&\sum_{m=1}^{p^\alpha}\sum_\chi\sum_{\bar\chi}|C(m,n;k;\chi;\bar\chi;p^\alpha)|^4\\
&=\sum_{m=1}^{p^\alpha}\varphi^2(p^\alpha)\sideset{}{'}\sum_{u=1}^{p^\alpha}\sideset{}{'}\sum_{a=1}^{p^\alpha}\Big|\sideset{}{'}\sum_{v=1}^{p^\alpha}\sideset{}{'}\sum_{b=1}^{p^\alpha}e\Big(\frac{mb^k(a^k-1)+vb^k(ua^k-1)+n\overline{b^k}(\overline{a^k}-1)}{p^\alpha}\Big)\Big|^2\\
&=\varphi^2(p^\alpha)\sum_{m=1}^{p^\alpha}\sideset{}{'}\sum_{u,a=1}^{p^\alpha}\sideset{}{'}\sum_{v,b,s,t=1}^{p^\alpha}\\
&\quad\times e\Big(\frac{m(b^k-s^k)(a^k-1)+(vb^k-ts^k)(ua^k-1)+n(\overline{b^k}-\overline{s^k})(\overline{a^k}-1)}{p^\alpha}\Big)\\
&=\varphi^2(p^\alpha)\sideset{}{'}\sum_{u=1}^{p^\alpha}\sideset{}{'}\sum_{s=1}^{p^\alpha}\sideset{}{'}\sum_{b=1}^{p^\alpha}\sum_{m=1}^{p^\alpha}e\Big(\frac{m(b^k-s^k)(a^k-1)+n(\overline{b^k}-\overline{s^k})(\overline{a^k}-1)}{p^\alpha}\Big)\\
&\quad\times\sideset{}{'}\sum_{u=1}^{p^\alpha}\sideset{}{'}\sum_{v=1}^{p^\alpha}\sideset{}{'}\sum_{t=1}^{p^\alpha}e\Big(\frac{(vb^k-ts^k)(ua^k-1)}{p^\alpha}\Big)\\
&=\varphi^2(p^\alpha)\times E_4\times E_5.
\end{aligned}$$

Now we compute E_4 and E_5. From the property of reduced residue systems we have

$$\begin{aligned}
E_4&=\sideset{}{'}\sum_{u=1}^{p^\alpha}\sideset{}{'}\sum_{s=1}^{p^\alpha}\sideset{}{'}\sum_{b=1}^{p^\alpha}\sum_{m=1}^{p^\alpha}e\Big(\frac{m(b^k-s^k)(a^k-1)+n(\overline{b^k}-\overline{s^k})(\overline{a^k}-1)}{p^\alpha}\Big)\\
&=\sideset{}{'}\sum_{u=1}^{p^\alpha}\sideset{}{'}\sum_{s=1}^{p^\alpha}\sideset{}{'}\sum_{b=1}^{p^\alpha}\sum_{m=1}^{p^\alpha}e\Big(\frac{ms^k(b^k-1)(a^k-1)+n\overline{b^ks^ka^k}(b^k-1)(a^k-1)}{p^\alpha}\Big)\\
&=\varphi(p^\alpha)p^\alpha\sideset{}{'}\sum_{\substack{a=1\\ p^\alpha|(a^k-1)}}^{p^\alpha}\sideset{}{'}\sum_{\substack{b=1\\ (b^k-1)}}^{p^\alpha}1.
\end{aligned}$$

By Lemma 2.4 we immediately deduce that

$$E_4 = \varphi(p^\alpha)p^\alpha(\varphi(p^\alpha)((\alpha-1)d^2+2d) - d^2p^{\alpha-1}),$$

where $d = (k, p-1)$.

For computing the E_5 we can use the definition of Ramanujan's sum and get

$$\begin{aligned} E_5 &= \sideset{}{'}\sum_{u=1}^{p^\alpha}\sideset{}{'}\sum_{v=1}^{p^\alpha}\sideset{}{'}\sum_{t=1}^{p^\alpha} e\Big(\frac{(vb^k - ts^k)(ua^k-1)}{p^\alpha}\Big) \\ &= \sideset{}{'}\sum_{u=1}^{p^\alpha}\Big(\sideset{}{'}\sum_{v=1}^{p^\alpha} e\Big(\frac{vb^k(ua^k-1)}{p^\alpha}\Big)\Big)\Big(\sideset{}{'}\sum_{t=1}^{p^\alpha} e\Big(-\frac{ts^k(ua^k-1)}{p^\alpha}\Big)\Big). \end{aligned}$$

Since $(b, p^\alpha) = 1$ and r runs through a reduced residue system modulo p^α, we have $(b^k, p^\alpha) = 1$ and $b^k v$ runs through a reduced system modulo p^α, (analogously $s^k t$ runs through a reduced residue system modulo p^α). Thus

$$E_5 = \sideset{}{'}\sum_{u=1}^{p^\alpha} C_{p^\alpha}(ua^k-1)\overline{C_{p^\alpha}(ua^k-1)} = \sideset{}{'}\sum_{u=1}^{p^\alpha} |C_{p^\alpha}(ua^k-1)|^2.$$

From Lemma 2.3 we have

$$\begin{aligned} E_5 &= \begin{cases} \displaystyle\sideset{}{'}\sum_{\substack{u=1 \\ p\mid(ua^k-1)}}^{p} \varphi^2(p) + \sideset{}{'}\sum_{\substack{u=1 \\ p\nmid(ua^k-1)}}^{p} 1 & \text{if } \alpha = 1, \\ \displaystyle\sideset{}{'}\sum_{\substack{u=1 \\ p^\alpha\mid(ua^k-1)}}^{p^\alpha} \varphi^2(p^\alpha) + \sideset{}{'}\sum_{\substack{u=1 \\ p^{\alpha-1}\parallel(ua^k-1)}}^{p^\alpha} p^{2(\alpha-1)} & \text{if } \alpha \geqslant 2, \end{cases} \\ &= \begin{cases} \varphi^2(p) + \varphi(p) - 1 & \text{if } \alpha = 1, \\ \varphi^2(p^\alpha) + p^{2(\alpha-1)}\varphi(p) & \text{if } \alpha \geqslant 2, \end{cases} \\ &= \begin{cases} \varphi(p^2) - 1 & \text{if } \alpha = 1, \\ \varphi^2(p^{2\alpha}) & \text{if } \alpha \geqslant 2. \end{cases} \end{aligned}$$

Combining the above formulas we immediately deduce the computational formula

$$\begin{aligned} &\sum_m \sum_\chi \sum_{\bar\chi} |C(m,n;k;\chi;\bar\chi;p^\alpha)|^4 \\ &= \begin{cases} \varphi^3(p)pd(2\varphi(p)-d)(\varphi(p^2)-1) & \text{if } \alpha = 1, \\ \varphi^3(p^\alpha)p^\alpha d(\varphi(p^\alpha)((\alpha-1)d+2) - dp^{\alpha-1})\varphi^2(p^{2\alpha}) & \text{if } \alpha \geqslant 2. \end{cases} \end{aligned}$$

This completes the proof of Lemma 2.5. □

3. Proof of main theorem

P r o o f. Now we prove the theorem. First, for any integer n with $(n,q)=1$ let q and m be written as $q=\prod\limits_{i=1}^{r}p_i^{\alpha_i}$ and $m=\sum\limits_{i=1}^{r}m_iq/p_i^{\alpha_i}$ respectively. Furthermore, if m_i $(i=1,2,\ldots,r)$ pass through a complete residue system modulo $p_i^{\alpha_i}$, then m runs through a reduced residue system modulo q. Finally, by Lemma 2.1 and Lemma 2.5 in the previous section we obtain

$$
\begin{aligned}
&\sum_{m=1}^{q}\sum_{\chi \bmod q}\sum_{\bar\chi \bmod q}|C(m,n;k;\chi;\bar\chi;q)|^4\\
&=\prod_{i=1}^{r}\Bigg(\sum_{m_i=1}^{p_i^{\alpha_i}}\sum_{\chi_i \bmod p_i^{\alpha_i}}\sum_{\bar\chi_i \bmod p_i^{\alpha_i}}\Big|C\Big(m_i\frac{q}{p_i^{\alpha_i}}\Big(\frac{q}{p_i^{\alpha_i}}\Big)^{k-1},n_i;k;\chi_i;\bar\chi_i;p_i^{\alpha_i}\Big)\Big|^4\Bigg)\\
&=\prod_{i=1}^{r}\Bigg(\sum_{m_i=1}^{p_i^{\alpha_i}}\sum_{\chi_i \bmod p_i^{\alpha_i}}\sum_{\bar\chi_i \bmod p_i^{\alpha_i}}|C(m_i,n_i;k;\chi_i;\bar\chi_i;p_i^{\alpha_i})|^4\Bigg)\\
&=\prod_{\substack{i=1\\ \alpha=1}}^{r}\Bigg(\sum_{m_i=1}^{p_i^{\alpha_i}}\sum_{\chi_i \bmod p_i^{\alpha_i}}\sum_{\bar\chi_i \bmod p_i^{\alpha_i}}\Big|C\Big(m_i,n_i\frac{q}{p_i^{\alpha_i}};k;\chi_i;\bar\chi_i;p_i^{\alpha_i}\Big)\Big|^4\Bigg)\\
&\quad\times\prod_{\substack{i=1\\ \alpha\geqslant 2}}^{r}\Bigg(\sum_{m_i=1}^{p_i^{\alpha_i}}\sum_{\chi_i \bmod p_i^{\alpha_i}}\sum_{\bar\chi_i \bmod p_i^{\alpha_i}}\Big|C\Big(m_i,n_i\frac{q}{p_i^{\alpha_i}};k;\chi_i;\bar\chi_i;p_i^{\alpha_i}\Big)\Big|^4\Bigg)\\
&=\prod_{p\|q}(\varphi^3(p)p(k,p-1)(2\varphi(p)-(k,p-1))(\varphi(p^2)-1))\\
&\quad\times\prod_{\substack{p^\alpha\|q\\ \alpha\geqslant 2}}(\varphi^3(p^\alpha)p^\alpha(k,p-1)(\varphi(p^\alpha)((\alpha-1)(k,p-1)+2)-(k,p-1)p^{\alpha-1})\varphi^2(p^{2\alpha}))\\
&=\varphi^3(q)q\prod_{p\|q}((k,p-1)(2\varphi(p)-(k,p-1))(\varphi(p^2)-1))\\
&\quad\times\prod_{\substack{p^\alpha\|q\\ \alpha\geqslant 2}}((k,p-1)(\varphi(p^\alpha)((\alpha-1)(k,p-1)+2)-(k,p-1)p^{\alpha-1})\varphi^2(p^{2\alpha}))\\
&=\varphi^3(q)q\prod_{p\|q}(k,p-1)(2p-(k,p-1)-1)(p^2-p-1)\\
&\quad\times\prod_{\substack{p^\alpha\|q\\ \alpha\geqslant 2}}(k,p-1)p^{3\alpha-2}(p-1)((\alpha p-p-\alpha)(k,p-1)+2p).
\end{aligned}
$$

This completes the proof of the theorem. □

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