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THE EFFICIENCY OF APPROXIMATING REAL NUMBERS BY LÜROTH EXPANSION

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Abstract. For any $x \in (0,1]$, let

$$x = \frac{1}{d_1} + \frac{1}{d_1(d_1-1)d_2} + \ldots + \frac{1}{d_1(d_1-1)\ldots d_{n-1}(d_{n-1}-1)d_n} + \ldots$$

be its Lüroth expansion. Denote by $P_n(x)/Q_n(x)$ the partial sum of the first n terms in the above series and call it the nth convergent of x in the Lüroth expansion. This paper is concerned with the efficiency of approximating real numbers by their convergents $\{P_n(x)/Q_n(x)\}_{n\geqslant 1}$ in the Lüroth expansion. It is shown that almost no points can have convergents as the optimal approximation for infinitely many times in the Lüroth expansion. Consequently, Hausdorff dimension is introduced to quantify the set of real numbers which can be well approximated by their convergents in the Lüroth expansion, namely the following Jarník-like set: $\{x \in (0,1] \colon |x - P_n(x)/Q_n(x)| < 1/Q_n(x)^{\nu+1}$ infinitely often$\}$ for any $\nu \geqslant 1$.

Keywords: Lüroth expansion, optimal approximation, Hausdorff dimension

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1. Introduction

How well an irrational number can be approximated by rationals is a long standing question in number theory. Many algorithms have been introduced to approximate real numbers by rationals, such as decimal expansion, continued fraction expansion, as well as the Lüroth expansion. In a common sense, one of the criteria for testing whether an algorithm is effective or not in approximating the real numbers by rational numbers is whether the convergent defined by the algorithm is the optimal approximation or not.

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Recall that a rational fraction a/b ($b > 0$) is said to be an optimal approximation to a real number α if for any $c/d \neq a/b$ and $0 < d \leqslant b$ holds that

$$|d\alpha - c| > |b\alpha - a|.$$

In this paper, we mainly talk about the efficiency of approximating real numbers by the convergents in the Lüroth expansion. Let us first recall the algorithm for the Lüroth expansion, which was first introduced by Lüroth [11] in 1883. For any $x \in (0,1]$, the Lüroth map $T\colon (0,1] \to (0,1]$ is defined by

$$T(x) := d_1(x)(d_1(x)-1)\Big(x - \frac{1}{d_1(x)}\Big), \quad \text{where } d_1(x) = \Big[\frac{1}{x}\Big] + 1. \tag{1.1}$$

Then we define the integer sequence $\{d_n(x), n \geqslant 1\}$ by

$$d_n(x) = d_1(T^{n-1}(x)), \quad n \geqslant 1, \tag{1.2}$$

where T^n denotes the nth iterate of T ($T^0 = \mathrm{Id}_{(0,1]}$).

By the algorithm (1.1) and (1.2), any $x \in (0,1]$ can be developed uniquely into an infinite series expansion of the form

$$x = \frac{1}{d_1(x)} + \sum_{j=2}^{n} \frac{1}{\prod_{i=1}^{j-1} d_i(x)(d_i(x)-1)d_j(x)} + \frac{T^n(x)}{\prod_{j=1}^{n} d_j(x)(d_j(x)-1)} \tag{1.3}$$

$$= \sum_{j=1}^{\infty} \frac{1}{d_1(x)(d_1(x)-1)\dots d_{j-1}(x)(d_{j-1}(x)-1)d_j(x)}, \tag{1.4}$$

which is called the Lüroth expansion of x and denoted by $x = [d_1(x), d_2(x), \dots, d_n(x), \dots]$ for short.

In this setting, the nth convergent $P_n(x)/Q_n(x)$ of x in the Lüroth expansion is defined as the partial sum of the first n terms of the series (1.4), i.e.

$$\frac{P_n(x)}{Q_n(x)} = \sum_{j=1}^{n} \frac{1}{d_1(x)(d_1(x)-1)\dots d_{j-1}(x)(d_{j-1}(x)-1)d_j(x)}.$$

Now we are interested in the set of points whose convergents in the Lüroth expansion are the optimal approximation for infinitely many times. It is natural to ask how large such a set is in the sense of Lebesgue measure. Our first result shows that it is of Lebesgue measure zero. Denote such a set by F, i.e.

$$F = \Big\{x \in (0,1]\colon \frac{P_n(x)}{Q_n(x)} \quad \text{is an optimal approximation to } x \text{ i.o.}\Big\}.$$

We prove

Theorem 1.1. *$m(F) = 0$, where m denotes the Lebesgue measure.*

Nevertheless, there should exist points which can be well approximated by the convergents in the Lüroth expansion. Hence we are led to quantify such null sets by their Hausdorff dimension, namely the following Jarník-like set: for any $\nu \geqslant 1$,

$$W_\nu = \Big\{x \in (0,1] \colon \Big|x - \frac{P_n(x)}{Q_n(x)}\Big| < \frac{1}{Q_n(x)^{\nu+1}} \quad \text{for infinitely many } n\Big\}.$$

Theorem 1.2. *For any $\nu \geqslant 1$, $\dim_{\mathrm{H}} W_\nu = 1/(\nu+1)$.*

It should be compared with approximating real numbers by convergents in continued fraction expansion. By Dirichlet's theorem, the set corresponding to W_1 is of full measure, while, by Jarník's theorem, the Hausdorff dimension of the set corresponding to W_ν is $2/(\nu+1)$.

The metric and ergodic properties of the sequence $\{d_n(x)\}_{n\geqslant 1}$ and the Lüroth map T defined by (1.1) have been extensively studied in [4] (see also [7], [8], [9], [12], [16]). The behavior of approximating real numbers by the Lüroth expansion was thoroughly investigated in [2], [3], where the authors studied the distribution of the approximation coefficients $\theta_n = \theta_n(x) = Q_n(x)x - P_n(x)$ for $n \geqslant 1$. The error-sum function of the Lüroth expansion defined by $S(x) = \sum_{n=1}^{\infty} (x - P_n(x)/Q_n(x))$ was studied in [13], where the authors investigated the properties of this function and determined the Hausdorff dimension of its graph. Since the Lüroth system can also be viewed as an infinite iterated function system, dimensional theory is also of great importance for Lüroth expansions. The spectrum analysis of the frequency of the digits of $\{d_n\}_{n\geqslant 1}$ was given in [1], [6]. The growth speed of the sequence $\{d_n(x)\}_{n\geqslant 1}$ was studied in [14].

Throughout this paper, we use $|\cdot|$ to denote the diameter of a subset of $(0,1]$, $\dim_H$ to denote the Hausdorff dimension and cl the closure of a set, respectively. Also, we may use a probability notation i.o. to mean "for infinitely many times".

2. Preliminaries

In this section, we fix some notation and briefly recall some basic properties of the Lüroth expansion.

Definition 2.1. A sequence of integers $\{d_n\}_{n\geqslant 1}$ is called an admissible sequence if there exists $x \in (0,1]$ such that

$$d_n(x) = d_n, \quad \forall\, n \in \mathbb{N}$$

in the Lüroth expansion of x.

Lemma 2.1 ([7]). *A sequence of integers* $\{d_n\}_{n\geqslant 1}$ *is admissible if and only if*

$$d_n \geqslant 2, \quad \forall\, n \geqslant 1.$$

For any $n \geqslant 1$, denote by $\mathbb{L}_n$ the collection of all admissible blocks of order n, i.e.

$$\mathbb{L}_n = \{(d_1, d_2, \ldots, d_n)\colon\ d_j \geqslant 2 \text{ for any } 1 \leqslant j \leqslant n\}.$$

For any $(d_1, d_2, \ldots, d_n) \in \mathbb{L}_n$, let

$$I_n(d_1, d_2, \ldots, d_n) = \mathrm{cl}\{x \in (0,1]\colon\ d_1(x) = d_1, d_2(x) = d_2, \ldots, d_n(x) = d_n\}.$$

It is clear from the algorithm (1.1) that its length is given by the following formula.

Lemma 2.2 ([7]). $|I_n(d_1, d_2, \ldots, d_n)| = \Big(\prod\limits_{j=1}^{n} d_j(d_j - 1)\Big)^{-1}$.

In order to compute the measure of the set F, we need some results related to the continued fraction expansion (see [10] for more details). As is well known, every real number $x \in (0,1]$ can be expanded as the continued fraction expansion

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \ddots}} := [0; a_1(x), a_2(x), \ldots],$$

where the digits $a_n(x)$ are all positive integers and called partial quotients of x. The convergents in the continued fraction expansion are defined as follows:

$$\frac{p_n(x)}{q_n(x)} = [0; a_1(x), a_2(x), \ldots, a_n(x)].$$

Lemma 2.3 ([10]). *For any* $x \in (0,1]$, $p_n(x)/q_n(x)$ *are the convergents in the continued fraction expansion of* x. *Then*

(a) *for any* $n \geqslant 1$, $p_{n-1}(x)q_n(x) - p_n(x)q_{n-1}(x) = (-1)^n$,
(b) $|x - p_n(x)/q_n(x)| < 1/q_n(x)q_{n+1}(x)$.

Lemma 2.4 ([10]). *Every optimal approximation is a convergent in the continued fraction expansion.*

To end this section, we state the mass distribution principle, which is a classical tool to determine the lower bound of the Hausdorff dimension of a set.

Lemma 2.5 (Mass distribution principle [5]). *Suppose $E \subset (0,1]$ and let μ be a measure with $\mu(E) > 0$. If there exist constants $c > 0$ and $\delta > 0$ such that*

$$\mu(D) \leqslant c|D|^s \tag{2.1}$$

for all sets D with the diameter $|D| \leqslant \delta$, then

$$\dim_{\mathrm{H}} E \geqslant s.$$

3. Efficiency of approximation by Lüroth expansion

Recall that $F = \{x \in (0,1]\colon\ P_n(x)/Q_n(x)$ is an optimal approximation to x i.o.$\}$. In this section, we prove Theorem 1.1, i.e. F is a set of Lebesgue measure zero, which indicates that almost no points can be well approximated by the convergents in the Lüroth expansion.

Proof of Theorem 1.1. Let $\{a_j(x)\}_{j\geqslant 1}$ and $\{p_j(x)/q_j(x)\}_{j\geqslant 1}$ be, respectively, the partial quotients and convergents in the continued fraction expansion of x. For any $0 < \varepsilon < 1$, assume $A = \{x\colon\ a_{j+1}(x) \geqslant q_j^{1-\varepsilon}(x)$ i.o.$\}$ and $B = \{x\colon\ a_{j+1}(x) \geqslant q_j^{1-\varepsilon}(x)$ for only a finite number of indices$\}$. Then we have

$$F = (F \cap A) \cup (F \cap B).$$

By the Borel-Cantelli lemma, the set A is of measure zero, hence we only need to prove that $F \cap B$ is of measure zero. Suppose $x \in F \cap B$ and for any $Q_n(x)$ there exists an integer i such that

$$q_i(x) \leqslant Q_n(x) < q_{i+1}(x).$$

Then if $P_n(x)/Q_n(x)$ is the optimal approximation, by Lemma 2.3 and Lemma 2.4 we know that $P_n(x)/Q_n(x) = p_i(x)/q_i(x)$. Thus we have

$$\Bigl|x - \frac{P_n(x)}{Q_n(x)}\Bigr| = \Bigl|x - \frac{p_i(x)}{q_i(x)}\Bigr| < \frac{1}{q_i(x)q_{i+1}(x)} < \frac{a_{i+1}(x)+1}{Q_n^2(x)} \leqslant \frac{1}{Q_n^{1+\varepsilon}(x)}.$$

While algorithm (1.3) gives

$$\Bigl|x - \frac{P_n(x)}{Q_n(x)}\Bigr| = \frac{T^n(x)}{d_1(x)(d_1(x)-1)\ldots d_n(x)(d_n(x)-1)}.$$

Since $T^n(x) > 1/d_{n+1}(x)$, we obtain

$$\begin{aligned} F \cap B &\subset \{x \in (0,1]\colon\ d_{n+1}(x)(d_n(x)-1) > Q_n^{\varepsilon}(x) \text{ i.o.}\} \\ &= \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{x \in (0,1]\colon\ d_{n+1}(x)(d_n(x)-1) > Q_n^{\varepsilon}(x)\} \\ &= \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \bigcup_{(d_1,d_2,\dots,d_n)\in \mathbb{L}_n} \bigcup_{d > Q_n^{\varepsilon}/(d_n-1)} I_{n+1}(d_1,d_2,\dots,d_n,d) \end{aligned}$$

where $Q_n = d_1(d_1-1)\dots d_{n-1}(d_{n-1}-1)d_n$.

From Lemma 2.2 we have

$$m\Big(\bigcup_{d > Q_n^{\varepsilon}/(d_n-1)} I_{n+1}(d_1,d_2,\dots,d_n,d)\Big) \leqslant \frac{2}{Q_n^{1+\varepsilon}}.$$

Thus

$$\begin{aligned} m(F\cap B) &\leqslant \liminf_{m\to\infty} \sum_{n\geqslant m} \sum_{(d_1,d_2,\dots,d_n)\in\mathbb{L}_n} \frac{2}{Q_n^{1+\varepsilon}} \\ &\leqslant \liminf_{m\to\infty} \sum_{n\geqslant m} \sum_{(d_1,d_2,\dots,d_{n-1})\in\mathbb{L}_{n-1}} \Big(\prod_{j=1}^{n-1} d_j(d_j-1)\Big)^{-(1+\varepsilon)} \sum_{d_n=2}^{\infty} \frac{2}{(d_n-1)^{1+\varepsilon}} \\ &\leqslant 2\zeta(1+\varepsilon) \liminf_{m\to\infty} \sum_{n\geqslant m} 2^{-(n-1)\varepsilon} \sum_{(d_1,d_2,\dots,d_{n-1})\in\mathbb{L}_{n-1}} \Big(\prod_{j=1}^{n-1} d_j(d_j-1)\Big)^{-1} \\ &\leqslant 2\zeta(1+\varepsilon) \liminf_{m\to\infty} \sum_{n\geqslant m} 2^{-(n-1)\varepsilon} = 0 \end{aligned}$$

where $\zeta(t) = \sum_{n=1}^{\infty} 1/n^t$ is the Riemann-Zeta function. □

4. The Hausdorff dimension of W_ν

Recall

$$W_\nu = \Big\{x \in (0,1]\colon\ \Big|x - \frac{P_n(x)}{Q_n(x)}\Big| < \frac{1}{Q_n(x)^{\nu+1}} \quad \text{for infinitely many } n\Big\}.$$

In this section, we will establish the Hausdorff dimension of W_ν. Let

$$E_\beta = \Big\{x \in (0,1]\colon\ d_j(x) \geqslant 2,\ \forall j \geqslant 1 \text{ and } d_{n+1}(x) \geqslant \beta \prod_{j=1}^{n-1} (d_j(x)(d_j(x)-1))^{\nu} d_n^{\nu-1}(x) \text{ i.o.}\Big\}$$

for any $\beta \geqslant 1$. Then we have:

Lemma 4.1. $E_\beta \subset W_\nu \subset E_1$ *with* $\beta > 4$.

P r o o f. This follows from the simple observation that by the algorithm (1.1), (1.2) and the formula (1.4) we have

$$\Bigl|x - \frac{P_n(x)}{Q_n(x)}\Bigr| = \frac{T^n(x)}{d_1(x)(d_1(x)-1)\dots d_n(x)(d_n(x)-1)}$$

and

$$\frac{1}{d_{n+1}(x)} < T^n x \leqslant \frac{1}{d_{n+1}(x)-1}.$$

□

As a result, it suffices to determine the dimension of E_β for any $\beta \geqslant 1$.

4.1. The upper bound.

Proposition 4.1. $\dim_{\rm H} E_\beta \leqslant 1/(\nu+1)$.

P r o o f. Let

$$J_n(d_1, d_2, \dots, d_n) = \bigcup_{d \geqslant \beta \prod_{j=1}^{n-1}(d_j(d_j-1))^\nu d_n^{\nu-1}} I_{n+1}(d_1, d_2, \dots, d_n, d).$$

Then we have

$$\begin{aligned} E_\beta &= \Bigl\{x \in (0,1]\colon\ d_j(x) \geqslant 2, \forall j \geqslant 1, d_{n+1}(x) \\ &\qquad\qquad \geqslant \beta \prod_{j=1}^{n-1}(d_j(x)(d_j(x)-1))^\nu d_n^{\nu-1}(x) \text{ i.o.}\Bigr\} \\ &= \bigcap_{m=1}^\infty \bigcup_{n=m}^\infty \Bigl\{x \in (0,1]\colon\ d_j(x) \geqslant 2, \forall j \geqslant 1, d_{n+1}(x) \\ &\qquad\qquad \geqslant \beta \prod_{j=1}^{n-1}(d_j(x)(d_j(x)-1))^\nu d_n^{\nu-1}(x)\Bigr\} \\ &= \bigcap_{m=1}^\infty \bigcup_{n=m}^\infty \bigcup_{(d_1,d_2,\dots,d_n)\in\mathbb{L}_n} \bigcup_{d \geqslant \beta \prod_{j=1}^{n-1}(d_j(d_j-1))^\nu d_n^{\nu-1}} I_{n+1}(d_1, d_2, \dots, d_n, d) \\ &= \bigcap_{m=1}^\infty \bigcup_{n=m}^\infty \bigcup_{(d_1,d_2,\dots,d_n)\in\mathbb{L}_n} J_n(d_1, d_2, \dots, d_n). \end{aligned}$$

From Lemma 2.2 we have

$$|J_n(d_1, d_2, \ldots, d_n)| = \frac{1}{\prod\limits_{j=1}^{n} d_j(d_j-1)\Big(\beta\Big(\prod\limits_{j=1}^{n-1} d_j(d_j-1)\Big)^{\nu} d_n^{\nu-1} - 1\Big)}$$
$$< \frac{2}{\beta \prod\limits_{j=1}^{n-1} (d_j(d_j-1))^{\nu+1} d_n^{\nu}(d_n-1)}.$$

Then for any $s > 1/(\nu+1)$,

$$\begin{aligned}
H^s(E_\beta) &\leqslant \liminf_{m\to\infty} \sum_{n\geqslant m} \sum_{(d_1,d_2,\ldots,d_n)\in\mathbb{L}_n} |J_n(d_1,d_2,\ldots,d_n)|^s \\
&\leqslant \liminf_{m\to\infty} \sum_{n\geqslant m} \sum_{(d_1,d_2,\ldots,d_n)\in\mathbb{L}_n} \beta^{-s}2^s \prod_{j=1}^{n-1} (d_j(d_j-1))^{-(\nu+1)s} \\
&\qquad \times (d_n^{\nu}(d_n-1))^{-s} \\
&\leqslant \beta^{-s}2^s \liminf_{m\to\infty} \sum_{n\geqslant m} \sum_{(d_1,d_2,\ldots,d_{n-1})\in\mathbb{L}_{n-1}} \prod_{j=1}^{n-1} (d_j(d_j-1))^{-(\nu+1)s} \\
&\qquad \times \sum_{d_n=2}^{\infty} (d_n-1)^{-(\nu+1)s} \\
&\leqslant \beta^{-s}2^s \zeta((\nu+1)s) \liminf_{m\to\infty} \sum_{n\geqslant m} \sum_{(d_1,\ldots,d_{n-1})\in\mathbb{L}_{n-1}} \prod_{j=1}^{n-1} (d_j(d_j-1))^{-(\nu+1)s} \\
&\leqslant \beta^{-s}2^s \zeta((\nu+1)s)
\end{aligned}$$

where $\zeta(t) = \sum\limits_{n=1}^{\infty} 1/n^t$. Then we have $\dim_{\mathrm{H}} W \leqslant s$, and since $s > 1/(\nu+1)$ is arbitrary, we get $\dim_{\mathrm{H}} W \leqslant 1/(\nu+1)$. □

4.2. The lower bound. In this subsection, we determine the lower bound of $\dim_{\mathrm{H}} E_\beta$ for $\beta > 4$. It should be mentioned that the method used here is similar to that in [15].

Proposition 4.2. $\dim_{\mathrm{H}} E_\beta \geqslant 1/(\nu+1)$.

Applying the Mass distribution principle to obtain the lower bound of $\dim_{\mathrm{H}} E_\beta$, first we construct a Cantor subset of E, secondly, we define a probability measure μ supported on the Cantor subset, and at last, we estimate the Hölder exponent of μ.

4.2.1. Cantor subset. On account of using the following formula frequently, for the clarity and concision of expression we write

$$\Theta(x,n,\{d_j\}) = \left[\beta \prod_{j=1}^{n-1} (d_j(x)(d_j(x)-1))^{\nu} d_n^{\nu-1}(x)\right].$$

For any given $B \geqslant 2$, we choose inductively a rapidly increasing sequence of integers satisfying $n_1 \geqslant 2$ and for any $k \geqslant 1$,

$$n_{k+1} \geqslant (2\beta)^{(\nu+1)k(3^{k+1}+k)} B^{(\nu+1)k \sum_{i=1}^{k} 3^i n_{k-i+1}}.$$

Let

$$E_B = \{x \in (0,1]\colon\ \Theta(x,n_k,\{d_j\}) + 1 \leqslant d_{n_k+1}(x) \leqslant 2\Theta(x,n_k,\{d_j\}) \text{ for any } k \geqslant 1, \text{ and } 2 \leqslant d_j(x) \leqslant B, \text{ for any } j \neq n_k+1\}.$$

In addition, to make the proof clearer and illustrate the structure of E_B, we will make use of a kind of symbolic space defined as follows. For any $n \geqslant 1$, define

$$D_n = \{(\sigma_1,\ldots,\sigma_n) \in \mathbb{N}^n\colon\ \Theta(x,n_k,\{\sigma_j\}) + 1 \leqslant \sigma_{n_k+1} \leqslant 2\Theta(x,n_k,\{\sigma_j\}), \ 2 \leqslant \sigma_j \leqslant B \text{ for any } j \neq n_k+1 \leqslant n\},$$

and $D = \bigcup_{n=0}^{\infty} D_n$ ($D_0 := \emptyset$).

For any $n \geqslant 1$ and $(\sigma_1,\ldots,\sigma_n) \in D_n$, we call $I_n(\sigma_1,\ldots,\sigma_n)$ a basic interval of order n and

$$J(\sigma_1,\ldots,\sigma_n) = \bigcup_{\sigma_{n+1}} I_{n+1}(\sigma_1,\ldots,\sigma_n,\sigma_{n+1}) \tag{4.1}$$

a fundamental interval of order n, where the union in (4.1) is taken over all σ_{n+1} such that $(\sigma_1,\ldots,\sigma_n,\sigma_{n+1}) \in D_{n+1}$. Then, if $n \neq n_k$ for any $k \geqslant 1$,

$$|J(\sigma_1,\sigma_2,\ldots,\sigma_n)| = \frac{B-1}{B} \prod_{j=1}^{n} \frac{1}{\sigma_j(\sigma_j-1)}; \tag{4.2}$$

if $n = n_k$ for some $k \geqslant 1$, we have

$$|J(\sigma_1,\sigma_2,\ldots,\sigma_n)| = \left(2\Theta(x,n,\{\sigma_j\}) \prod_{j=1}^{n} \sigma_j(\sigma_j-1)\right)^{-1}. \tag{4.3}$$

It is clear that

$$E_B = \bigcap_{n\geqslant 1} \bigcup_{(\sigma_1,\dots,\sigma_n)\in D_n} J_n(\sigma_1, \sigma_2, \dots, \sigma_n). \tag{4.4}$$

4.2.2. A probability measure supported on E_B. For any given $B \geqslant 2$, let s_B be the unique solution of

$$\sum_{d=2}^{B} \Big(\frac{1}{d(d-1)}\Big)^{(\nu+1)\varrho} = 1.$$

Since

$$\sum_{d=2}^{\infty} \frac{1}{d(d-1)} = 1,$$

we know that $0 \leqslant s_B \leqslant 1/(\nu+1)$.

To constitute a probability measure supported on E_B, we first define a set function $\mu\colon \{J(\sigma), \sigma \in D \setminus D_0\} \to \mathbb{R}^+$ given as follows.

For any $(\sigma_1, \dots, \sigma_{n_1}) \in D_{n_1}$, let

$$\mu(J(\sigma_1, \dots, \sigma_{n_1})) = \prod_{j=1}^{n_1} (\sigma_j(\sigma_j - 1))^{-(\nu+1)s_B}$$

and let

$$\mu(J(\sigma_1, \dots, \sigma_{n_1}, \sigma_{n_1+1})) = \Theta(x, n_1, \{\sigma_j\})^{-1} \mu(J(\sigma_1, \dots, \sigma_{n_1}))$$

for any $(\sigma_1, \dots, \sigma_{n_1}, \sigma_{n_1+1}) \in D_{n_1+1}$.

For any $(\sigma_1, \dots, \sigma_{n_2}) \in D_{n_2}$, let

$$\mu(J(\sigma_1, \dots, \sigma_{n_2})) = \mu(J(\sigma_1, \dots, \sigma_{n_1}, \sigma_{n_1+1})) \prod_{j=n_1+2}^{n_2} (\sigma_j(\sigma_j - 1))^{-(\nu+1)s_B}$$

and let

$$\mu(J(\sigma_1, \dots, \sigma_{n_2}, \sigma_{n_2+1})) = \Theta(x, n_2, \{\sigma_j\})^{-1} \mu(J(\sigma_1, \dots, \sigma_{n_2}))$$

for any $(\sigma_1, \dots, \sigma_{n_2}, \sigma_{n_2+1}) \in D_{n_2+1}$.

For any $n_1 + 1 < n < n_2$ and $(\sigma_1, \dots, \sigma_n) \in D_n$, let

$$\begin{aligned} \mu(J(\sigma_1, \dots, \sigma_n)) &= \sum_{2\leqslant \sigma_{n+1},\dots,\sigma_{n_2}\leqslant B} \mu(J(\sigma_1, \dots, \sigma_{n_2})) \\ &= \mu(J(\sigma_1, \dots, \sigma_{n_1}, \sigma_{n_1+1})) \prod_{j=n_1+2}^{n} (\sigma_j(\sigma_j - 1))^{-(\nu+1)s_B}. \end{aligned}$$

Suppose that for some $k \geqslant 3$, $\mu(J(\sigma_1, \ldots, \sigma_{n_k}))$ has been defined for any $(\sigma_1, \ldots, \sigma_{n_k}) \in D_{n_k}$, and let

$$\mu(J(\sigma_1, \ldots, \sigma_{n_k}, \sigma_{n_k+1})) = \Theta(x, n_k, \{\sigma_j\})^{-1} \mu(J(\sigma_1, \ldots, \sigma_{n_k})).$$

For any $n_{k-1} + 1 < n < n_k$ and $(\sigma_1, \ldots, \sigma_n) \in D_n$, let

$$\begin{aligned}\mu(J(\sigma_1, \ldots, \sigma_n)) &= \sum_{2 \leqslant \sigma_{n+1}, \ldots, \sigma_{n_k} \leqslant B} \mu(J(\sigma_1, \ldots, \sigma_{n_k})) \\ &= \mu(J(\sigma_1, \ldots, \sigma_{n_{k-1}}, \sigma_{n_{k-1}+1})) \prod_{j=n_{k-1}+2}^{n} (\sigma_j(\sigma_j - 1))^{-(\nu+1)s_B}\end{aligned}$$

and let

$$\mu(J(\sigma_1, \ldots, \sigma_{n_{k+1}})) = \mu(J(\sigma_1, \ldots, \sigma_{n_k+1})) \prod_{j=n_k+2}^{n_{k+1}} (\sigma_j(\sigma_j - 1))^{-(\nu+1)s_B}.$$

Until now, the set function $\mu\colon \{J(\sigma), \sigma \in D \setminus D_0\} \to \mathbb{R}^+$ is well defined. It is easy to check that for any $n \geqslant 1$ and $(\sigma_1, \ldots, \sigma_n) \in D_n$ we have

$$\mu(J(\sigma_1, \ldots, \sigma_n)) = \sum_{\sigma_{n+1}} \mu(J(\sigma_1, \ldots, \sigma_{n+1})),$$

where the summation is taken over all σ_{n+1} such that $(\sigma_1, \ldots, \sigma_{n+1}) \in D_{n+1}$. Notice that

$$\sum_{\sigma_1 \in D_1} \mu(J(\sigma_1)) = 1.$$

By the Kolmogorov extension theorem, the set function μ can be extended into a probability measure supported on E_B, which is still denoted by μ.

4.2.3. Hölder exponent of μ. Given a fundamental interval $J(\sigma_1, \ldots, \sigma_n)$ where $(\sigma_1, \ldots, \sigma_n) \in D_n$, denote by $g^r(\sigma_1, \ldots, \sigma_n)$ the distance between $J(\sigma_1, \ldots, \sigma_n)$ and the fundamental interval of order n which lies to the right of $J(\sigma_1, \ldots, \sigma_n)$ and is closest to it, and by $g^l(\sigma_1, \ldots, \sigma_n)$ the distance between $J(\sigma_1, \ldots, \sigma_n)$ and the fundamental interval of order n which lies to the left of $J(\sigma_1, \ldots, \sigma_n)$ and is closest to it. Let

$$g(\sigma_1, \ldots, \sigma_n) = \min\{g^r(\sigma_1, \ldots, \sigma_n), g^l(\sigma_1, \ldots, \sigma_n)\}.$$

Then we get the following lemma.

Lemma 4.2.

$$g(\sigma_1, \ldots, \sigma_n) \geqslant \frac{1}{B^2 - 1}|J(\sigma_1, \ldots, \sigma_n)|.$$

Proof. We will divide the proof into two parts.

Case I: $n \neq n_k$ *for any* $k \geqslant 1$.

In this case, the fundamental interval of order n which is closest to and lies to the left of $J(\sigma_1, \ldots, \sigma_n)$ is separated from $J(\sigma_1, \ldots, \sigma_n)$ by $I_{n+1}(\sigma_1, \ldots, \sigma_n, B+1)$. For the right side, we can always assume that these two fundamental intervals are both contained in the $J(\sigma_1, \ldots, \sigma_{n-1})$; if not, it means that they are contained in different fundamental intervals of order $n-1$, and then

$$g^r(\sigma_1, \ldots, \sigma_n) \geqslant g^r(\sigma_1, \ldots, \sigma_{n-1}).$$

Then the fundamental interval of order n which is closest to and lies to the right of $J(\sigma_1, \ldots, \sigma_n)$ is separated from $J(\sigma_1, \ldots, \sigma_n)$ by $I_{n+1}(\sigma_1, \ldots, \sigma_n - 1, B+1)$. Thus

$$\begin{aligned} g(\sigma_1, \ldots, \sigma_n) &\geqslant \min\{|I_{n+1}(\sigma_1, \ldots, \sigma_n, B+1)|, |I_{n+1}(\sigma_1, \ldots, \sigma_n - 1, B+1)|\} \\ &\geqslant \prod_{j=1}^{n} \frac{1}{\sigma_j(\sigma_j - 1)} \frac{1}{B(B+1)} \geqslant \frac{1}{B^2 - 1}|J(\sigma_1, \ldots, \sigma_n)|. \end{aligned}$$

Case II: $n = n_k$ *for some* $k \geqslant 1$.

In this case, $J(\sigma_1, \ldots, \sigma_n)$ lies in the interior of $I_n(\sigma_1, \ldots, \sigma_n)$, so it suffices to consider the distance between the two endpoints of $J(\sigma_1, \ldots, \sigma_n)$ and the corresponding two endpoints of $I_n(\sigma_1, \ldots, \sigma_n)$. Then we have

$$\begin{aligned} g^l(\sigma_1, \ldots, \sigma_n) &\geqslant \sum_{\sigma_{n+1} = 2\Theta(x, n, \{\sigma_j\})+1}^{\infty} |I_{n+1}(\sigma_1, \ldots, \sigma_{n+1})| \\ &\geqslant \left(2\Theta(x, n, \{\sigma_j\}) \prod_{j=1}^{n} (\sigma_j(\sigma_j - 1))\right)^{-1}, \\ g^r(\sigma, \ldots, \sigma_n) &\geqslant \sum_{\sigma_{n+1}=2}^{\Theta(x, n, \{\sigma_j\})} |I_{n+1}(\sigma_1, \ldots, \sigma_{n+1})| \geqslant |I_{n+1}(\sigma_1, \ldots, \sigma_n, 2)|. \end{aligned}$$

Thus

$$g(\sigma_1, \ldots, \sigma_n) \geqslant \frac{1}{B^2 - 1}|J(\sigma_1, \ldots, \sigma_n)|.$$

This completes the proof. □

Lemma 4.3. *For any $\varepsilon > 0$ and any $(\sigma_1, \dots, \sigma_n) \in D_n$ with n sufficiently large, we have*

$$\mu(J(\sigma_1, \dots, \sigma_n)) \ll |J(\sigma_1, \dots, \sigma_n)|^{t-\varepsilon}, \quad \textit{with } t = s_B - \varepsilon,$$

where the constant in "$\ll$" is an absolute constant.

Proof. First, choose k_0 sufficiently large such that $1/k < \varepsilon$ for any $k > k_0$. For any $n \geqslant n_{k_0}$ and $(\sigma_1, \dots, \sigma_n) \in D_n$, we estimate $\mu(J(\sigma_1, \dots, \sigma_n))$.

Case I: $n = n_k$ for some $k > k_0$.

$$\begin{aligned}
&\mu(J(\sigma_1, \dots, \sigma_{n_k})) \\
&= \prod_{j=n_{k-1}+2}^{n_k} (\sigma_j(\sigma_j - 1))^{-(\nu+1)s_B} \mu(J(\sigma_1, \dots, \sigma_{n_{k-1}+1})) \\
&= \prod_{i=1}^{k-1} \Theta(x, n_i, \{\sigma_j\})^{-1} \prod_{j=n_{i-1}+2}^{n_i} (\sigma_j(\sigma_j - 1))^{-(\nu+1)s_B} \\
&\leqslant 2^{k-1} \prod_{i=1}^{k-1} (\sigma_{n_i+1})^{-1} \prod_{j=n_{i-1}+2}^{n_i} (\sigma_j(\sigma_j - 1))^{-(\nu+1)s_B} \quad (\text{since } \sigma_{n+1} \leqslant 2\Theta(x, n, \{\sigma_j\})) \\
&\leqslant 2^{k-1} \prod_{i=1}^{k-1} (\sigma_{n_i+1}(\sigma_{n_i+1} - 1))^{-t} \prod_{j=n_{i-1}+2}^{n_i} (\sigma_j(\sigma_j - 1))^{-(\nu+1)s_B} \quad (\text{since } t \leqslant 1/2) \\
&\ll 2^{k-1} |J(\sigma_1, \dots, \sigma_n)|^t \left(2\beta \prod_{j=1}^{k-1} \sigma_{n_j+1}(\sigma_{n_j+1} - 1)\right)^{t\nu} \sigma_n^{-t}(\sigma_n - 1)^{-t\nu} \\
&\ll 2^{k-1} |J(\sigma_1, \dots, \sigma_n)|^t \prod_{j=1}^{k-1} \sigma_{n_j+1}^{2t\nu} \\
&\ll |J(\sigma_1, \dots, \sigma_n)|^{t-\varepsilon} \quad (\text{by the choice of } \{n_k\}_{k \geqslant 1}).
\end{aligned}$$

Case II: $n = n_k + 1$ for some $k > k_0$.

$$\begin{aligned}
\mu(J(\sigma_1, \dots, \sigma_{n_k+1})) &= \Theta(x, n_k, \{\sigma_j\})^{-1} \mu(J(\sigma_1, \dots, \sigma_{n_k})) \\
&\ll \Theta(x, n_k, \{\sigma_j\})^{-1} |J(\sigma_1, \dots, \sigma_{n_k})|^{t-\varepsilon} \\
&\ll (\sigma_{n_i+1}(\sigma_{n_i+1} - 1))^{-(t-\varepsilon)} |J(\sigma_1, \dots, \sigma_{n_k})|^{t-\varepsilon} \\
&\ll |J(\sigma_1, \dots, \sigma_n)|^{t-\varepsilon}.
\end{aligned}$$

Case III: $n_k+1 < n < n_{k+1}$ for some $k > k_0$.

$$\begin{aligned}\mu(J(\sigma_1,\dots,\sigma_n)) &= \sum_{2\leqslant\sigma_{n+1},\dots,\sigma_{n_{k+1}}\leqslant B} \mu(J(\sigma_1,\dots,\sigma_{n_{k+1}}))\\ &= \mu(J(\sigma_1,\dots,\sigma_{n_k},\sigma_{n_k+1}))\prod_{j=n_k+2}^{n}(\sigma_j(\sigma_j-1))^{-(\nu+1)s_B}\\ &\ll |J(\sigma_1,\dots,\sigma_{n_k+1})|^{t-\varepsilon}\prod_{j=n_k+2}^{n}(\sigma_j(\sigma_j-1))^{-(t-\varepsilon)}\\ &\ll |J(\sigma_1,\dots,\sigma_n)|^{t-\varepsilon}.\end{aligned}$$

The proof is completed. □

Now, we will give the estimation of $\mu(B(x,r))$.

Lemma 4.4. *For any $x\in E_B$ and sufficiently small $r>0$,*

$$\mu(B(x,r)) \ll r^{t-\varepsilon}.$$

Proof. Take $r_0 = \min\limits_{1\leqslant j\leqslant n_{k_0}} \min\limits_{(\sigma_1,\dots,\sigma_j)\in D_j} g(\sigma_1,\dots,\sigma_j)$. Fix $x\in E_B$ and $0<r<r_0$, then there exists a unique sequence $\sigma_1,\dots,\sigma_k,\dots$ such that $x\in J(\sigma_1,\dots,\sigma_k)$ for all $k\geqslant 1$ and for some $n\geqslant n_{k_0}$,

$$g(\sigma_1,\dots,\sigma_{n+1}) \leqslant r < g(\sigma_1,\dots,\sigma_n).$$

From the definition of $g(\sigma_1,\dots,\sigma_n)$ we know that the ball $B(x,r)$ can intersect only one fundamental interval of order n, which is $J(\sigma_1,\dots,\sigma_n)$.

Case I: $n=n_k$ for some $k\geqslant k_0$.

(i) $r\leqslant \frac13|I_{n_k+1}(\sigma_1,\dots,\sigma_{n_k+1})|$. In this case, the ball $B(x,r)$ can intersect at most three basic intervals of order n_k+1, which are $I_{n_k+1}(\sigma_1,\dots,\sigma_{n_k+1}-1)$, $I_{n_k+1}(\sigma_1,\dots,\sigma_{n_k+1})$ and $I_{n_k+1}(\sigma_1,\dots,\sigma_{n_k+1}+1)$. Then by Lemma 4.3 we have

$$\begin{aligned}\mu(B(x,r)) &\leqslant 3\mu(J(\sigma_1,\dots,\sigma_{n_k+1}))\leqslant 3|J(\sigma_1,\dots,\sigma_{n_k+1})|^{t-\varepsilon}\\ &\leqslant 3(B^2-1)|g(\sigma_1,\dots,\sigma_{n_k+1})|^{t-\varepsilon}\leqslant 3(B^2-1)r^{t-\varepsilon}.\end{aligned}$$

(ii) $r>\frac13|I_{n_k+1}(\sigma_1,\dots,\sigma_{n_k+1})|$. In this case, since

$$|I_{n_k+1}(\sigma_1,\dots,\sigma_{n_k+1})| = \prod_{j=1}^{n_k+1}(\sigma_j(\sigma_j-1))^{-1} \geqslant (2\Theta(x,n_k,\{\sigma_j\}))^{-2}\prod_{j=1}^{n_k}(\sigma_j(\sigma_j-1))^{-1},$$

the number of fundamental intervals of order n_k+1 contained in $J(\sigma_1,\dots,\sigma_{n_k})$ that the ball $B(x,r)$ intersects is at most

$$l=\frac{6r}{(2\Theta(x,n_k,\{\sigma_j\}))^{-2}\prod_{j=1}^{n_k}(\sigma_j(\sigma_j-1))^{-1}}\leqslant 6r(2\Theta(x,n_k,\{\sigma_j\}))^2\prod_{j=1}^{n_k}(\sigma_j(\sigma_j-1)).$$

Therefore

$$\begin{aligned}\mu(B(x,r))&\leqslant\min\{\mu(J(\sigma_1,\dots,\sigma_{n_k})),\mu(J(\sigma_1,\dots,\sigma_{n_k+1}))l\}\\&\leqslant\mu(J(\sigma_1,\dots,\sigma_{n_k}))\min\Big\{1,24r\Theta(x,n_k,\{\sigma_j\})\prod_{j=1}^{n_k}\sigma_j(\sigma_j-1)\Big\}\\&\leqslant|J(\sigma_1,\dots,\sigma_{n_k+1})|^{t-\varepsilon}(12r)^{t-\varepsilon}(2\Theta(x,n_k,\{\sigma_j\})\prod_{j=1}^{n_k}\sigma_j(\sigma_j-1))^{t-\varepsilon}\ll r^{t-\varepsilon}.\end{aligned}$$

Case II: $n\neq n_k$ for any $k\geqslant k_0$.

By the definition of μ, for any $2\leqslant\tau\neq\eta\leqslant B$ we have

$$\frac{\mu(J(\sigma_1,\dots,\sigma_n,\eta))}{\mu(J(\sigma_1,\dots,\sigma_n,\tau))}\leqslant B^2,$$

hence

$$\mu(J(\sigma_1,\dots,\sigma_n))\leqslant(B-1)B^2\mu(J(\sigma_1,\dots,\sigma_n,\sigma_{n+1})),$$

thus

$$\begin{aligned}\mu(B(x,r))\leqslant\mu(J(\sigma_1,\dots,\sigma_n))&\leqslant(B-1)B^2\mu(J(\sigma_1,\dots,\sigma_n,\sigma_{n+1}))\\&\leqslant(B-1)B^2(B^2-1)|J(\sigma_1,\dots,\sigma_n,\sigma_{n+1})|^{t-\varepsilon}\\&\leqslant(B-1)B^2(B^2-1)|g(\sigma_1,\dots,\sigma_n,\sigma_{n+1})|^{t-\varepsilon}\\&\leqslant(B-1)B^2(B^2-1)r^{t-\varepsilon}.\end{aligned}$$

This completes the proof. □

Proof of Proposition 4.2. By Lemma 4.4 and the mass distribution principle, we have

$$\dim_{\mathrm H}E_B\geqslant t-\varepsilon=s_B-2\varepsilon.$$

Since $\varepsilon\geqslant 0$ is arbitrary, we have

$$\dim_{\mathrm H}E_B\geqslant s_B.$$

Since B is arbitrary, and when B tends to infinity, s_B tends to $1/(\nu+1)$, we have

$$\dim_{\mathrm H}E_\beta\geqslant\frac{1}{\nu+1}.$$

□

Combining this with Proposition 4.1 completes the proof of Theorem 1.2.

Moreover, we can find there is a close relation between the set F and the set E_β for $\nu = 1$ in the proof of Theorem 1.2.

Lemma 4.5. *Let* $F = \{x \in (0,1]\colon P_n(x)/Q_n(x)$ *is optimal approximation to* x i.o.$\}$, *then we have* $E_\beta \subset F$ *for any* $\beta > 8$ *and* $\nu = 1$.

P r o o f. Supposing $x \in E_\beta$, as in the proof of Lemma 4.1 there exists a sequence of integers $\{n_k\}_{k\geqslant 1}$ such that

$$\Bigl|x - \frac{P_{n_k}(x)}{Q_{n_k}(x)}\Bigr| < \frac{1}{2Q^2_{n_k}(x)}.$$

We will prove that the fraction $P_{n_k}(x)/Q_{n_k}(x)$ is the optimal approximation for the number x. Let

$$|dx - c| \leqslant |Q_{n_k}(x)x - P_{n_k}(x)| < \frac{1}{2Q_{n_k}(x)},$$

where $d > 0$ and $c/d \neq P_{n_k}(x)/Q_{n_k}(x)$, then

$$\Bigl|x - \frac{c}{d}\Bigr| < \frac{1}{2dQ_{n_k}}$$

and consequently

$$\Bigl|\frac{c}{d} - \frac{P_{n_k}(x)}{Q_{n_k}(x)}\Bigr| \leqslant \Bigl|x - \frac{c}{d}\Bigr| + \Bigl|x - \frac{P_{n_k}(x)}{Q_{n_k}(x)}\Bigr| < \frac{1}{2dQ_{n_k}(x)} + \frac{1}{2Q^2_{n_k}(x)} = \frac{Q_{n_k}(x) + d}{2Q^2_{n_k}(x)d}. \tag{4.5}$$

On the other hand, since $c/d \neq P_{n_k}(x)/Q_{n_k}(x)$, we have

$$\Bigl|\frac{c}{d} - \frac{P_{n_k}(x)}{Q_{n_k}(x)}\Bigr| \geqslant \frac{1}{Q_{n_k}(x)d},$$

and combining it with (4.5), we obtain $d > Q_{n_k}(x)$. Thus the fraction $P_{n_k}(x)/Q_{n_k}(x)$ is indeed the optimal approximation. This completes the proof. □

From Proposition 4.2, we can get the following result.

Corollary 4.1. $\dim_{\mathrm H} F \geqslant \frac{1}{2}$.

From the above theorem, we know that there exists an uncountable number of points in F.

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