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RINGS OF CONSTANTS OF GENERIC 4D LOTKA-VOLTERRA SYSTEMS

JANUSZ ZIELIŃSKI, PIOTR OSSOWSKI, Toruń

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Abstract. We show that the rings of constants of generic four-variable Lotka-Volterra derivations are finitely generated polynomial rings. We explicitly determine these rings, and we give a description of all polynomial first integrals of their corresponding systems of differential equations. Besides, we characterize cofactors of Darboux polynomials of arbitrary four-variable Lotka-Volterra systems. These cofactors are linear forms with coefficients in the set of nonnegative integers. Lotka-Volterra systems have various applications in such branches of science as population biology and plasma physics, among many others.

Keywords: Lotka-Volterra derivation, polynomial constant, polynomial first integral, Darboux polynomial

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1. Introduction

Throughout this paper, \( k \) is a field of characteristic zero. By \( k[X] \) we denote \( k[x_1, \ldots, x_n] \), the polynomial ring in \( n \) variables. For \( n \leq 3 \) the ring of constants of any derivation of \( k[X] \) is finitely generated (see [7]). For \( n = 4 \) the ring of constants may not be finitely generated. An example was given in [3]. There is no general procedure for determining the ring of constants, nor even deciding whether it is finitely generated. Even for a given specific derivation of \( k[X] \) the problem may be difficult, see various counterexamples to Hilbert’s fourteenth problem (for example [3]) and the three-variable Lotka-Volterra derivation (for example [5]). Such problems are closely linked to the invariant theory, namely for every connected algebraic group \( G \subseteq \text{Gl}_n(k) \) there exists a derivation \( d \) such that \( k[X]^G = k[X]^d \) (see, for instance, [6]).

It is well known that Lotka-Volterra systems play a significant role in population biology. They also have many applications in other branches of science, for
instance in plasma physics (for more details we refer the reader to [1] and its extensive bibliography). Moreover, they play an important part in the derivation theory itself. A derivation $d: k[X] \to k[X]$ is said to be factorizable if $d(x_i) = x_if_i$, where the polynomials $f_i$ are of degree 1 for $i = 1, \ldots, n$. Examples of such derivations are Lotka-Volterra derivations. How to associate a factorizable derivation with any given derivation is shown in [10]. The construction helps to establish new facts on constants of the initial derivation (see, for instance, [8]). We have thus a special interest in describing constants of factorizable derivations.

Section 3 provides some facts on Darboux polynomials of Lotka-Volterra derivations in 4 variables with arbitrary coefficients. Section 4 contains several properties of Lotka-Volterra derivations for $n$ variables, which supply potential tools for further studies. In Section 5, we prove Theorem 5.1, which gives a full description of the ring of polynomial constants of the derivation $d: k[x_1, \ldots, x_4] \to k[x_1, \ldots, x_4]$ defined by

$$d = \sum_{i=1}^{4} x_i(x_{i-1} - C_ix_{i+1}) \frac{\partial}{\partial x_i},$$

for $C_i$ not belonging to the set of positive rationals. It is the main result of the paper. As a consequence we obtain that a generic four-variable Lotka-Volterra system has a finitely generated ring of constants.

2. Notation and preliminaries

If $R$ is a commutative $k$-algebra, then a $k$-linear map $d: R \to R$ is called a derivation of $R$ if $d(ab) = ad(b) + d(a)b$ for all $a, b \in R$. We call $R^d = \ker d$ the ring of constants of the derivation $d$. If $f_1, \ldots, f_n \in k[X]$, then there exists exactly one derivation $d: k[X] \to k[X]$ such that $d(x_1) = f_1, \ldots, d(x_n) = f_n$. The set $k[X]^d \setminus k$ is equal to the set of all polynomial first integrals of the corresponding system of ordinary differential equations (see [6] for more details).

A derivation $d: k[X] \to k[X]$ is called homogeneous of degree $s$ if the image of a homogeneous form of degree $t$ under $d$ is a homogeneous form of degree $s + t$ for all $t \in \mathbb{N}$. Since $k$ is a field of characteristic zero, we have $\mathbb{Q} \subseteq k$. Let $\mathbb{Q}_+$ denote the set of positive rationals and $\mathbb{N}$ denote the set of nonnegative integers. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, we denote by $X^\alpha$ the monomial $x_1^{\alpha_1} \ldots x_n^{\alpha_n} \in k[X]$ and by $|\alpha|$ the sum $\alpha_1 + \ldots + \alpha_n$.

Let $n \geq 3$. Throughout the rest of this paper, $R = k[x_1, \ldots, x_n]$ and $d: R \to R$ is a derivation of the form

$$d(x_i) = x_i(x_{i-1} - C_ix_{i+1}),$$

for $C_i$ not belonging to the set of positive rationals. It is the main result of the paper. As a consequence we obtain that a generic four-variable Lotka-Volterra system has a finitely generated ring of constants.
for \( i = 1, \ldots, n \), and we adhere to the convention that \( x_{n+1} = x_1 \) and \( x_0 = x_n \). All our considerations are in the cyclic sense; for example, \( \{i, i + 1\} \) admits also \( \{n, 1\} \).

We write a minus sign before \( C_i \) just to simplify further computations. Denote by \( R_{(m)} \) the homogeneous component of \( R \) of degree \( m \). Let \( R^d_{(m)} = R_{(m)} \cap R^d \).

Since \( d \) is homogeneous, we have \( R^d = \bigoplus_{m=0}^{\infty} R^d_{(m)} \) and we need only to determine the homogeneous constants.

3. Darboux polynomials

A nonzero polynomial \( f \) is said to be a Darboux polynomial of a derivation \( \delta: R \to R \) if \( \delta(f) = \Lambda f \) for some \( \Lambda \in R \). We will call \( \Lambda \) a cofactor of \( f \). Since \( R \) is a domain, \( \Lambda \) is unique. The product \( f_1f_2 \) of Darboux polynomials is a Darboux polynomial and its cofactor equals the sum of the cofactors of \( f_1 \) and \( f_2 \).

Proposition 3.1 is well known (see [6], Proposition 2.2.1). It is true for \( k \) being any unique factorization domain and any derivation \( \delta \) of \( k[x_1, \ldots, x_n] \).

Proposition 3.1. If \( f \in R \) is a Darboux polynomial of \( \delta \), then all factors of \( f \) are also Darboux polynomials of \( \delta \).

We call a polynomial \( g \in R \) strict if it is nonzero, homogeneous and not divisible by the variables \( x_1, \ldots, x_n \). Every nonzero homogeneous polynomial \( f \in R \) has a unique presentation \( f = X^\alpha g \), where \( X^\alpha \) is a monomial and \( g \) is strict.

If \( f \) is a Darboux polynomial of a homogeneous derivation \( \delta \) with a cofactor \( \Lambda \), then every homogeneous part of \( f \) is a Darboux polynomial of \( \delta \) with the same cofactor \( \Lambda \) (see [6], Proposition 2.2.3).

If \( f = X^\alpha g \) is a Darboux polynomial of the derivation \( d \), then it is easy to compute the cofactor of the monomial \( X^\alpha \) (see the proof of Lemma 3.4). Thus we are going to characterize cofactors of strict Darboux polynomials (Lemma 3.2 and Corollary 3.3). Such a characterization for 3 variables was done in [4]. Since \( d \) is a homogeneous derivation of degree 1, the cofactor of any homogeneous Darboux polynomial is a homogeneous form of degree 1.

Lemma 3.2. Let \( n = 4 \). Let \( g \in R_{(m)} \) be a Darboux polynomial of \( d \) with the cofactor \( \lambda_1x_1 + \ldots + \lambda_4x_4 \). Let \( i \in \{1, 2, 3, 4\} \). If \( g \) is not divisible by \( x_i \), then \( \lambda_{i+1} \in \mathbb{N} \). More precisely, if \( g(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_4) = x_{i+1}^{\beta_{i+2}} G \) and \( x_{i+2} \mid G \), then \( \lambda_{i+1} = \beta_{i+2} \) and \( \lambda_{i+3} = -C_{i+2}\lambda_{i+1} \).
Proof. Without loss of generality we can assume that \( i = 4 \). Since \( g \) is a Darboux polynomial, we have

\[
\sum_{i=1}^{4} x_i(x_{i-1} - C_i x_{i+1}) \frac{\partial g}{\partial x_i} = (\lambda_1 x_1 + \ldots + \lambda_4 x_4) g.
\]

We put \( x_4 = 0 \) in the equation above and obtain

\[
-x_1 C_1 x_2 \frac{\partial G}{\partial x_1} + x_2(x_1 - C_2 x_3) \frac{\partial G}{\partial x_2} + x_3 x_2 \frac{\partial G}{\partial x_3} = (\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3) G,
\]

where \( G = g(x_1, x_2, x_3, 0) \neq 0 \), since \( x_4 \nmid g \).

Let \( G = x_2^\beta G \), where \( x_2 \nmid G \) and \( \beta \in \mathbb{N} \). Then

\[
(3.1) \quad -C_1 x_1 x_2 x_2^\beta \frac{\partial G}{\partial x_1} + x_2(x_1 - C_2 x_3) \frac{\partial G}{\partial x_2} + x_3 x_2 x_2^\beta \frac{\partial G}{\partial x_3} = (\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3) x_2^\beta G
\]

(if \( \beta = 0 \), then we assume that expression \( \beta_2 x_2^{\beta_2-1} \) is equal to 0). We divide both sides of (3.1) by \( x_2^\beta \), then we add \((C_2 x_3 - x_1) \beta_2 G \) to both sides of (3.1) and we obtain

\[
(3.2) \quad -C_1 x_1 x_2 \frac{\partial G}{\partial x_1} + x_2(x_1 - C_2 x_3) \frac{\partial G}{\partial x_2} + x_3 x_2 \frac{\partial G}{\partial x_3} = ((\lambda_1 - \beta_2)x_1 + \lambda_2 x_2 + (\lambda_3 + C_2 \beta_2) x_3) G.
\]

The left-hand side of (3.2) is the divisible by \( x_2 \), so also is the right-hand side of (3.2). Since \( x_2 \nmid G \), we get

\[
x_2 \mid (\lambda_1 - \beta_2)x_1 + \lambda_2 x_2 + (\lambda_3 + C_2 \beta_2) x_3.
\]

Hence \( \lambda_1 - \beta_2 = 0 \) and \( \lambda_3 + C_2 \beta_2 = 0 \). Finally, \( \lambda_1 = \beta_2 \) and \( \lambda_3 = -C_2 \beta_2 = -C_2 \lambda_1 \).

\[\square\]

Corollary 3.3. Let \( n = 4 \). If \( g \in R_{(m)} \) is a strict Darboux polynomial, then its cofactor is a linear form with coefficients in \( \mathbb{N} \).

Lemma 3.4. Let \( n = 4 \). If \( d(f) = 0 \) and \( f = X^\alpha g \), where \( g \) is strict, then \( d(X^\alpha) = 0 \) and \( d(g) = 0 \).
Proof. If \( d(f) = 0 \), then \( f \) is a Darboux polynomial. In view of Proposition 3.1, also \( X^\alpha \) and \( g \) are Darboux polynomials. If \( \alpha = (\alpha_1, \ldots, \alpha_4) \), then a short computation shows that the cofactor of \( X^\alpha \) equals \((\alpha_2 - \alpha_4 C_4)x_1 + (\alpha_3 - \alpha_1 C_1)x_2 + (\alpha_4 - \alpha_2 C_2)x_3 + (\alpha_1 - \alpha_3 C_3)x_4 \). The polynomial \( g \) is strict, therefore by Lemma 3.2, if \( \lambda_1 x_1 + \ldots + \lambda_4 x_4 \) is the cofactor of \( g \), then \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{N} \) and \( \lambda_1 = -C_4 \lambda_3 \), \( \lambda_2 = -C_1 \lambda_4 \), \( \lambda_3 = -C_2 \lambda_1 \), \( \lambda_4 = -C_3 \lambda_2 \). The cofactor of the product \( X^\alpha g \) is the sum of the cofactors of \( X^\alpha \) and \( g \), that is, equals
\[
(\alpha_2 - \alpha_4 C_4 + \lambda_1)x_1 + (\alpha_3 - \alpha_1 C_1 + \lambda_2)x_2 + (\alpha_4 - \alpha_2 C_2 + \lambda_3)x_3 + (\alpha_1 - \alpha_3 C_3 + \lambda_4)x_4.
\]
On the other hand, by assumption, this cofactor is equal to 0. Thus
\[
\alpha_2 - \alpha_4 C_4 + \lambda_1 = 0, \\
\alpha_3 - \alpha_1 C_1 + \lambda_2 = 0, \\
\alpha_4 - \alpha_2 C_2 + \lambda_3 = 0, \\
\alpha_1 - \alpha_3 C_3 + \lambda_4 = 0.
\]
Suppose \( g \) is not a constant of \( d \). Then \( \lambda_i \neq 0 \) for some \( i \in \{1, \ldots, 4\} \). There is no loss of generality in assuming that \( i = 1 \). Then \( \lambda_1 = -C_4 \lambda_3 \) implies that also \( \lambda_3 \neq 0 \). Hence \( C_4 = -\lambda_1/\lambda_3 < 0 \). Then \( \alpha_2 \geq 0 \), \( -\alpha_4 C_4 \geq 0 \) and \( \lambda_1 > 0 \). Therefore \( \alpha_2 - \alpha_4 C_4 + \lambda_1 > 0 \), which is a contradiction. This proves that \( d(g) = 0 \).

If \( d(X^\alpha g) = 0 \) and \( d(g) = 0 \), then obviously \( d(X^\alpha) = 0 \).  

\[ \square \]

4. Restrictions of polynomials

Let \( \varphi \in R \) and \( 1 \leq q \leq n \). Then for every subset \( \{i_1, \ldots, i_q\} \subseteq \{1, \ldots, n\} \) we denote by \( \varphi^{(i_1, \ldots, i_q)} \) the sum of terms of \( \varphi \) that depend on variables \( x_{i_1}, \ldots, x_{i_q} \), that is, \( \varphi^{(i_1, \ldots, i_q)} = \varphi|_{x_j = 0 \text{ for } j \notin \{i_1, \ldots, i_q\}} \). We noticed that for inductive purposes it is more convenient to deal with polynomials \( \varphi \) such that \( d(\varphi^A)^A = 0 \) for a given \( A \subseteq \{1, \ldots, n\} \), than with the constants themselves.

The first three results, that is 4.1, 4.2 and 4.3, are similar to those for \( C_1 = \ldots = C_n = 1 \) of our paper [9]. As an obvious consequence of the fact that \( x_i \mid d(x_i) \), for \( i = 1, \ldots, n \), we obtain the following proposition.

Proposition 4.1. If \( A \subseteq \{1, \ldots, n\} \), then for every homogeneous polynomial \( \varphi \in R_{(m)} \), we have \( d(\varphi^A)^A = d(\varphi)^A \).

Corollary 4.2. If \( A \subseteq \{1, \ldots, n\} \), then for every \( \varphi \in R_{(m)}^d \) we have \( d(\varphi^A)^A = 0 \).
Lemma 4.3. If $B \subseteq A \subseteq \{1, \ldots, n\}$ and $d(\varphi^A)^A = 0$, then also $d(\varphi^B)^B = 0$.

Proof. Let $\varphi^A = \varphi^B + \psi$, where each monomial in $\psi$ has $x_j$ in a positive power for some $j \in A \setminus B$. Then $d(\varphi^A) = d(\varphi^B) + d(\psi)$. If $d(\varphi^A)^A = 0$, then clearly $d(\varphi^A)^B = 0$. Therefore $0 = d(\varphi^A)^B = d(\varphi^B)^B + d(\psi)^B$. Moreover $d(\psi)^B = 0$, because every monomial in $d(\psi)$ has $x_j$ in positive a power for some $j \in A \setminus B$, by the definition of $d$. Finally, $d(\varphi^B)^B = 0$. \qed

We formulated Lemma 4.4 in [9] without a proof. Note that there is no assumption on the coefficients $C_i$ in this lemma.

Lemma 4.4. Let $\varphi \in R_m$ and $A = \{i, i + 1\} \subset \{1, \ldots, n\}$. If $d(\varphi^A)^A = 0$, then $\varphi^A = a(x_i + C_i x_{i+1})^m$, for $a \in k$.

Proof. Let $\varphi^A = \sum b_r x_i^{m-r} x_{i+1}^r$. Then

$$d(\varphi^A) = \sum_{r=0}^m b_r \left((m-r)x_i^{m-r} + x_i^{m-r} x_{i+1}^r\right) = \sum_{r=0}^m b_r x_i^{m-r} x_{i+1}^r \left((m-r)x_{i-1} - C_i x_{i+1}\right) + r(x_i - C_{i+1} x_{i+2})\right).$$

Therefore,

$$d(\varphi^A)^A = \sum_{r=0}^m b_r (r x_i^{m-r+1} x_{i+1}^r - C_i (m-r) x_i^{m-r} x_{i+1}^r) = \sum_{r=1}^m r b_r x_i^{m-r+1} x_{i+1}^r - C_i \sum_{r=0}^{m-1} (m-r) b_r x_i^{m-r} x_{i+1}^r = \sum_{r=1}^m r b_r x_i^{m-r+1} x_{i+1}^r - C_i \sum_{r=1}^m (m-r+1) b_{r-1} x_i^{m-r+1} x_{i+1}^r = \sum_{r=1}^m (r b_r - C_i (m-r+1) b_{r-1}) x_i^{m-r+1} x_{i+1}^r = 0.$$

Hence for $r = 1, \ldots, m$ we have $r b_r = C_i (m-r+1) b_{r-1}$, that is, $b_r = \frac{m-r+1}{r} C_i b_{r-1}$. Thus an easy induction on $r$ shows that $b_r = \binom{m}{r} C_i^r b_0$ for $r = 0, \ldots, m$. Consequently, $\varphi^A = b_0 (x_i + C_i x_{i+1})^m$.\qed

Note that the above $a = b_0$ may be equal to 0. Here and throughout, by the support of $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ we mean the set $\text{supp}(\alpha) = \{i: \alpha_i \neq 0\}$. Observe that there is an assumption on only one coefficient $C_i$ in Lemma 4.5.

Lemma 4.5. Let $n \geq 4$, $\varphi \in R_m$ and $A = \{i, i+1, i+2\} \subset \{1, \ldots, n\}$. If $d(\varphi^A)^A = 0$ and $C_i \notin \mathbb{Q}_+$, then $\varphi^A \in k[x_i + C_i x_{i+1} + C_i C_{i+1} x_{i+2}]$. 534
Proof. Let \( m = 1 \). By assumption and Lemma 4.3, \( d(\varphi^{(i,i+1)})^{(i,i+1)} = 0 \). In view of Lemma 4.4, we have \( \varphi^{(i,i+1)} = a_1(x_i + C_ix_{i+1}) \). Similarly, we obtain \( \varphi^{(i+1,i+2)} = a_2(x_{i+1} + C_{i+1}x_{i+2}) \). Thus \( a_2 = a_1C_i \) and \( \varphi^A = a_1(x_i + C_ix_{i+1} + C_iC_{i+1}x_{i+2}) \). Now let \( m = 2 \). Since \( d(\varphi^{(i,i+1)})^{(i,i+1)} = 0 \), it follows that \( \varphi^{(i,i+1)} = a_1(x_i + C_ix_{i+1})^2 \). Analogously \( \varphi^{(i+1,i+2)} = a_2(x_{i+1} + C_{i+1}x_{i+2})^2 \). Hence \( a_2 = a_1C_i^2 \) and \( \varphi^{(i+1,i+2)} = a_1(C_ix_{i+1} + C_iC_{i+1}x_{i+2})^2 \). Therefore, \( \varphi^A = a_1(x_i + C_ix_{i+1} + C_iC_{i+1}x_{i+2})^2 + bx_ix_{i+2} \) for some \( b \in k \). Applying first \( d(\cdot) \) and then \( (\cdot)^A \) to both sides of the last equation we get \( 0 = b(1 - C_i)x_ix_{i+1}x_{i+2} \). Since \( C_i \neq 1 \), we have \( b = 0 \).

Assume \( m \geq 3 \). Then \( \varphi^A \) is a linear combination of monomials \( X^\alpha \) such that \( |\alpha| = m \) and \( \text{supp}(\alpha) \subseteq \{i,i+1,i+2\} \). We have \( \varphi^{(i,i+1)} = a_1(x_i + C_ix_{i+1})^m \) and \( \varphi^{(i+1,i+2)} = a_2(x_{i+1} + C_{i+1}x_{i+2})^m \), for \( a_1, a_2 \in k \). Thus \( a_2 = a_1C_i^m \) and \( \varphi^{(i+1,i+2)} = a_1(C_ix_{i+1} + C_iC_{i+1}x_{i+2})^m \). The terms of the form \( x_i^{m-r}x_{i+1}^{m-r} \) and \( x_{i+1}^{m-r}x_{i+2} \) for \( r = 0, \ldots, m \) have the same coefficients in \( \varphi^A \) and in \( a_1(x_i + C_ix_{i+1} + C_iC_{i+1}x_{i+2})^m \). Therefore

\[
\varphi^A = a_1(x_i + C_ix_{i+1} + C_iC_{i+1}x_{i+2})^m + \sum_{\text{supp}(\alpha) = \{i,i+1,i+2\}} b_\alpha X^\alpha + \sum_{\text{supp}(\alpha) = \{i,i+1,i+2\}} b_\alpha X^\alpha,
\]

that is, \( \varphi^A = a_1(x_i + C_ix_{i+1} + C_iC_{i+1}x_{i+2})^m + x_ix_{i+2}b \), where \( b \in R_{(m-2)} \) and \( b \) depends on the variables \( x_i, x_{i+1}, x_{i+2} \) only. We show that \( \psi = 0 \). First,

\[
d(\varphi^A) = a_1 d((x_i + C_ix_{i+1} + C_iC_{i+1}x_{i+2})^m) + d(x_ix_{i+2}) \psi + x_ix_{i+2}d(\psi) = a_1m(x_i + C_ix_{i+1} + C_iC_{i+1}x_{i+2})^{m-1}(x_ix_{i-1} - C_iC_{i+1}C_{i+2}x_{i+2}x_{i+3}) + (x_{i-1} + (1 - C_i)x_{i+1} - C_{i+2}x_{i+3})x_ix_{i+2} \psi + x_ix_{i+2}d(\psi).
\]

Obviously, \( \psi^A = \psi \). Therefore,

\[
0 = d(\varphi^A)^A = (1 - C_i)x_ix_{i+1}x_{i+2} \psi + x_ix_{i+2}d(\psi)^A.
\]

Hence \( d(\psi)^A = (C_i - 1)x_{i+1} \psi \).

Suppose \( \psi \neq 0 \). Let \( s = \deg_{x_{i+1}} \psi \). Let \( bx_i^{s+1}x_{i+1}^{s+2} \) be a term of \( \psi \) with \( b \in k \setminus \{0\} \) (we fix one of the terms of \( \psi \) that are divisible by \( x_{i+1}^s \). Then the coefficient of the monomial \( x_i^{s+1}x_{i+1}^{s+2} \) in the expansion of \( (C_i - 1)x_{i+1} \psi \) equals \( (C_i - 1)b \). The coefficient of \( x_i^{s+1}x_{i+1}^{s+2} \) in the expansion of \( d(\psi)^A \) is equal to \( b(t - rC_i) \) (because in all terms of the \( d \)-image of any term the exponent of only one variable may be increased). Therefore \( C_i = (t + 1)/(r + 1) \in \mathbb{Q}_+ \). The contradiction obtained proves that \( \psi = 0 \).

Thus \( \varphi^A = a_1(x_i + C_ix_{i+1} + C_iC_{i+1}x_{i+2})^m \in k[x_i + C_ix_{i+1} + C_iC_{i+1}x_{i+2}] \).

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5. Rings of constants

**Theorem 5.1.** Let $R = k[x_1, \ldots, x_4]$ and $C_1, \ldots, C_4 \notin \mathbb{Q}_+$. Let $d: R \rightarrow R$ be a derivation of the form

$$d(x_i) = x_i(x_{i-1} - C_ix_{i+1}),$$

for $i = 1, \ldots, 4$. If $C_1C_2C_3C_4 = 1$, then

$$R^d = k[x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4].$$

If $C_1C_2C_3C_4 \neq 1$, then $R^d = k$.

**Proof.** First we show that $R^d_{(m)} \subseteq k[x_1C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4]$, for all $m \geq 0$. Let $A_1 = \{2, 3, 4\}, A_2 = \{1, 3, 4\}, A_3 = \{1, 2, 4\}, A_4 = \{1, 2, 3\}$ and let $\varphi \in R^d_{(m)}$. By Corollary 4.2 and Lemma 4.5, $\varphi^{A_i} = a_{i+1}(x_{i+1} + C_{i+1}x_{i+2} + C_{i+1}C_{i+2}x_{i+3})^m$, for $i = 1, \ldots, 4$. Comparison of the coefficients of $x_2^m$ in $\varphi^{A_1}$ and $\varphi^{A_2}$ gives $a_2 = a_1C_1^m$. Analogously, $a_3 = a_2C_2^m = a_1C_1^mC_2^m$ and $a_4 = a_3C_3^m = a_1C_1^mC_2^mC_3^m$. Let $\psi = a_1(x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4)^m$. Then $\varphi^{A_i} = \psi^{A_i}$, for $i = 1, \ldots, 4$. This means that the polynomials $\varphi$ and $\psi$ have the same terms that depend on at most three variables. Therefore

$$\varphi = a_1(x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4)^m + \eta,$$

where each term of the polynomial $\eta$ has all four variables in positive powers, that is, $\eta$ is divisible by $x_1x_2x_3x_4$.

We show that $\eta$ is a constant of the derivation $d$. If $m < 4$, then $\eta = 0$, since $x_1x_2x_3x_4 | \eta$. Assume, then, that $m \geq 4$. If $C_1C_2C_3C_4 = 1$, then $\varphi$ and $x_4 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4$ are constants of $d$, so also is $\eta$. If $C_1C_2C_3C_4 \neq 1$, then

$$0 = d(\varphi) = a_1m(x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4)^{m-1}x_4(1 - C_1C_2C_3C_4) + d(\eta).$$

The derivation $d$ is factorizable, hence $x_1x_2x_3x_4 | \eta$ implies $x_1x_2x_3x_4 | d(\eta)$. Therefore, the coefficient of $x_1^4x_4$ in $d(\varphi)$ equals 0, on the one hand, and is equal to $a_1m(1 - C_1C_2C_3C_4)$, on the other hand. Thus $a_1 = 0$ and $\varphi = \eta$. In particular, $\eta$ is a constant of $d$.

We show that $\eta = 0$. Suppose that $\eta$ is a monomial. Let $\eta = cx_1^r x_2^s x_3^t x_4^u$, where $r, s, t, u \geq 1$. Then

$$0 = d(\eta) = cx_1^r x_2^s x_3^t x_4^u((s - uC_4)x_1 + (t - rC_1)x_2 + (u - sC_2)x_3 + (r - tC_3)x_4).$$

If $c \neq 0$, then $C_4 = s/u \in \mathbb{Q}_+$, which is a contradiction. Then $c = 0$ and $\eta = 0$.

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Suppose that $\eta$ is not a term. Then $\eta = X^\alpha g$, where $X^\alpha$ is a monomial and $g$ is strict. Since $\eta$ is divisible by $x_1x_2x_3x_4$, the monomial $X^\alpha$ has positive exponents. Since $\eta$ is a constant, by Lemma 3.4 also $X^\alpha$ and $g$ are. However, the considerations above prove that no monomial of positive exponents is a constant of $C$.

Thus $\eta = 0$ and $\varphi = a_1(x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4)^m \in k[x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4]$. Consequently, $R^d \subseteq k[x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4]$.

Case $C_1C_2C_3C_4 = 1$. Since $d(x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4) = x_1x_4 - C_1C_2C_3C_4x_1x_4 = 0$, we have $k[x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4] \subseteq R^d$.

Case $C_1C_2C_3C_4 \neq 1$. Let $a \in k \setminus \{0\}$ and $m \in \{1, 2, \ldots\}$. Then

$$d(a(x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4)^m) = am(x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4)^{m-1}(x_1x_4 - C_1C_2C_3C_4x_1x_4) \neq 0.$$ 

Thus $a = 0$ or $m = 0$. Hence, $R^d = k$. \qed

**Corollary 5.2.** If $k = \mathbb{R}$ or $k = \mathbb{C}$, then in the generic case a four-variable Lotka-Volterra derivation has a finitely generated (even trivial) ring of constants.

Lotka-Volterra derivations with positive rational coefficients are investigated for instance in [4], [5], [9], [11].

Note that if we consider a field $k$ of a positive characteristic $p$, then all elements of the form $x^n$ are constants of any polynomial derivation. For more information on this case we refer the reader to [2] and its bibliography.

**References**


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Authors’ address: Janusz Zieliński (corresponding author), Piotr Ossowski, Faculty of Mathematics and Computer Science, N. Copernicus University, ul. Chopina 12/18, 87-100 Toruń, Poland, e-mails: ubukrol@mat.uni.torun.pl, ossowski@mat.uni.torun.pl.