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IDENTIFICATION PROBLEMS FOR DEGENERATE PARABOLIC EQUATIONS

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Abstract. This paper deals with multivalued identification problems for parabolic equations. The problem consists of recovering a source term from the knowledge of an additional observation of the solution by exploiting some accessible measurements. Semigroup approach and perturbation theory for linear operators are used to treat the solvability in the strong sense of the problem. As an important application we derive the corresponding existence, uniqueness, and continuous dependence results for different degenerate identification problems. Applications to identification problems for the Stokes system, Poisson-heat equation, and Maxwell system are given to illustrate the theory.

Keywords: identification problem; perturbation theory for linear operators; degenerate differential equation

MSC 2010: 34G99, 34A55, 47A55.

1. Introduction

Using partial differential equations to model physical systems is one of the oldest activities in applied mathematics. A complete model requires certain state inputs in the form of initial and/or boundary data together with what might be called structure inputs such as coefficients or source terms which are related to the physical properties of the system. When some of the required inputs are not available we may be able to determine the missing inputs from outputs that are measured rather than computed by formulating and solving an appropriate inverse problem. In particular, when the missing inputs are one or more unknown coefficients in the partial differential equation, the problem is called a coefficient identification problem, and when a source term is missing it is a source identification problem (see [4], [10], [14], [21], [22]).

Inverse problems lie at the heart of scientific inquiry and technological development. Applications include a number of medical as well as other imaging techniques, location of oil and mineral deposits in the earth's substructure, creation of astrophysical images from telescope data, finding cracks and interfaces within materials, shape optimization, model identification in growth processes and, more recently, modelling in the life sciences. During the last 10 years or so there have been significant developments both in the mathematical theory and applications of inverse problems.

We point out that the problem of identifying a linear source in nondegenerate parabolic equations is very popular and widely studied in the literature concerning inverse problems for PDEs. The question of uniqueness has been solved in [7], [8], [13] by a method based on local Carleman estimates. There have been many papers dealing with the Lipschitz stability for parabolic problems (see for instance [12]). All the Lipschitz stability results were obtained using global Carleman estimates, which were first introduced to prove observability inequalities and null controllability results. Semigroup theory and fixed point arguments are also applied in the field of inverse problems by Prilepko et al. [23, Chapter 7], Orlovsky [19], [20], Awawdeh [4], [5], [6], Lorenzi [15], [16], [17], [18].

However, to the authors' knowledge, degenerate inverse problems for parabolic systems have not been studied thoroughly yet, even though this class of operators occurs in interesting theoretical and applied problems such as heat conduction processes, geophysics, controllability, oil search and underground filtration (see, e.g., [1], [2], [3] and [7], [8]). The main difficulties for these inverse problems come from the fact that the operators associated with the problem may have no bounded inverse and so the classical theory of semigroups does not apply.

Let X be a Banach space endowed with the norm $\|\cdot\|$, let M and L be two single-valued linear operators in X, and M^* be the adjoint operator of M. Let $z \in X$, let $\varphi\colon X \to \mathbb{R}_+$ be a C^1 functional and let us consider the following identification problems for degenerate equations:

(IP1) Find $f \in C^1([0,\tau];\mathbb{R})$ and a strict solution $u \in C^1([0,\tau];X)$ of the degenerate problem

$$\begin{cases} M\dfrac{\mathrm{d}u(t)}{\mathrm{d}t} = Lu(t) + Mf(t)z, \quad 0 \leqslant t \leqslant \tau, \\ u(0) = u_0, \end{cases}$$

satisfying the additional condition

$$\varphi[u(t)] = g(t), \quad 0 \leqslant t \leqslant \tau.$$

(IP2) Find $f \in C^1([0,\tau];\mathbb{R})$ and a strict solution $v \in C^1([0,\tau];X)$ of the degenerate problem

$$\begin{cases} M^*\dfrac{\mathrm{d}Mv(t)}{\mathrm{d}t} = Lv(t) + M^*f(t)z, \quad 0 \leqslant t \leqslant \tau, \\ Mv(0) = u_0, \end{cases}$$

satisfying the additional condition

$$\varphi[Mv(t)] = g(t), \quad 0 \leqslant t \leqslant \tau.$$

Roughly speaking, we will use the semigroup approach and perturbation theory for linear operators to treat problems of the form (IP1)–(IP2). This approach reduces the problems to the multivalued problem:

(IP3) Find $f \in C^1([0,\tau];\mathbb{R})$ and a strict solution $u \in C^1([0,\tau];X)$ of the problem

$$\begin{cases} \dfrac{\mathrm{d}u(t)}{\mathrm{d}t} \in Au(t) + f(t)z, \quad 0 \leqslant t \leqslant \tau, \\ u(0) = u_0, \\ \varphi[u(t)] = g(t), \quad 0 \leqslant t \leqslant \tau, \end{cases}$$

where A is a multivalued linear operator that generates a c_0-semigroup on X. It shows all its power when A is dissipative, because then we are able to treat even nonlinear operators A. However, maximal dissipativity of A is in general achieved only with respect to weak norms, for example, norms of distribution spaces H^{-m}. Here we exploit the linearity of the operator involved to develop a theory for the multivalued problem that seems much more satisfactory from this point of view.

The rest of the paper is organized as follows. In the next section, some definitions and preliminary results are introduced. Section 3 is devoted to the global existence in time and the uniqueness results of the identification problems (IP1)–(IP3). Section 4 is devoted to different applications of our results including identification problems related to the Navier-Stokes system, Maxwell equations, and Poisson-heat equations.

2. Preliminary Results

We denote by X a Banach space with norm $\|\cdot\|$ and $A\colon D(A) \to X$ is the infinitesimal generator of a c_0-semigroup of bounded linear operators $T(t)$, $t \geqslant 0$, on X. It is well known that A is closed and its domain $D(A)$ equipped with the graph norm

$$\|x\|_A = \|x\| + \|Ax\|$$

becomes a Banach space, which we shall denote by X_A.

Let L and M be two single-valued, closed linear operators in X with $D(L) \subset D(M)$. We are concerned with the resolvent of the multivalued linear operator LM^{-1}. In order to represent $(\lambda - LM^{-1})^{-1}$ by L and M, we introduce the notion $\varrho_M(L)$ of the M resolvent set of L by:

$$\varrho_M(L) = \{\lambda \in \mathbb{C}\colon \ \lambda M - L \text{ has a single-valued and bounded inverse on } X\}$$

and the bounded operator $(\lambda M - L)^{-1}$ is called the M resolvent of L. In subsequent sections we shall need the following assertion concerning multivalued linear operators.

Theorem 1 ([11]). *Let A be a multivalued linear operator on X such that $A - \beta$ is maximal dissipative with some real number β, i.e., A satisfies*

$$\mathrm{Re}(f, u)_X \leqslant \beta \|u\|_X^2 \quad \textit{for all } f \in Au \tag{1}$$

with the range condition

$$R(\lambda_0 - A) = X \quad \textit{for some } \lambda_0 > \beta. \tag{2}$$

Then, $\varrho(A) \supset (\beta, \infty)$ and A satisfies

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(X)} \leqslant \frac{1}{\lambda - \beta}, \quad \lambda > \beta.$$

By virtue of Theorem 1, if A is a multivalued linear operator on X with a maximal dissipative $A - \beta$, $\beta \in \mathbb{R}$, a semigroup $T(t) = \mathrm{e}^{tA}$ is generated by A on the whole space X.

In the sequel, special attention is being paid to necessary and sufficient conditions under which one can determine a unique solution of the multivalued Cauchy problem:

$$u'(t) \in Au(t) + f(t), \quad 0 \leqslant t \leqslant T, \tag{3}$$

$$u(0) = u_0. \tag{4}$$

This type of situation is covered by the following result.

Theorem 2 ([11]). *Let X be a Banach space and A be a multivalued linear operator generating a c_0-semigroup $T(t)$ on X. For any $f \in C^1([0,T]; X)$ and any $u_0 \in D(A)$, the function*

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)\,\mathrm{d}s$$

is the unique solution to the multivalued problem (3)–(4).

The following result from perturbation theory for linear operators will be helpful in the sequel.

Theorem 3 ([9]). *Let X be a Banach space and let A be the infinitesimal generator of a c_0-semigroup $T(t)$ on X. If $B\colon\ X_A \longrightarrow X_A$ is a continuous linear operator, then $A+B$ is the infinitesimal generator of a c_0-semigroup on X.*

3. Main Results

Let us first consider the multivalued identification problem (IP3) in a Banach space X:

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} \in Au(t) + f(t)z, \quad 0 \leqslant t \leqslant \tau, \tag{5}$$

$$u(0) = u_0, \tag{6}$$

$$\varphi[u(t)] = g(t), \quad 0 \leqslant t \leqslant \tau. \tag{7}$$

The following result allows to formulate some conditions under which one can find a unique strong solution to the identification problem (5)–(7).

Theorem 4. *Let A be a multivalued linear operator that generates a c_0-semigroup on X, $z \in X$, $u_0 \in D(A)$, $g \in C^1([0,\tau];\mathbb{R})$, $\varphi \in X^*$, and $\varphi[z] \neq 0$. Then the identification problem* (5)–(7) *possesses a unique solution in the class of functions*

$$u \in C^1([0,\tau]; D(A)), \quad f \in C^1([0,\tau];\mathbb{R}).$$

P r o o f. By applying the linear functional φ to both sides of (5) and using (7), we have

$$g'(t) \in \varphi[Au(t)] + f(t)\varphi[z],$$

and if $\varphi[z] \neq 0$, we obtain

$$f(t) \in \frac{1}{\varphi[z]}(g'(t) - \varphi[Au(t)]). \tag{8}$$

Substituting (8) in (5), we get

$$u'(t) \in Au(t) + \frac{1}{\varphi[z]}(g'(t) - \varphi[Au(t)])z. \tag{9}$$

By defining the operator

$$Bx = \frac{-1}{\varphi[z]}(\varphi[Ax])z, \tag{10}$$

then (9) becomes

$$u'(t) \in (A+B)u(t) + \frac{1}{\varphi[z]} g'(t)z. \tag{11}$$

The boundedness of the operator B in X_A follows from the estimate

$$\begin{aligned} \|B\|_A &= \sup_{\|x\|_A=1} \|Bx\| \\ &= \sup_{\|x\|_A=1} \left\| \frac{-1}{\varphi[z]} (\varphi[A(x)])z \right\| \\ &\leqslant \sup_{\|x\|_A=1} \frac{1}{|\varphi[z]|} \|z\| \|\varphi\| \|Ax\| \\ &\leqslant \frac{1}{|\varphi[z]|} \|z\| \|\varphi\|. \end{aligned}$$

This proves that B is a bounded linear operator on X_A. By virtue of Theorem 3, $A+B$ is the infinitesimal generator of a semigroup $S(t)$, $t \geqslant 0$. Since $u_0 \in D(A)$, Theorem 2 implies that the Cauchy problem (5)–(6) has a unique solution

$$u(t) = S(t)u_0 + \frac{1}{\varphi[z]} \int_0^t S(t-s)g'(s)z \, \mathrm{d}s, \tag{12}$$

and by (8) and (12), $f(t)$ is uniquely determined. Therefore, the problem (5)–(7) has a unique solution (u, f) and the proof is completed. □

We now turn our attention to degenerate identification problems. Consider the degenerate identification problem (IP1) in the Banach space X, where M and L are single-valued linear operators. Note that M may have no bounded inverse and so the classical theory of semigroups does not apply here.

We assume the resolvent set $\varrho_M(L)$ contains a region

$$\Sigma_\gamma = \{\lambda \in \mathbb{C} \colon \operatorname{Re}(\lambda - \gamma) \geqslant -c(|\operatorname{Im}\lambda| + 1)^\alpha\}, \quad \gamma \in \mathbb{R}, \tag{13}$$

and the M resolvent satisfies

$$\|(\lambda M - L)^{-1} M\|_{\mathcal{L}(X)} \leqslant \frac{C}{(|\lambda - \gamma| + 1)^\beta}, \quad \lambda \in \Sigma_\gamma, \tag{14}$$

with some exponents $0 < \beta \leqslant \alpha \leqslant 1$ and constants $c, C > 0$.

We can now prove the existence and uniqueness theorem of solutions to the identification problem (IP1).

Theorem 5. *Let X, Y be two Banach spaces such that $X \subset Y$, M is a bounded linear operator from X to Y and L is a single-valued closed linear operator in Y with $D(L) \subset X$, $z \in X$, $u_0 \in X$, $g \in C^1([0,\tau];\mathbb{R})$, $\varphi \in X^*$, $\varphi[z] \neq 0$ and let* (13) *and* (14) *be satisfied. Then, problem* (IP1) *possesses a unique solution* (u, f) *such that*

$$u \in C^1((0,\tau];X), \quad Lu \in C((0,\tau];Y), \quad f \in C([0,\tau];\mathbb{R}).$$

Proof. We rewrite the identification problem (IP1) into the multivalued form

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} \in M^{-1}Lu(t) + f(t)z, \quad 0 \leqslant t \leqslant \tau, \\ u(0) = u_0, \\ \varphi[u(t)] = g(t) \end{cases} \tag{15}$$

in the space X. Notice that $M^{-1}L$ acts in X. In addition, the change of the unknown function to $u_\gamma(t) = \mathrm{e}^{-\gamma t}u(t)$ yields that (15) is regarded as a multivalued problem of the form (5)–(7) with the operator coefficient $A = M^{-1}L - \gamma$. We can verify that $\varrho_M(L) \subset \varrho(LM^{-1})$ and $M(\lambda M - L)^{-1} = (\lambda - LM^{-1})^{-1}$ (see [11]). It follows for $\lambda + \gamma \in \varrho_M(L)$ that

$$(\lambda - A)^{-1} = ((\lambda + \gamma)M - L)^{-1}M.$$

In this line, (13) and (14) yields directly that the resolvent set $\varrho(A)$ contains a region

$$\Sigma = \{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda \geqslant -c(|\operatorname{Im}\lambda| + 1)^\alpha\}, \quad \gamma \in \mathbb{R}, \tag{16}$$

and the resolvent $(\lambda - A)^{-1}$ satisfies

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(X)} \leqslant \frac{C}{(|\lambda| + 1)^\beta}, \quad \lambda \in \Sigma, \tag{17}$$

with some exponents $0 < \beta \leqslant \alpha \leqslant 1$ and constants $c, C > 0$. Formulae (16) and (17) ensure that the multivalued linear operator A generates an infinitely differentiable semigroup on X (see [2]). Therefore, the reduced multivalued problem (5)–(7) possesses a unique strict solution (u, f). Clearly u_γ is a strict solution to (5)–(7) if and only if u is a strict solution to the identification problem (IP1) in the sense

$$u \in C^1((0,\tau];X), \quad Lu \in C((0,\tau];Y), \quad f \in C([0,\tau];\mathbb{R}).$$

□

Continuity of the solution $u(t)$ at $t = 0$ in the topology of X is achieved as follows.

Theorem 6. *Let $u_0 \in L^{-1}(R(M))$. In the case when $\alpha = \beta = 1$, let $u_0 \in \overline{L^{-1}(R(M))}$. Then, the solution u obtained in Theorem 5 is continuous at $t = 0$ in the norm of X, i.e. $u \in C([0, \tau]; X)$ with $u(0) = \lim_{t\to 0} u(t) = u_0$.*

P r o o f. Since in this case $D(A) = D(M^{-1}L) = L^{-1}(R(M))$, the continuity of $u(t)$ at $t = 0$ in the topology of X is obtained. $\square$

The identification problem (IP2) can be handled in the same manner, provided that we assume:

$$\operatorname{Re}(Lv, v)_X \leqslant \beta \|Mv\|_X^2 \quad \text{for all } v \in D(L), \tag{18}$$

$$R(\lambda_0 M^* M - L) \supset R(M^*), \tag{19}$$

and

$$(\lambda_0 M^* M - L)^{-1} \text{ is single-valued on } R(M^*) \text{ with some } \lambda_0 > \beta. \tag{20}$$

Then we prove the following theorem.

Theorem 7. *Let M be a bounded linear operator on the Banach space X, M^* is the adjoint operator of M, L is a single-valued linear operator in X, $z \in X$, $g \in C^1([0, \tau]; \mathbb{R})$, $\varphi \in X^*$, $\varphi[z] \neq 0$ and let* (18)–(20) *be satisfied. Then, for any u_0 satisfying*

$$u_0 = Mv_0, \quad Lv_0 \in R(M^*) \quad \textit{with some} \quad v_0 \in D(L), \tag{21}$$

problem (IP2) *possesses a unique solution (v, f) such that*

$$Mv \in C^1((0, \tau]; X), \quad Lv \in C((0, \tau]; X), \quad f \in C([0, \tau]; \mathbb{R}).$$

P r o o f. By changing the unknown function to $u(t) = Mv(t)$, we rewrite the identification problem (IP2) into a multivalued equation of the form (5)–(7) with the coefficient operator $A = (M^*)^{-1} L M^{-1}$. Then $A - \beta$ is shown to be maximal dissipative in X. Indeed, if $h \in Au$, then $M^* h = Lv$ and $Mv = u$ with some $v \in D(L)$, so

$$(h, u)_X = (h, Mv)_X = (Lv, v)_X.$$

This shows that (1) follows from (18). On the other hand, for any $h \in X$, we have by (18) that

$$M^* h = (\lambda_0 M^* M - L)v \quad \text{with some } v \in D(L).$$

If we put $u = Mv$, then $v \in M^{-1}u$ and $\lambda_0 u - h \in (M^*)^{-1}Lv$. That is, $h \in (\lambda_0 - A)u$ and (2) is satisfied. According to Theorem 1, $A - \beta$ is maximal dissipative in X and so A is the infinitesimal generator of a c_0-semigroup on X. It is easy to verify that $u_0 \in D(A)$ is equivalent to (21). Therefore, the reduced multivalued problem (5)–(7) possesses a unique strict solution (u, f). It is also easy to verify that $u = Mv$ is a strict solution to (5)–(7) if and only if v is a strict solution to the identification problem (IP2) in the sense

$$Mv \in C^1((0,\tau];X), \quad Lv \in C((0,\tau];X), \quad f \in C([0,\tau];\mathbb{R}).$$

The uniqueness of the solution v follows from the invertibility of $(\lambda_0 M^*M - L)^{-1}$, as assumed in (20). □

4. Applications

4.1. Identification problem of the system of Navier-Stokes equations. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. We focus our attention on the system consisting of the linearized Navier-Stokes equations

$$\frac{\partial u}{\partial t} = \nu\Delta u - \nabla p + f(t)h(x), \quad (x,t) \in \Omega \times (0,T], \tag{22}$$

and the incompressibility equation

$$\operatorname{div} u = 0, \quad (x,t) \in \Omega \times (0,T]. \tag{23}$$

The direct problem here consists of finding a vector-valued function

$$u\colon\ \Omega \times [0,T] \to \mathbb{R}^n$$

and a scalar-valued function

$$p\colon\ \Omega \times [0,T] \to \mathbb{R},$$

satisfying the system (22)–(23) with the supplementary boundary and initial conditions

$$u = 0, \quad (x,t) \in \partial\Omega \times (0,T], \tag{24}$$

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{25}$$

Here $h(x)$ is a given function, $u_0(x)$ is an initial function and $\nu = \text{const} > 0$. The system (22)–(25) permits us to describe the motion of a viscous incompressible fluid in the domain Ω, where the velocity of the fluid is well-characterized by the function u, while the function p is associated with the pressure. The coefficient ν is called the coefficient of kinematic viscosity. The fluid is supposed to be homogeneous with unit density. We impose a normalization condition on the pressure values. For example, one way of proceeding is to require that

$$\int_\Omega p(x,t)\,\mathrm{d}x = 0. \tag{26}$$

We proceed further by completely posing the inverse problem for the Navier-Stokes equation assuming that the external force function f is unknown. In this line, we impose the subsidiary information in the form of integral overdetermination

$$\int_\Omega (u(x,t), w(x))\,\mathrm{d}x = g(t) \quad \forall t \in [0,T], \tag{27}$$

where the vector-valued function $w(x)$ and the scalar-valued function $g(t)$ are given.

We show that the identification problem (22)–(27) can be formulated by using a multivalued problem of the form (5)–(7).

Take $X = \{L^2(\Omega)\}^n$. It is well known that X is the orthogonal direct sum of two subspaces:

$$\begin{cases} X_s = \text{The closure of } \{u \in C_0^\infty(\Omega)\colon \operatorname{div} u = 0 \text{ in } \Omega\} \text{ in } X, \\ X_g = \{\nabla p\colon \; p \in H^1(\Omega)\}. \end{cases}$$

In X we define the multivalued linear operator A as follows:

$$\begin{cases} D(A) = \{H^2(\Omega)\}^n \cap \{H_0^1(\Omega)\}^n \cap X_s, \\ Au = \Delta u + X_s. \end{cases}$$

Then (22)–(27) is written in the form

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} \in \nu Au + f(t)h(x), \quad 0 < t \leqslant T, \\ u(0) = u_0, \\ \varphi[u] = g(t) \end{cases}$$

with $u_0 \in X_s$.

As a matter of fact, this formulation is essentially equivalent to the classical one which is written as

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = \nu A_s u + Pf(t)h(x), \quad 0 < t \leqslant T, \\ u(0) = u_0, \\ \varphi[u] = g(t) \end{cases}$$

in the subspace X_s by using a linear section A_s (called the Stokes operator) of A such that

$$\begin{cases} D(A_s) = D(A), \\ A_s = P\Delta u, \end{cases}$$

P being the orthogonal projection on X_s, $P\colon\ X \to X_s$.

For $u \in D(A)$, $f \in (\lambda - \nu A)u$ if and only if $(\lambda - \nu A_s)u = Pf$. Therefore, $(\lambda - \nu A)^{-1} = (\lambda - \nu A_s)^{-1}P$ with $\varrho(A) = \varrho(A_s)$. Since νA_s is the generator of an analytic semigroup on X_s, the same is true for νA on X with

$$\mathrm{e}^{t\nu A} = \mathrm{e}^{t\nu A_s}P, \quad t > 0.$$

In particular, it follows for $g \in X_s = \overline{D(A)}$ that

$$\mathrm{e}^{t\nu A}g = \mathrm{e}^{t\nu A_s}g, \quad t > 0,$$

and for $h \in X_g$

$$\mathrm{e}^{t\nu A}h = 0, \quad t > 0.$$

4.2. The identification problem related to the system of Maxwell equations.

Consider the system of Maxwell equations in a bounded domain $\Omega \subset \mathbb{R}^3$:

$$\text{(28)} \qquad \begin{aligned} \operatorname{rot} E &= -\frac{\partial B}{\partial t}, \\ \operatorname{rot} H &= \frac{\partial D}{\partial t} + J, \end{aligned}$$

where E is the vector of the electric field strength, H is the vector of the magnetic field strength, D and B denote the electric and magnetic induction vectors, respectively. In what follows, we denote by J the current density.

In the sequel, we deal with a linear medium in which the vectors of strengths are proportional to those of inductions in accordance with the governing laws:

$$\text{(29)} \qquad D = \varepsilon E, \quad B = \mu H,$$

and we assume, in addition, that Ohm's law

$$J = \sigma E + I \tag{30}$$

is satisfied in the domain Ω, where ε is the dielectric permeability of the medium, μ is the magnetic permeability, σ is the electric conductance, and I the density of the extraneous current. Further development is connected with the initial conditions for the vectors of the electric and magnetic inductions:

$$\begin{aligned} D(x,0) &= D_0(x), \\ B(x,0) &= B_0(x). \end{aligned} \tag{31}$$

The direct problem here consists of finding the functions E, D, H, B from the system (28)–(31) for the given functions $\varepsilon, \mu, \sigma, I, D_0$, and B_0. The formulation of an inverse problem involves the density of the extraneous current as an unknown of the structure

$$I(x,t) = f(t)p(x), \tag{32}$$

where the matrix $p(x)$ of size 3×3 is known for all $x \in \Omega$, while the unknown vector-valued function $f(t)$ is sought. To complete such a setting of the problem, we take the integral overdetermination in the form

$$\int_\Omega E(x,t)w(x) = g(t), \quad 0 \leqslant t \leqslant T, \tag{33}$$

where the function $w(x)$ is known in advance. The system of equations (28)–(33) is treated as the inverse problem for the Maxwell system related to the unknown functions E, H and f.

By setting

$$v = \begin{pmatrix} E \\ H \end{pmatrix}, \quad c(x) = \begin{pmatrix} \varepsilon(x) & 0 \\ 0 & \mu(x) \end{pmatrix}, \\ b(x) = -\begin{pmatrix} \sigma(x) & 0 \\ 0 & 0 \end{pmatrix}, \quad f(x,t) = -\begin{pmatrix} I(x,t) \\ 0 \end{pmatrix}, \tag{34}$$

the system (28)–(31) can be written as

$$\frac{\partial c(x)v}{\partial t} = \sum_{i=1}^{3} a_i \frac{\partial v}{\partial x_i} + b(x)v + f(x,t), \quad (x,t) \in \mathbb{R}^3 \times [0,T], \tag{35}$$

with certain 6×6 matrices a_i, $i = 1, 2, 3$.

The quantities $\varepsilon(x)$, $\mu(x)$ and $\sigma(x)$ are assumed to be real matrices whose components are bounded measurable functions in $\mathbb{R}^3$. In addition, we assume:

$$\varepsilon(x) \text{ is symmetric and } \varepsilon(x) \geqslant 0 \text{ for all } x \in \mathbb{R}^3; \tag{36}$$

$$\mu(x) \text{ is symmetric and } \mu(x) \geqslant \delta, \text{ for some } \delta > 0 \text{ uniformly in } x \in \mathbb{R}^3; \tag{37}$$

$$(\{\gamma\varepsilon(x) + \sigma(x)\}\xi, \xi) \geqslant \delta\|\xi\|^2, \ \xi \in \mathbb{R}^3, \tag{38}$$

for some $\delta > 0$ and $\gamma \geqslant 0$ uniformly in $x \in \mathbb{R}^3$. Further treatment of the system (28)–(33) as an abstract problem is connected with introducing the Lebesgue space

$$X = (L^2(\mathbb{R}^3))^3 \times (L^2(\mathbb{R}^3))^3 = (L^2(\mathbb{R}^3))^6,$$

using the bounded operator M of multiplication by $c(x)$ acting in X (the adjoint operator M^* of M satisfies $M^* = M$) and the closed linear operator L satisfies, for each $v \in D(L)$ the set of relations

$$D(L) = \left\{ v \in X \colon \sum_{i=1}^{3} a_i \frac{\partial v}{\partial x_i} \in X \right\},$$

$$Lv = \sum_{i=1}^{3} a_i \frac{\partial v}{\partial x_i} + b(x)v.$$

Accordingly, the symbols $v(t)$ and $f(t)z$ will refer to the same functions $v(x,t)$ and $f(t)p(x)$ but viewed as abstract functions of the variable t with values in the space X. The symbol u_0 will be used in treating the function $u_0(x) = (D_0(x), B_0(x))$ as the element of the space X. With these ingredients, the system (28)–(33) reduces to the inverse problem (IP2) in the Banach space X.

Condition (18) is verified as follows. Let $v \in Y = (H^1(\mathbb{R}^3))^6 \subset D(L)$. Then,

$$(Lv, v)_X = -\left(v, \sum_{i=1}^{3} a_i \frac{\partial v}{\partial x_i} + b(x)v \right)_X + (b(x)v, v)_X + (v, b(x)v)_X;$$

so

$$\operatorname{Re}(Lv, v)_X = \operatorname{Re}(b(x)v, v)_X = -\operatorname{Re}(\sigma E, E)_{L^2}.$$

In this line, (37) and (38) yield that

$$\begin{aligned} \operatorname{Re}(Lv, v)_X &\leqslant -\delta(\|E\|_{L^2}^2 + \|H\|_{L^2}^2) + \gamma(\varepsilon(x)E, E)_{L^2} + (\mu(x)H, H)_{L^2} \\ &\leqslant -\delta\|v\|_X^2 + \lambda((\varepsilon(x)E, E)_{L^2} + (\mu(x)H, H)_{L^2}) \\ &= -\delta\|v\|_X^2 + \lambda\|Mv\|_X^2, \end{aligned} \tag{39}$$

where $\lambda \geqslant \max\{\gamma, 1\}$. This estimate is in fact verified for $v \in D(L)$ also, because there exists a sequence $v_n \in Y$ such that $v_n \to v$ and $Lv_n \to Lv$ in X. Thus, (1) holds with $\beta = \max\{\gamma, 1\}$.

Let us next verify (19) and (20). Since (39) yields that

$$\|(\lambda M^*M - L)v\|_X \geqslant \delta\|v\|_X, \quad v \in D(L),$$

$(\lambda M^*M - L)$ is seen to be one-to-one and to have a closed range. Therefore, it suffices to verify that $R(\lambda M^*M - L)^\perp = \{0\}$. Let $w \in R(\lambda M^*M - L)^\perp$. Then $w \in D(L^*)$ and $(\lambda M^*M - L^*)w = 0$. On the other hand, since the principal part of L is symmetric, $w \in D(L)$. Therefore, (39) yields

$$-\delta\|w\|_X^2 + \lambda\|Mw\|_X^2 \geqslant \mathrm{Re}(Lw, w)_X = \mathrm{Re}(w, L^*w)_X = \lambda\|Mw\|_X^2,$$

and so $w = 0$.

As a result, if conditions (36)–(38) are satisfied, then a solution E, H, p of the identification problem (28)–(33) exists and it is unique in the class of functions

$$E, H \in C([0,T]; (L^2(\mathbb{R}^3))^3), \quad p \in C([0,T]; \mathbb{R}^3).$$

4.3. Identification problem of the Poisson-Heat equation.

In several applications, when the temperature of a thermal body, subjected to an external supply of heat, is to be determined, the source itself is often unknown or scarcely known. So, we are faced with recovering both the temperature and the unknown source. To compensate for the lack of information, suitable measurements involving the temperature are given, as well as suitable assumptions on the source are made. For instance, it is assumed to depend on a single space variable, i.e. on time only, or to be the product of two functions, the first depending on the temperature and the second on the space variable.

Consider the Poisson-heat equation:

$$\begin{aligned} m(x)\frac{\partial u}{\partial t} &= \Delta u + m(x)f(t)h(x), \quad (x,t) \in \Omega \times (0,T], \\ v &= 0, \quad (x,t) \in \partial\Omega \times (0,T], \\ u(x,0) &= u_0(x), \quad x \in \Omega, \end{aligned} \tag{40}$$

with the supplementary condition

$$\int_\Omega \eta(x)u(x,t)\,\mathrm{d}x = g(t) \quad \forall t \in [0,T] \tag{41}$$

in a bounded region $\Omega \subset \mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Here $m(x) \geqslant 0$ in Ω is a given function in $L^\infty(\Omega)$, $u_0, \eta, h \in H^{-1}(\Omega)$, and g is a continuous function on $[0,T]$.

This problem is regarded as a problem of the form (IP1) in which M is the multiplication operator by $m(x)$ and L is Δ with the Dirichlet boundary conditions. As the underlying space Y, we take $H^{-1}(\Omega)$. Then $L\colon\ H_0^1(\Omega) \to H^{-1}(\Omega)$. We can take $X = H_0^1(\Omega)$, then $M\colon\ X \to L^2(\Omega) \subset Y$. We can verify that $(\lambda M - L)^{-1}$, $\lambda \in \Sigma$, exists as a bounded operator from X to Y. Consider the scalar product

$$\langle(\lambda M - L)^{-1} M u, \varphi\rangle_{H_0^1 \times H^{-1}} = \langle u, M(\lambda M - L)^{-1}, \varphi\rangle_{H_0^1 \times H^{-1}}.$$

Moreover, noting the identity

$$\lambda M(\lambda M - L)^{-1} = 1 + L(\lambda M - L)^{-1},$$

we obtain that

$$\|M(\lambda M - L)^{-1}\varphi\|_{H^{-1}} \leqslant C|\lambda|^{-1}\|\varphi\|_{H^{-1}}, \quad \varphi \in H^{-1}(\Omega). \tag{42}$$

Then it follows from (42) that

$$|\langle(\lambda M - L)^{-1} M u, \varphi\rangle_{H_0^1 \times H^{-1}}| \leqslant C|\lambda|^{-1}\|\varphi\|_{H^{-1}}\|u\|_{H_0^1},$$

and this immediately yields that (13) and (14) are valid with $\alpha = \beta = 1$, $\gamma = 0$.

Therefore for any $f \in C^\sigma([0,T]; H_0^1(\Omega))$, $\sigma > 0$, and any $u_0 \in H_0^1(\Omega)$, the problem (40)–(41) possesses a unique solution

$$u \in C^1([0,T]; H_0^1(\Omega)) \cap C([0,T]; H_0^1(\Omega) \cap H^2(\Omega)), \quad f \in C([0,T]; \mathbb{R}).$$

From Theorem 6, u is continuous at $t = 0$ if $\Delta u_0 = m u_1$ with some $u_1 \in L^2(\Omega)$ (note that $m u_1$ belongs to the closure of $R(M)$ in Y).

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