Ali R. Soheili; Mahdieh Arezoomandan

Approximation of stochastic advection diffusion equations with stochastic alternating direction explicit methods

Applications of Mathematics, Vol. 58 (2013), No. 4, 439--471

Persistent URL: http://dml.cz/dmlcz/143340

APPROXIMATION OF STOCHASTIC ADVECTION DIFFUSION EQUATIONS WITH STOCHASTIC ALTERNATING DIRECTION EXPLICIT METHODS

Ali R. Soheili, Mashad, Mahdieh Arezoomandan, Zahedan

(Received May 23, 2011)

Abstract. The numerical solutions of stochastic partial differential equations of Itô type with time white noise process, using stable stochastic explicit finite difference methods are considered in the paper. Basically, Stochastic Alternating Direction Explicit (SADE) finite difference schemes for solving stochastic time dependent advection-diffusion and diffusion equations are represented and the main properties of these stochastic numerical methods, e.g. stability, consistency and convergence are analyzed. In particular, it is proved that when stable alternating direction explicit schemes for solving linear parabolic PDEs are developed to the stochastic case, they retain their unconditional stability properties applying to stochastic advection-diffusion and diffusion SPDEs. Numerically, unconditional stable SADE techniques are significant for approximating the solutions of the proposed SPDEs because they do not impose any restrictions for refining the computational domains. The performance of the proposed methods is tested for stochastic diffusion and advection-diffusion problems, and the accuracy and efficiency of the numerical methods are demonstrated.

Keywords: stochastic partial differential equation, finite difference method, alternating direction method, Saul'yev method, Liu method, convergence, consistency, stability

MSC 2010: 60H15, 65M06

1. Introduction and preliminaries

The extensive application of random effects in describing some practical sciences like engineering and mathematical finance has developed with the theory of stochastic partial differential equations, or SPDEs. Thus, providing applicable numerical

This research was supported by a grant from Ferdowsi University of Mashhad (No. MA90262ARS)

techniques and high accuracy computational methods is of great importance for approximating the solution of stochastic problems. Many effective researches for solving stochastic differential equations as well as their strong and weak approximation have been implemented by Kloeden and Platen [6], Komori and Mitsui [7], Milstein [10] and Rößler [11].

In recent years, some main numerical methods for solving PDEs like finite difference and finite element schemes [1], [4], [9] and some practical techniques like the method of lines [12], [9] for boundary value problems have been applied to the linear stochastic partial differential equations, and the results of these approaches have been experimented numerically.

In [17], the authors considered the approximation of stochastic parabolic equations with real valued Brownian motion using two various finite difference methods, and investigated their numerical results. In other words, the fundamental notions of deterministic methods have been applied to the stochastic case with approximation of one dimensional white noise. This article provides several alternating direction explicit methods for solving stochastic heat-diffusion and time dependent advection-diffusion equations with one dimensional white noise process. These schemes are unconditionally stable and explicit in nature and can considerably reduce the necessary computation in higher dimensions in comparison with other unconditionally stable methods which are mostly implicit.

Our concern in the current work is on approximating the solutions of the Itô stochastic partial differential equations of the following form:

$$\tag{1} \begin{gathered} \frac{\partial u}{\partial t}(x,t) + \nu \frac{\partial u}{\partial x}(x,t) = \gamma \frac{\partial^2 u}{\partial x^2}(x,t) + \sigma u(x,t)\dot{W}(t), \quad 0 \leqslant t \leqslant T, \\ u(x,0) = u_0(x), \quad 0 \leqslant x \leqslant 1, \end{gathered}$$

with respect to an R^1-valued Wiener process $(W(t), F_t)_{t\in[0,T]}$ defined on a complete probability space (Ω, F, P), adapted to the standard filtration $(F_t)_{t\in[0,T]}$. The parameter γ is the viscosity coefficient and ν is the phase speed, and both are assumed to be positive.

Clearly, in the absence of the advection term the stochastic partial differential equation will be reduced to the stochastic diffusion equation of the form

$$\tag{2} \frac{\partial u}{\partial t}(x,t) = \gamma \frac{\partial^2 u}{\partial x^2}(x,t) + \sigma u(x,t)\dot{W}(t), \quad 0 \leqslant t \leqslant T.$$

The performance of stochastic ADE methods will be illustrated for both the diffusion and advection-diffusion equations and their main qualifications will be studied.

This paper is organized as follows: the deterministic theory of several alternating direction methods, in particular, Saul'yev, Liu and Saul'yev/Robert and Weiss schemes for solving deterministic heat-diffusion and advection-diffusion equations are reviewed in Section 2. In Section 3, stochastic Saul'yev, Liu and Saul'yev/Robert and Weiss schemes for approximating the diffusion and advection-diffusion SPDEs are presented. In the next section some general considerations for stochastic difference methods are given. Sections 5, 6, and 7 contain the analysis of the stability, consistency and convergence of the proposed stochastic alternating direction methods, respectively. The computational performance of the stochastic difference methods and their numerical results are demonstrated in Section 8. Section 9 contains some concluding remarks.

2. Alternating direction deterministic schemes

Basically, partial differential equations have considerable applications in different areas of science and engineering. In this section, we consider the approximation of the solution of deterministic heat diffusion and advection diffusion equations using deterministic ADE finite difference schemes. Numerically, finite difference methods have vast applications in approximating the solution of partial differential equations (PDEs). These schemes discretize continuous space and time into an evenly distributed grid system, and the values of the state variables are evaluated at each node of the grid. Considering a uniform space grid Δx and time grid Δt in the time-space lattice, we can estimate the solution of the equation at the points of this lattice. The value of the approximate solution at the point $(k\Delta x, n\Delta t)$ will be denoted by u_k^n, where n, k are integers [19].

Alternating direction explicit finite-difference methods are essentially based on the two approximations that are implemented for computations proceeding in alternating directions, e.g., from left to right and from right to left [2], [3]. These methods were first introduced by Saul'yev [15], [16] for solving initial value problems involving the one-dimensional heat diffusion equation and then developed to the other cases. In these schemes, the computation process in two opposite directions follows the implementation of explicit finite difference methods. The Saul'yev and Liu techniques for solving the linear parabolic heat equation are unconditionally stable explicit methods that do not require the solution of large systems of simultaneous equations at each time step like most other unconditionally stable methods.

An important characteristic of these approaches, in addition to their unconditional stability, is that they have truncations error of opposite sign. So, by appropriate combination it may be possible to construct combined solutions having the properties of accuracy, stability and simplicity. In the following subsection, the formulations of Saul'yev and Liu schemes for the diffusion equation and Saul'yev/Robert and Weiss schemes for the advection-diffusion equations are reviewed:

2.1. Alternating direction methods for diffusion equation.

2.1.1. Saul'yev schemes. The Saul'yev scheme approximates the diffusion equation by

$$(1+\gamma\varrho)U_k^{n+1} = (1-\gamma\varrho)U_k^n + \gamma\varrho(U_{k-1}^{n+1} + U_{k+1}^n), \tag{3}$$

with the calculation proceeding from the left boundary to the right (L-R Saul'yev), and

$$(1+\gamma\varrho)U_k^{n+1} = (1-\gamma\varrho)U_k^n + \gamma\varrho(U_{k+1}^{n+1} + U_{k-1}^n), \tag{4}$$

with the calculation proceeding from the right boundary to the left (R-L Saul'yev), where $\varrho = \Delta t/\Delta x^2$. The Saul'yev schemes can be shown to be unconditionally stable using the Von Neumann method of stability analysis [2], [3].

2.1.2. Liu schemes. Liu [8] used a higher order approximation and obtained the L-R Liu difference equation

$$(3\gamma\varrho+2)U_k^{n+1} = 2(1-2\gamma\varrho)U_k^n + \gamma\varrho(U_{k-1}^n + 3U_{k+1}^n - U_{k-2}^{n+1} + 4U_{k-1}^{n+1}), \tag{5}$$

and the R-L Liu algorithm by

$$(3\gamma\varrho+2)U_k^{n+1} = 2(1-2\gamma\varrho)U_k^n + \gamma\varrho(U_{k+1}^n + 3U_{k-1}^n - U_{k+2}^{n+1} + 4U_{k+1}^{n+1}). \tag{6}$$

They are analogous to the Saul'yev schemes except that the first point on any line (either from left to right or in the reverse direction) must be obtained by some other methods. Liu schemes in both cases are unconditionally stable for approximating the solution of the heat equation. Basically, separate use of left to right and right to left Saul'yev algorithm at a certain time and the following calculation of the average of the solutions at the gridpoints of the lattice can lead to a better approximation because of the truncation error cancellation. Of course, such a combination can also be applied to the Liu algorithm [2].

2.2. Alternating direction methods for advection-diffusion equation.

2.2.1. The Saul'yev/Robert and Weiss schemes. This method is based on an alternative approximation for the first derivative in the advection diffusion equation. Combined with the Saul'yev's discretization for the diffusion term, the first space derivative is approximated using the alternating direction explicit schemes of Robert and Weiss [3]. The discretization is

$$\frac{\partial u}{\partial x} \approx \frac{u_j^{n+1} + u_{j+1}^n - u_{j-1}^{n+1} - u_j^n}{2\Delta x} \tag{7}$$

in the left to right step and

$$\frac{\partial u}{\partial x} \approx \frac{u_{j+1}^{n+1} + u_j^n - u_j^{n+1} - u_{j-1}^n}{2\Delta x} \tag{8}$$

in the right to left step.

Applying the Robert and Weiss and Saul'yev discretizations to the first and second space derivatives respectively, we have

$$\Big(1+\gamma\varrho+\frac{\nu\lambda}{2}\Big)u_j^{n+1} = \Big(1-\gamma\varrho+\frac{\nu\lambda}{2}\Big)u_j^n + \Big(\gamma\varrho+\frac{\nu\lambda}{2}\Big)u_{j-1}^{n+1} + \Big(\gamma\varrho-\frac{\nu\lambda}{2}\Big)u_{j+1}^n \tag{9}$$

in the L-R step and

$$\Big(1+\gamma\varrho-\frac{\nu\lambda}{2}\Big)u_j^{n+1} = \Big(1-\gamma\varrho-\frac{\nu\lambda}{2}\Big)u_j^n + \Big(\gamma\varrho-\frac{\nu\lambda}{2}\Big)u_{j+1}^{n+1} + \Big(\gamma\varrho+\frac{\nu\lambda}{2}\Big)u_{j-1}^n \tag{10}$$

in the R-L step where $\lambda = \Delta t/\Delta x$ and $\varrho = \Delta t/\Delta x^2$. In the stability analysis, the L-R Saul'yev/Robert and Weiss scheme is unconditionally stable and can be used with no stability restriction on the time step size. But, the R-L Saul'yev/Robert and Weiss scheme is conditionally stable and a sufficient condition for stability is $\nu\lambda \leqslant 1$ applying to advection-diffusion equations [3].

3. Stochastic alternating direction schemes

3.1. Stochastic Saul'yev and Liu schemes.

We extend the proposed unconditional stable finite difference schemes for approximating the solutions of the stochastic diffusion equations of the form (2) and investigate their performance in the stochastic cases.

First, we introduce L-R stochastic Saul'yev methods using the approximation of the white noise process $\dot{W}(t)$ for stochastic parabolic equation (2):

$$u_k^{n+1} = u_k^n + \gamma\frac{\Delta t}{\Delta x^2}[u_{k+1}^n - u_k^n - u_k^{n+1} + u_{k-1}^{n+1}] + \sigma u_k^n \Delta W_n, \tag{11}$$

where $\Delta W_n = W((n+1)\Delta t) - W(n\Delta t)$. This equation can also be written as

$$(1+\gamma\varrho)u_k^{n+1} = \gamma\varrho u_{k+1}^n + (1-\gamma\varrho)u_k^n + \gamma\varrho u_{k-1}^{n+1} + \sigma u_k^n \Delta W_n, \tag{12}$$

where $\varrho = \Delta t/\Delta x^2$ and u_k^n is intended as an approximation to $u(k\Delta x, n\Delta t)$. A similar stochastic difference method can be rewritten for the stochastic R-L Saul'yev technique. In the same way, the stochastic Liu scheme can be considered as

$$(3\gamma\varrho+2)u_k^{n+1} = 2(1-2\gamma\varrho)u_k^n + \gamma\varrho(u_{k-1}^n + 3u_{k+1}^n - u_{k-2}^{n+1} + 4u_{k-1}^{n+1}) + \sigma u_k^n \Delta W_n. \tag{13}$$

We want to approximate the solution of (2) by the random variable u_k^n defined by (12) and (13), which are respectively the stochastic version of the Saul'yev and Liu methods. For all proposed schemes, the increments of Wiener process are assumed independent of the state u_k^n.

3.2. Stochastic Saul'yev/Robert and Weiss schemes.

Applying the Saul'yev/Robert and Weiss schemes for discretizing the advection and diffusion terms in equation (1) and using the discrete time approximation of continuous time white noise, we obtain

$$\begin{aligned}\Big(1+\gamma\varrho+\frac{\nu\lambda}{2}\Big)u_k^{n+1} = \Big(1-\gamma\varrho+\frac{\nu\lambda}{2}\Big)u_k^n + \Big(\gamma\varrho+\frac{\nu\lambda}{2}\Big)u_{k-1}^{n+1} \\ + \Big(\gamma\varrho-\frac{\nu\lambda}{2}\Big)u_{k+1}^n + \sigma u_k^n \Delta W_n\end{aligned} \tag{14}$$

for the L-R stochastic Saul'yev/Robert and Weiss scheme, and

$$\begin{aligned}\Big(1+\gamma\varrho-\frac{\nu\lambda}{2}\Big)u_k^{n+1} = \Big(1-\gamma\varrho-\frac{\nu\lambda}{2}\Big)u_k^n + \Big(\gamma\varrho-\frac{\nu\lambda}{2}\Big)u_{k+1}^{n+1} \\ + \Big(\gamma\varrho+\frac{\nu\lambda}{2}\Big)u_{k-1}^n + \sigma u_k^n \Delta W_n\end{aligned} \tag{15}$$

for the R-L stochastic Saul'yev/Robert and Weiss scheme, where

$$\Delta W_n = W((n+1)\Delta t) - W(n\Delta t), \quad \lambda = \frac{\Delta t}{\Delta x} \quad \text{and} \quad \varrho = \frac{\Delta t}{\Delta x^2}.$$

4. Some general considerations for stochastic schemes

In this section, we present some basic definitions which are indeed stochastic versions of the main concepts of the deterministic finite difference schemes.

Essentially, it is extremely important for the solution of stochastic difference schemes (SDS) to converge to the solution of the stochastic partial differential equations or SPDEs. First, consider a stochastic partial differential equation:

$$Lv = G,$$

where L denotes the differential operator and $G \in L^2(R)$ is an inhomogeneity. Assuming u_k^n is the solution that is approximated by a stochastic finite difference scheme denoted by L_k^n, and applying the stochastic scheme to the SPDE, we have $L_k^n u_k^n = G_k^n$, whereby G_k^n is the approximation of the inhomogeneity.

We refer to Roth [13], [14] for the following definitions, but first we introduce for sequences $u = \{\ldots, u_{-1}, u_0, u_1, \ldots\}$ the $\ell_{2,\Delta x}$-norm $|u|_{2,\Delta x} = \sqrt{\sum_{k=-\infty}^{+\infty} |u_k|^2 \Delta x}$ and the sup-norm $|u|_\infty = \sqrt{\sup_k |x_k|^2}$.

Definition 1 (Convergence of an SDS). A stochastic difference scheme $L_k^n u_k^n = G_k^n$ approximating the stochastic partial differential equation $Lv = G$ is convergent in mean square at time t if, as $(n+1)\Delta t$ converges to t,

$$E\|u_{n+1} - v_{n+1}\|^2 \to 0 \quad \text{for } (n+1)\Delta t = t, \text{ and } \Delta x \to 0, \tag{16}$$

where u^{n+1} and v^{n+1} are infinite dimensional vectors

$$\begin{aligned} u^{n+1} &= (\ldots, u_{k-2}^{n+1}, u_{k-1}^{n+1}, u_k^{n+1}, u_{k+1}^{n+1}, u_{k+2}^{n+1}, \ldots)^{\mathrm{T}}, \\ v^{n+1} &= (\ldots, v_{k-2}^{n+1}, v_{k-1}^{n+1}, v_k^{n+1}, v_{k+1}^{n+1}, v_{k+2}^{n+1}, \ldots)^{\mathrm{T}}. \end{aligned}$$

Definition 2 (Consistency of an SDS). The finite stochastic difference scheme $L_k^n u_k^n = G_k^n$ is pointwise consistent with the stochastic partial differential equation $Lv = G$ at a point (x,t), if for any continuously differentiable function $\Phi = \Phi(x,t)$,

$$E\|(L\Phi - G)_k^n - [L_k^n \Phi(k\Delta x, n\Delta t) - G_k^n]\|^2 \to 0 \tag{17}$$

in mean square as $\Delta x \to 0, \Delta t = t$, and $(k\Delta x, (n+1)\Delta t)$ converges to (x,t).

Definition 3 (Stability of an SDS). A stochastic difference scheme is said to be stable with respect to a norm in mean square if there exist positive constants $\overline{\Delta x_0}$ and $\overline{\Delta t_0}$ and non negative constants K and β such that

$$E\|u^{n+1}\|^2 \leqslant K \mathrm{e}^{\beta t} E\|u^0\|^2 \tag{18}$$

for all $0 \leqslant t = (n+1)\Delta t$, $0 \leqslant \Delta x \leqslant \overline{\Delta x_0}$, $0 \leqslant \Delta t \leqslant \overline{\Delta t_0}$.

R e m a r k 1 (Stability analysis of a stochastic scheme using the Fourier-transformation).

Von Neumann [18] introduces a method to prove stability using Fourier analysis so that it can give a necessary and sufficient condition for the stability of deterministic finite difference schemes.

Assuming $\hat{u}^{n+1}$ is the Fourier transformation of u^{n+1}, the Fourier-inversion-formula gives us

$$u_m^{n+1} = \frac{1}{\sqrt{2\pi}} \int_{-\pi/\Delta x}^{\pi/\Delta x} \mathrm{e}^{\mathrm{i}m\Delta x\xi} \hat{u}^{n+1}(\xi)\,\mathrm{d}\xi, \tag{19}$$

where

$$\hat{u}^{n+1} = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{m=\infty} \mathrm{e}^{-\mathrm{i}m\Delta x\xi} u_m^{n+1} \Delta x, \tag{20}$$

and ξ is a real variable. Substituting in a stochastic difference scheme, we have

$$\hat{u}^{n+1}(\xi) = g(\Delta x\xi, \Delta t, \Delta x)\hat{u}^n(\xi), \tag{21}$$

where $g(\Delta x\xi, \Delta t, \Delta x)$ is the amplification factor of the stochastic difference scheme. The decision whether a scheme is stable or not, can be simplified by the aid of amplification factor. Like the deterministic case, we get the following necessary and sufficient condition, for a scheme's stability via its amplification factor, see Roth [13]:

$$E|g(\Delta x\xi, \Delta t, \Delta x)|^2 \leqslant 1 + K\Delta t. \tag{22}$$

5. Stability analysis of stochastic ADE schemes

5.1. Stability of stochastic Saul'yev and Liu schemes for stochastic diffusion equations.

Theorem 1. *The stochastic Saul'yev schemes are unconditionally stable according to the Fourier-transformation analysis for the stochastic diffusion equation.*

P r o o f. According to the Fourier-inversion-formula, u_m^n has the transformation

$$u_m^n = \frac{1}{\sqrt{2\pi}} \int_{-\pi/\Delta x}^{\pi/\Delta x} \mathrm{e}^{\mathrm{i}m\Delta x\xi} \hat{u}^n(\xi)\,\mathrm{d}\xi.$$

Substituting in L-R stochastic Saul'yev scheme, we have

$$(1+\gamma\varrho)\hat{u}^{n+1}(\xi) - (\gamma\varrho)\mathrm{e}^{-\mathrm{i}\Delta x\xi}\hat{u}^{n+1}(\xi) = (\gamma\varrho)\mathrm{e}^{\mathrm{i}\Delta x\xi}\hat{u}^n(\xi) + (1-\gamma\varrho)\hat{u}^n(\xi) + \sigma\hat{u}^n(\xi)\Delta W_n.$$

Then we get

$$[(1+\gamma\varrho) - (\gamma\varrho)\mathrm{e}^{-\mathrm{i}\Delta x\xi}]\hat{u}^{n+1}(\xi) = [(\gamma\varrho)\mathrm{e}^{\mathrm{i}\Delta x\xi} + (1-\gamma\varrho)]\hat{u}^n(\xi) + \sigma\hat{u}^n(\xi)\Delta W_n,$$

and

$$\hat{u}^{n+1}(\xi) = \Big\{\frac{1-\gamma\varrho+\gamma\varrho\mathrm{e}^{\mathrm{i}\Delta x\xi}}{1+\gamma\varrho-\gamma\varrho\mathrm{e}^{-\mathrm{i}\Delta x\xi}} + \frac{\sigma\Delta W_n}{1+\gamma\varrho-\gamma\varrho\mathrm{e}^{-\mathrm{i}\xi}}\Big\}\hat{u}^n(\xi).$$

Therefore, the amplification factor of the stochastic Saul'yev scheme is

$$g(\Delta x\xi,\Delta t,\Delta x) := \Big\{\frac{1-\gamma\varrho+\gamma\varrho\mathrm{e}^{\mathrm{i}\Delta x\xi}}{1+\gamma\varrho-\gamma\varrho\mathrm{e}^{-\mathrm{i}\Delta x\xi}} + \frac{\sigma\Delta W_n}{1+\gamma\varrho-\gamma\varrho\mathrm{e}^{-\mathrm{i}\Delta x\xi}}\Big\}.$$

So, we get

$$\begin{aligned}
E|g(\Delta x\xi,\Delta t,\Delta x)|^2 &= E\Big|\frac{1-\gamma\varrho+\gamma\varrho\mathrm{e}^{\mathrm{i}\Delta x\xi}}{1+\gamma\varrho-\gamma\varrho\mathrm{e}^{-\mathrm{i}\Delta x\xi}} + \frac{\sigma\Delta W_n}{1+\gamma\varrho-\gamma\varrho\mathrm{e}^{-\mathrm{i}\Delta x\xi}}\Big|^2 \\
&= E\Big|\frac{1-\gamma\varrho+\gamma\varrho\mathrm{e}^{\mathrm{i}\Delta x\xi}}{1+\gamma\varrho-\gamma\varrho\mathrm{e}^{-\mathrm{i}\Delta x\xi}}\Big|^2 + E\Big|\frac{\sigma\Delta W_n}{1+\gamma\varrho-\gamma\varrho\mathrm{e}^{-\mathrm{i}\Delta x\xi}}\Big|^2 \\
&\quad + 2E\Big|\frac{1-\gamma\varrho+\gamma\varrho\mathrm{e}^{\mathrm{i}\Delta x\xi}}{1+\gamma\varrho-\gamma\varrho\mathrm{e}^{-\mathrm{i}\Delta x\xi}} \times \frac{\sigma\Delta W_n}{1+\gamma\varrho-\gamma\varrho\mathrm{e}^{-\mathrm{i}\Delta x\xi}}\Big|.
\end{aligned}$$

Because of the independence of the Wiener process, we have

$$E|g(\Delta x\xi,\Delta t,\Delta x)|^2 = \Big(\frac{1-\gamma\varrho+\gamma\varrho\mathrm{e}^{\mathrm{i}\Delta x\xi}}{1+\gamma\varrho-\gamma\varrho\mathrm{e}^{-\mathrm{i}\Delta x\xi}}\Big)^2 + \Big(\frac{\sigma}{1+\gamma\varrho-\gamma\varrho\mathrm{e}^{-\mathrm{i}\Delta x\xi}}\Big)^2\Delta t.$$

Since for every γ, ϱ and Δx we have

$$\Big|\frac{1-\gamma\varrho+\gamma\varrho\mathrm{e}^{\mathrm{i}\Delta x\xi}}{1+\gamma\varrho-\gamma\varrho\mathrm{e}^{-\mathrm{i}\Delta x\xi}}\Big| \leqslant 1, \quad \Big(\frac{\sigma}{1+\gamma\varrho-\gamma\varrho\mathrm{e}^{-\mathrm{i}\Delta x\xi}}\Big)^2 \leqslant K,$$

therefore

$$E|g(\Delta x\xi,\Delta t,\Delta x)|^2 \leqslant 1 + K\Delta t.$$

So the stochastic Saul'yev scheme is unconditionally stable. □

By a similar argument it can be proved that the stochastic right to left Saul'yev method is unconditionally stable when applied to the stochastic diffusion equations.

Theorem 2. *The stochastic Liu schemes are unconditionally stable according to the Fourier-transformation analysis for the stochastic diffusion equation.*

Proof. We give a proof for the stability condition of the L-R Liu stochastic scheme. Applying the stochastic L-R Liu scheme to the equation (2), we have

$$\begin{aligned}
&(2+3\gamma\varrho)\hat{u}^{n+1} + (\gamma\varrho)\mathrm{e}^{-2\mathrm{i}\Delta x\xi}\hat{u}^{n+1} - 4(\gamma\varrho)\mathrm{e}^{-\mathrm{i}\Delta x\xi}\hat{u}^{n+1} \\
&\qquad = 2(1-2\gamma\varrho)\hat{u}^n + \gamma\varrho(\mathrm{e}^{-\mathrm{i}\Delta x\xi} + 3\mathrm{e}^{\mathrm{i}\Delta x\xi})\hat{u}^n + \sigma\hat{u}^n(\xi)\Delta W_n.
\end{aligned}$$

Therefore,

$$\begin{aligned}&[(2+3\gamma\varrho)+(\gamma\varrho)\mathrm{e}^{-2\mathrm{i}\Delta x\xi}-4\gamma\varrho\mathrm{e}^{-\mathrm{i}\Delta x\xi}]\hat{u}^{n+1}\\&\qquad=[2(1-2\gamma\varrho)+\gamma\varrho\mathrm{e}^{-\mathrm{i}\Delta x\xi}+3(\gamma\varrho)\mathrm{e}^{\mathrm{i}\Delta x\xi}]\hat{u}^{n}+\sigma\hat{u}^{n}(\xi)\Delta W_n,\end{aligned}$$

and

$$\begin{aligned}\hat{u}^{n+1}(\xi)=\Big\{&\frac{2(1-2\gamma\varrho)+\gamma\varrho\mathrm{e}^{-\mathrm{i}\Delta x\xi}+3\gamma\varrho\mathrm{e}^{\mathrm{i}\Delta x\xi}}{(2+3\gamma\varrho)+\gamma\varrho\mathrm{e}^{-2\mathrm{i}\Delta x\xi}-4\gamma\varrho\mathrm{e}^{-\mathrm{i}\Delta x\xi}}\\&+\frac{\sigma\Delta W_n}{(2+3\gamma\varrho)+\gamma\varrho\mathrm{e}^{-2\mathrm{i}\Delta x\xi}-4\gamma\varrho\mathrm{e}^{-\mathrm{i}\Delta x\xi}}\Big\}\hat{u}^{n}(\xi).\end{aligned}$$

Therefore, the amplification factor of the stochastic L-R Liu scheme is

$$\begin{aligned}g(\Delta x\xi,\Delta t,\Delta x)=&\frac{2(1-2\gamma\varrho)+\gamma\varrho\mathrm{e}^{-\mathrm{i}\Delta x\xi}+3\gamma\varrho\mathrm{e}^{\mathrm{i}\Delta x\xi}}{(2+3\gamma\varrho)+\gamma\varrho\mathrm{e}^{-2\mathrm{i}\Delta x\xi}-4\gamma\varrho\mathrm{e}^{-\mathrm{i}\Delta x\xi}}\\&+\frac{\sigma\Delta W_n}{(2+3\gamma\varrho)+\gamma\varrho\mathrm{e}^{-2\mathrm{i}\Delta x\xi}-4\gamma\varrho\mathrm{e}^{-\mathrm{i}\Delta x\xi}}.\end{aligned}$$

Applying $E|\cdot|^2$ to the amplification factor, we have

$$\begin{aligned}E|g(\Delta x\xi,\Delta t,\Delta x)|^2=E\Big|&\frac{2(1-2\gamma\varrho)+\gamma\varrho\mathrm{e}^{-\mathrm{i}\Delta x\xi}+3\gamma\varrho\mathrm{e}^{\mathrm{i}\Delta x\xi}}{(2+3\gamma\varrho)+\gamma\varrho\mathrm{e}^{-2\mathrm{i}\Delta x\xi}-4\gamma\varrho\mathrm{e}^{-\mathrm{i}\Delta x\xi}}\\&+\frac{\sigma\Delta W_n}{(2+3\gamma\varrho)+\gamma\varrho\mathrm{e}^{-2\mathrm{i}\Delta x\xi}-4\gamma\varrho\mathrm{e}^{-\mathrm{i}\Delta x\xi}}\Big|^2.\end{aligned}$$

So,we get

$$\begin{aligned}E|g(\Delta x\xi,\Delta t,\Delta x)|^2=&E\Big|\frac{2(1-2\gamma\varrho)+\gamma\varrho\mathrm{e}^{-\mathrm{i}\Delta x\xi}+3\gamma\varrho\mathrm{e}^{\mathrm{i}\Delta x\xi}}{(2+3\gamma\varrho)+\gamma\varrho\mathrm{e}^{-2\mathrm{i}\Delta x\xi}-4\gamma\varrho\mathrm{e}^{-\mathrm{i}\Delta x\xi}}\Big|^2\\&+E\Big|\frac{\sigma\Delta W_n}{(2+3\gamma\varrho)+\gamma\varrho\mathrm{e}^{-2\mathrm{i}\Delta x\xi}-4\gamma\varrho\mathrm{e}^{-\mathrm{i}\Delta x\xi}}\Big|^2\\&+2E\Big|\frac{2(1-2\gamma\varrho)+\gamma\varrho\mathrm{e}^{-\mathrm{i}\Delta x\xi}+3\gamma\varrho\mathrm{e}^{\mathrm{i}\Delta x\xi}}{(2+3\gamma\varrho)+\gamma\varrho\mathrm{e}^{-2\mathrm{i}\Delta x\xi}-4\gamma\varrho\mathrm{e}^{-\mathrm{i}\Delta x\xi}}\\&\quad\times\frac{\sigma\Delta W_n}{(2+3\gamma\varrho)+\gamma\varrho\mathrm{e}^{-2\mathrm{i}\Delta x\xi}-4\gamma\varrho\mathrm{e}^{-\mathrm{i}\Delta x\xi}}\Big|.\end{aligned}$$

Since the increments of a Wiener process are independent, we have

$$\begin{aligned}E|g(\Delta x\xi,\Delta t,\Delta x)|^2=&\Big(\frac{2(1-2\gamma\varrho)+\gamma\varrho\mathrm{e}^{-\mathrm{i}\Delta x\xi}+3\gamma\varrho\mathrm{e}^{\mathrm{i}\Delta x\xi}}{(2+3\gamma\varrho)+\gamma\varrho\mathrm{e}^{-2\mathrm{i}\Delta x\xi}-4\gamma\varrho\mathrm{e}^{-\mathrm{i}\Delta x\xi}}\Big)^2\\&+\Big(\frac{\sigma}{(2+3\gamma\varrho)+\gamma\varrho\mathrm{e}^{-2\mathrm{i}\Delta x\xi}-4\gamma\varrho\mathrm{e}^{-\mathrm{i}\Delta x\xi}}\Big)^2\Delta t.\end{aligned}$$

For every γ, ϱ and Δx,

$$\Big|\frac{2(1-2\gamma\varrho)+\gamma\varrho\mathrm{e}^{-\mathrm{i}\Delta x\xi}+3\gamma\varrho\mathrm{e}^{\mathrm{i}\Delta x\xi}}{(2+3\gamma\varrho)+\gamma\varrho\mathrm{e}^{-2\mathrm{i}\Delta x\xi}-4\gamma\varrho\mathrm{e}^{-\mathrm{i}\Delta x\xi}}\Big| \leqslant 1,$$

$$\Big|\frac{\sigma}{(2+3\gamma\varrho)+\gamma\varrho\mathrm{e}^{-2\mathrm{i}\Delta x\xi}-4\gamma\varrho\mathrm{e}^{-\mathrm{i}\Delta x\xi}}\Big|^2 \leqslant K.$$

So,

$$E|g(\Delta x\xi, \Delta t, \Delta x)|^2 \leqslant 1 + K\Delta t.$$

Therefore, the stochastic L-R Liu scheme is unconditionally stable for the stochastic diffusion equation. It can be proved that the stochastic R-L Liu method is also unconditionally stable for approximating the solutions of SPDE (2). □

5.2. Stability of stochastic Saul'yev/Robert and Weiss schemes for stochastic advection-diffusion equation.

Theorem 3. *The L-R stochastic Saul'yev/Robert and Weiss scheme is unconditionally stable according to the Fourier-transformation analysis for the stochastic advection-diffusion equation.*

P r o o f. Substituting the Fourier transformation of u^{n+1} in the L-R stochastic Saul'yev/Robert and Weiss scheme, we get

$$\Big(1+\gamma\varrho+\frac{\lambda\nu}{2}\Big)\hat{u}^{n+1}(\xi) = \Big(1-\gamma\varrho+\frac{\lambda\nu}{2}\Big)\hat{u}^{n}(\xi) + \Big(\gamma\varrho+\frac{\lambda\nu}{2}\Big)\mathrm{e}^{-\mathrm{i}\Delta x\xi}\hat{u}^{n+1}(\xi) + \Big(\gamma\varrho-\frac{\lambda\nu}{2}\Big)\mathrm{e}^{\mathrm{i}\Delta x\xi}\hat{u}^{n}(\xi) + \sigma\hat{u}^{n}(\xi)\Delta W_n.$$

Consequently, we have

$$\Big[\Big(1+\gamma\varrho+\frac{\lambda\nu}{2}\Big) - \Big(\gamma\varrho+\frac{\lambda\nu}{2}\Big)\mathrm{e}^{-\mathrm{i}\Delta x\xi}\Big]\hat{u}^{n+1}(\xi) = \Big[\Big(1-\gamma\varrho+\frac{\lambda\nu}{2}\Big) + \Big(\gamma\varrho-\frac{\lambda\nu}{2}\Big)\mathrm{e}^{\mathrm{i}\Delta x\xi}\Big]\hat{u}^{n}(\xi) + \sigma\hat{u}^{n}(\xi)\Delta W_n,$$

and

$$\hat{u}^{n+1}(\xi) = \Big\{\frac{(1-\gamma\varrho+\lambda\nu/2)+(\gamma\varrho-\lambda\nu/2)\mathrm{e}^{\mathrm{i}\Delta x\xi}}{(1+\gamma\varrho+\lambda\nu/2)-(\gamma\varrho+\lambda\nu/2)\mathrm{e}^{-\mathrm{i}\Delta x\xi}} + \frac{\sigma\Delta W_n}{(1+\gamma\varrho+\lambda\nu/2)-(\gamma\varrho+\lambda\nu/2)\mathrm{e}^{-\mathrm{i}\Delta x\xi}}\Big\}\hat{u}^{n}(\xi).$$

Therefore, the amplification factor of the L-R stochastic Saul'yev/Robert and Weiss scheme is

$$g(\Delta x\xi, \Delta t, \Delta x) = \Big\{\frac{(1-\gamma\varrho+\lambda\nu/2)+(\gamma\varrho-\lambda\nu/2)\mathrm{e}^{\mathrm{i}\Delta x\xi}}{(1+\gamma\varrho+\lambda\nu/2)-(\gamma\varrho+\lambda\nu/2)\mathrm{e}^{-\mathrm{i}\Delta x\xi}} + \frac{\sigma\Delta W_n}{(1+\gamma\varrho+\lambda\nu/2)-(\gamma\varrho+\lambda\nu/2)\mathrm{e}^{-\mathrm{i}\Delta x\xi}}\Big\}.$$

Further,

$$\begin{aligned} E|g(\Delta x\xi, \Delta t, \Delta x)|^2 &= E\Big|\frac{(1-\gamma\varrho+\lambda\nu/2)+(\gamma\varrho-\lambda\nu/2)\mathrm{e}^{\mathrm{i}\Delta x\xi}}{(1+\gamma\varrho+\lambda\nu/2)-(\gamma\varrho+\lambda\nu/2)\mathrm{e}^{-\mathrm{i}\Delta x\xi}} \\ &\quad + \frac{\sigma\Delta W_n}{(1+\gamma\varrho+\lambda\nu/2)-(\gamma\varrho+\lambda\nu/2)\mathrm{e}^{-\mathrm{i}\Delta x\xi}}\Big|^2 \\ &= E\Big|\frac{(1-\gamma\varrho+\lambda\nu/2)+(\gamma\varrho-\lambda\nu/2)\mathrm{e}^{\mathrm{i}\Delta x\xi}}{(1+\gamma\varrho+\lambda\nu/2)-(\gamma\varrho+\lambda\nu/2)\mathrm{e}^{-\mathrm{i}\Delta x\xi}}\Big|^2 \\ &\quad + E\Big|\frac{\sigma\Delta W_n}{(1+\gamma\varrho+\lambda\nu/2)-(\gamma\varrho+\lambda\nu/2)\mathrm{e}^{-\mathrm{i}\Delta x\xi}}\Big|^2 \\ &= \Big[\frac{(1-\gamma\varrho+\lambda\nu/2)+(\gamma\varrho-\lambda\nu/2)\mathrm{e}^{\mathrm{i}\Delta x\xi}}{(1+\gamma\varrho+\lambda\nu/2)-(\gamma\varrho+\lambda\nu/2)\mathrm{e}^{-\mathrm{i}\Delta x\xi}}\Big]^2 \\ &\quad + \frac{\sigma^2\Delta t}{[(1+\gamma\varrho+\lambda\nu/2)-(\gamma\varrho+\lambda\nu/2)\mathrm{e}^{-\mathrm{i}\Delta x\xi}]^2}. \end{aligned}$$

The condition

$$\Big|\frac{(1-\gamma\varrho+\lambda\nu/2)+(\gamma\varrho-\lambda\nu/2)\mathrm{e}^{\mathrm{i}\Delta x\xi}}{(1+\gamma\varrho+\lambda\nu/2)-(\gamma\varrho+\lambda\nu/2)\mathrm{e}^{-\mathrm{i}\Delta x\xi}}\Big|^2 \leqslant 1$$

is satisfied if and only if

$$\Big|1+\gamma\varrho(\mathrm{e}^{\mathrm{i}\Delta x\xi}-1)+\frac{\lambda\nu}{2}(1-\mathrm{e}^{-\mathrm{i}\Delta x\xi})\Big| \leqslant \Big|1+\gamma\varrho(1-\mathrm{e}^{-\mathrm{i}\Delta x\xi})+\frac{\lambda\nu}{2}(1-\mathrm{e}^{-\mathrm{i}\Delta x\xi})\Big|,$$

and we have

$$\begin{aligned} -1-\gamma\varrho(1-\mathrm{e}^{-\mathrm{i}\Delta x\xi})-\frac{\lambda\nu}{2}(1-\mathrm{e}^{-\mathrm{i}\Delta x\xi}) &\leqslant 1+\gamma\varrho(\mathrm{e}^{\mathrm{i}\Delta x\xi}-1)+\frac{\lambda\nu}{2}(1-\mathrm{e}^{-\mathrm{i}\Delta x\xi}) \\ &\leqslant 1+\gamma\varrho(1-\mathrm{e}^{-\mathrm{i}\Delta x\xi})+\frac{\lambda\nu}{2}(1-\mathrm{e}^{-\mathrm{i}\Delta x\xi}). \end{aligned}$$

The first inequality does not impose any condition on Δt and Δx, and for the second inequality we get

$$\gamma\varrho(\mathrm{e}^{\mathrm{i}\Delta x\xi}-1)-\gamma\varrho(1-\mathrm{e}^{-\mathrm{i}\Delta x\xi}) \leqslant \frac{\lambda\nu}{2}(1-\mathrm{e}^{-\mathrm{i}\Delta x\xi})-\frac{\lambda\nu}{2}(1-\mathrm{e}^{-\mathrm{i}\Delta x\xi})$$

which means that

$$-\gamma\varrho[2 - 2\cos\Delta x\xi] \leqslant 0.$$

This inequality is satisfied for all Δx, also we have

$$\frac{1}{[(1+\gamma\varrho+\lambda\nu/2) - (\gamma\varrho+\lambda\nu/2)\mathrm{e}^{-\mathrm{i}\Delta x\xi}]^2} \leqslant 1.$$

Therefore,

$$E|g(\Delta x\xi, \Delta t, \Delta x)|^2 \leqslant 1 + \sigma^2\Delta t.$$

The L-R stochastic Saul'yev/Robert and Weiss scheme is unconditionally stable for solving SPDE (2), with $K = \sigma^2$. □

Theorem 4. *The R-L stochastic Saul'yev/Robert and Weiss scheme with* $\lambda\nu < \gamma\varrho + \lambda\nu/2 < 1$ *is stable in mean square with respect to the* $|\cdot|_\infty = \sqrt{\sup_k |\cdot|^2}$*-norm for the stochastic advection-diffusion equation.*

Proof. Applying $E|\cdot|^2$ to the R-L stochastic Saul'yev/Robert and Weiss scheme, we get

$$\begin{aligned} E\Big|\Big(1+\gamma\varrho-\frac{\lambda\nu}{2}\Big)u_k^{n+1} - \Big(\gamma\varrho-\frac{\lambda\nu}{2}\Big)u_{k+1}^{n+1}\Big|^2 \\ = E\Big|\Big(1-\gamma\varrho-\frac{\lambda\nu}{2}\Big)u_k^n + \Big(\gamma\varrho+\frac{\lambda\nu}{2}\Big)u_{k-1}^n + \sigma u_k^n\Delta W_n\Big|^2. \end{aligned}$$

Using the assumption $\gamma\varrho + \lambda\nu/2 \leqslant 1$ and the independence of the Wiener process increments, we have

$$\begin{aligned} & E\Big|\Big(1+\gamma\varrho-\frac{\lambda\nu}{2}\Big)u_k^{n+1} - \Big(\gamma\varrho-\frac{\lambda\nu}{2}\Big)u_{k+1}^{n+1}\Big|^2 \\ & \quad = E\Big|\Big(1-\gamma\varrho-\frac{\lambda\nu}{2}\Big)u_k^n + \Big(\gamma\varrho+\frac{\lambda\nu}{2}\Big)u_{k-1}^n\Big|^2 + \sigma^2\Delta t E|u_k^n|^2 \\ & \quad \leqslant \Big(\Big|1-\gamma\varrho-\frac{\lambda\nu}{2}\Big| + \Big|\gamma\varrho+\frac{\lambda\nu}{2}\Big|\Big)^2 \sup_k E|u_k^n|^2 + \sigma^2\Delta t \sup_k E|u_k^n|^2 \\ & \quad \leqslant (1+\sigma^2\Delta t)\sup_k E|u_k^n|^2. \end{aligned}$$

Since this holds for all k, we have

$$\sup_k E\Big|\Big(1+\gamma\varrho-\frac{\lambda\nu}{2}\Big)u_k^{n+1} - \Big(\gamma\varrho-\frac{\lambda\nu}{2}\Big)u_{k+1}^{n+1}\Big|^2 \leqslant (1+\sigma^2\Delta t)\sup_k E|u_k^n|^2.$$

Assuming $1+\gamma\varrho-\lambda\nu/2\geqslant 0$ and $\gamma\varrho-\lambda\nu/2\geqslant 0$, we get

$$\begin{aligned}\sup_k E\Big|\Big(1+\gamma\varrho-\frac{\lambda\nu}{2}\Big)u_k^{n+1}-\Big(\gamma\varrho-\frac{\lambda\nu}{2}\Big)u_{k+1}^{n+1}\Big|^2&\\ \geqslant\Big(\Big|1+\gamma\varrho-\frac{\lambda\nu}{2}\Big|-\Big|\gamma\varrho-\frac{\lambda\nu}{2}\Big|\Big)\sup_k E|u_k^{n+1}|^2&\\ \geqslant\sup_k E|u_k^{n+1}|^2.&\end{aligned}$$

So we have

$$\begin{aligned}\|u^{n+1}\|_\infty^2&\leqslant(1+\sigma^2\Delta t)\|u^n\|_\infty^2,\\ \|u^{n+1}\|_\infty^2&\leqslant\Big(1+\sigma^2\frac{t}{n+1}\Big)^{n+1}\|u^0\|_\infty^2,\\ \|u^{n+1}\|_\infty&\leqslant\Big(1+\frac{\sigma^2 t}{n+1}\Big)^{(n+1)/2}\|u^0\|_\infty,\\ \|u^{n+1}\|_\infty&\leqslant \mathrm{e}^{\sigma^2 t/2}\|u^0\|_\infty.\end{aligned}$$

So the R-L stochastic Saul'yev/Robert and Weiss scheme is stable for

$$\lambda\nu<\gamma\varrho+\frac{\lambda\nu}{2}<1.$$

$\square$

6. Consistency conditions of stochastic ADE schemes

Theorem 5. *The stochastic L-R Liu scheme is consistent in mean square for approximating the solution of stochastic diffusion equation* (2).

Proof. If $\varphi(x,t)$ is a smooth function, then we have

$$\begin{aligned}L(\Phi)|_k^n&=\Phi(k\Delta x,(n+1)\Delta t)-\Phi(k\Delta x,n\Delta t)\\ &\quad-\gamma\int_{n\Delta t}^{(n+1)\Delta t}\Phi_{xx}(k\Delta x,s)\,\mathrm{d}s-\sigma\int_{n\Delta t}^{(n+1)\Delta t}\Phi(k\Delta x,s)\,\mathrm{d}W(s)\end{aligned}$$

and

$$\begin{aligned}L_k^n(\Phi)&=\Phi(k\Delta x,(n+1)\Delta t)-\Phi(k\Delta x,n\Delta t)\\ &\quad-\gamma\frac{\Delta t}{2\Delta x^2}[\Phi((k-1)\Delta x,n\Delta t)+3\Phi((k+1)\Delta x,n\Delta t)\\ &\quad-\Phi((k-2)\Delta x,(n+1)\Delta t)+4\Phi((k-1)\Delta x,(n+1)\Delta t)\\ &\quad-3\Phi(k\Delta x,(n+1)\Delta t)-4\Phi(k\Delta x,n\Delta t)]-\sigma\Phi(k\Delta x,n\Delta t)\Delta W_n.\end{aligned}$$

Therefore, we get in mean square

$$
\begin{aligned}
E|L(\Phi)|_k^n - L_k^n(\Phi)|^2 \leqslant 2\gamma^2 E\bigg| \int_{n\Delta t}^{(n+1)\Delta t} & \Phi_{xx}(k\Delta x, s) \\
& - \frac{1}{2\Delta x^2}[\Phi((k-1)\Delta x, n\Delta t) + 3\Phi((k+1)\Delta x, n\Delta t) \\
& - \Phi((k-2)\Delta x, (n+1)\Delta t) + 4\Phi((k-1)\Delta x, (n+1)\Delta t) \\
& - 3\Phi(k\Delta x, (n+1)\Delta t) - 4\Phi(k\Delta x, n\Delta t)]\, \mathrm{d}s\bigg|^2 \\
& + 2\sigma^2 E\bigg| \int_{n\Delta t}^{(n+1)\Delta t} (\Phi(k\Delta x, s) - \Phi(k\Delta x, n\Delta t))\, \mathrm{d}W(s)\bigg|^2,
\end{aligned}
$$

$$
\begin{aligned}
E|L(\Phi)|_k^n - L_k^n(\Phi)|^2 \leqslant 2\gamma^2 E\bigg| \int_{n\Delta t}^{(n+1)\Delta t} & \Phi_{xx}(k\Delta x, s) \\
& - \frac{1}{2\Delta x^2}[\Phi((k-1)\Delta x, n\Delta t) + 3\Phi((k+1)\Delta x, n\Delta t) \\
& - \Phi((k-2)\Delta x, (n+1)\Delta t) + 4\Phi((k-1)\Delta x, (n+1)\Delta t) \\
& - 3\Phi(k\Delta x, (n+1)\Delta t) - 4\Phi(k\Delta x, n\Delta t)]\, \mathrm{d}s\bigg|^2 \\
& + 2\sigma^2 \int_{n\Delta t}^{(n+1)\Delta t} |\Phi(k\Delta x, s) - \Phi(k\Delta x, n\Delta t)|^2\, \mathrm{d}s;
\end{aligned}
$$

if $\Phi(x,t)$ is a deterministic function, we get

$$
E|L(\Phi)|_k^n - L_k^n(\Phi)| \to 0,
$$

when $n, k \to \infty$. This proves the consistency. So, the stochastic L-R Liu scheme is consistent in mean square. □

Theorem 6. *The L-R stochastic Saul'yev/Robert and Weiss scheme is consistent in mean square for approximating the solution of stochastic advection-diffusion equation* (1).

Proof. Assuming $\varphi(x,t)$ is a smooth function, we have

$$
\begin{aligned}
L(\Phi)|_k^n = \Phi(k\Delta x, (n+1)\Delta t) & - \Phi(k\Delta x, n\Delta t) \\
& - \nu \int_{n\Delta t}^{(n+1)\Delta t} \Phi_x(k\Delta x, s)\, \mathrm{d}s - \gamma \int_{n\Delta t}^{(n+1)\Delta t} \Phi_{xx}(k\Delta x, s)\, \mathrm{d}s \\
& - \sigma \int_{n\Delta t}^{(n+1)\Delta t} \Phi(k\Delta x, s)\, \mathrm{d}W(s)
\end{aligned}
$$

and

$$
\begin{aligned}
L_k^n(\Phi) &= \Phi(k\Delta x, (n+1)\Delta t) - \Phi(k\Delta x, n\Delta t) \\
&\quad - \nu \frac{\Delta t}{2\Delta x}[\Phi(k\Delta x, (n+1)\Delta t) + \Phi((k+1)\Delta x, n\Delta t) \\
&\quad - \Phi((k-1)\Delta x, (n+1)\Delta t) - \Phi(k\Delta x, n\Delta t)] \\
&\quad - \gamma \frac{\Delta t}{\Delta x^2}[\Phi((k+1)\Delta x, n\Delta t) - \Phi(k\Delta x, n\Delta t) \\
&\quad - \Phi(k\Delta x, (n+1)\Delta t) + \Phi((k-1)\Delta x, (n+1)\Delta t)] \\
&\quad - \sigma\Phi(k\Delta x, n\Delta t)\Delta W_n.
\end{aligned}
$$

In mean square, we obtain

$$
\begin{aligned}
E|L(\Phi)|_k^n - L_k^n(\Phi)|^2 &\leqslant 2\nu^2 E\bigg| \int_{n\Delta t}^{(n+1)\Delta t} \Phi_x(k\Delta x, s) \\
&\quad - \frac{1}{2\Delta x}[\Phi(k\Delta x, (n+1)\Delta t) + \Phi((k+1)\Delta x, n\Delta t) \\
&\quad - \Phi((k-1)\Delta x, (n+1)\Delta t) - \Phi(k\Delta x, n\Delta t)]\,\mathrm{d}s\bigg|^2 \\
&\quad + 2\gamma^2 E\bigg| \int_{n\Delta t}^{(n+1)\Delta t} \Phi_{xx}(k\Delta x, s) \\
&\quad - \frac{1}{\Delta x^2}[\Phi((k+1)\Delta x, n\Delta t) - \Phi(k\Delta x, n\Delta t) \\
&\quad - \Phi(k\Delta x, (n+1)\Delta t) + \Phi((k-1)\Delta x, (n+1)\Delta t)]\,\mathrm{d}s\bigg|^2 \\
&\quad + 2\sigma^2 \int_{n\Delta t}^{(n+1)\Delta t} |\Phi(k\Delta x, s) - \Phi(k\Delta x, n\Delta t)|^2\,\mathrm{d}s.
\end{aligned}
$$

Since $\varphi(x,t)$ is assumed to be deterministic, for $\Delta t, \Delta x \to 0$ we have

$$
E|L(\Phi)|_k^n - L_k^n(\Phi)|^2 \to 0.
$$

□

7. Convergence of stochastic ADE schemes

7.1. Convergence of stochastic Liu schemes for stochastic diffusion equations.

Theorem 7. *Let $v \in H^1, H^3$. The stochastic L-R Liu scheme is convergent for the $\|.\|_\infty$-norm for $0 \leqslant \gamma(\Delta t/\Delta x^2) =: \gamma\varrho \leqslant \frac{1}{2}$ for approximating the solution of the stochastic diffusion equation* (2).

P r o o f. The stochastic L-R Liu scheme is given by

$$(3\gamma\varrho+2)u_k^{n+1}=2(1-2\gamma\varrho)u_k^n+\gamma\varrho(u_{k-1}^n+3u_{k+1}^n-u_{k-2}^{n+1}+4u_{k-1}^{n+1})+\sigma u_k^n\Delta W_n,$$

which can be represented by

$$u_k^{n+1}=u_k^n+\frac{\gamma\varrho}{2}\{-u_{k-2}^{n+1}+4u_{k-1}^{n+1}-3u_k^{n+1}+u_{k-1}^n-4u_k^n+3u_{k+1}^n\}+\sigma u_k^n\Delta W_n.$$

Considering the Taylor expansion $v_{xx}(x,s)$ with respect to the space variable, the solution v_k^{n+1} can be written as

$$\begin{aligned}
v_k^{n+1}&=v_k^n+\gamma\int_{n\Delta t}^{(n+1)\Delta t}v_{xx}(x,s)|_{x=x_k}\,\mathrm{d}s+\sigma\int_{n\Delta t}^{(n+1)\Delta t}v(x,s)|_{x=x_k}\,\mathrm{d}W(s)\\
&=v_k^n+\gamma\int_{n\Delta t}^{(n+1)\Delta t}\frac{-v_{k-2}^{n+1}+4v_{k-1}^{n+1}-3v_k^{n+1}+v_{k-1}^n-4v_k^n+3v_{k+1}^n}{2\Delta x^2}\\
&\quad+\frac{\Delta x}{2\times 3!}((3v_{xxx}((k+\alpha_1)\Delta x,s)-v_{xxx}((k+\alpha_2)\Delta x,s)\\
&\quad-4v_{xxx}((k+\beta_1)\Delta x,s+\Delta t))+8v_{xxx}((k+\beta_2)\Delta x,s+\Delta t)\\
&\quad-\frac{\Delta t}{\Delta x^2}v_{xt}(k\Delta x,s+\eta\Delta t))\,\mathrm{d}s+\sigma\int_{n\Delta t}^{(n+1)\Delta t}v(x,s)|_{x=x_k}\,\mathrm{d}W(s),
\end{aligned}$$

where $\alpha_1,\alpha_2,\beta_1,\beta_2\in(0,1)$. Assuming $r_k^n=v_k^n-u_k^n$, we have

$$\begin{aligned}
r_k^{n+1}&=r_k^n+\gamma\int_{n\Delta t}^{(n+1)\Delta t}\biggl\{\frac{-v_{k-2}^{n+1}+4v_{k-1}^{n+1}-3v_k^{n+1}+v_{k-1}^n-4v_k^n+3v_{k+1}^n}{2\Delta x^2}\\
&\quad-\frac{-u_{k-2}^{n+1}+4u_{k-1}^{n+1}-3u_k^{n+1}+u_{k-1}^n-4u_k^n+3u_{k+1}^n}{2\Delta x^2}\\
&\quad+\frac{\Delta x}{2\times 3!}((3v_{xxx}((k+\alpha_1)\Delta x,s)-v_{xxx}((k+\alpha_2)\Delta x,s)\\
&\quad-4v_{xxx}((k+\beta_1)\Delta x,s+\Delta t))+8v_{xxx}((k+\beta_2)\Delta x,s+\Delta t)\\
&\quad-\gamma\varrho v_{xxx}(k\Delta x,s+\eta\Delta t))\biggr\}\,\mathrm{d}s\\
&\quad-\frac{\Delta x}{2\times 3!}\sigma\gamma\varrho\int_{n\Delta t}^{(n+1)\Delta t}v_x(x,s)|_{x=x_k}\,\mathrm{d}W(s)\\
&\quad+\sigma\int_{n\Delta t}^{(n+1)\Delta t}(v(x,s)|_{x=x_k}-u_k^n)\,\mathrm{d}W(s),
\end{aligned}$$

where $\varrho = \Delta t/\Delta x^2$. Consequently, we get

$$
\begin{aligned}
\Big(1 - \frac{3\gamma\varrho}{2}\Big) r_k^{n+1} + 2\gamma\varrho r_{k-1}^{n+1} - \frac{\gamma\varrho}{2} r_{k-2}^{n+1} &= r_k^n + \frac{\gamma\varrho}{2}(u_{k-1}^n - 4u_k^n + 3u_{k+1}^n) \\
&+ \frac{\Delta x}{2\times 3!}\int_{n\Delta t}^{(n+1)\Delta t} \{((3v_{xxx}((k+\alpha_1)\Delta x, s) - v_{xxx}((k+\alpha_2)\Delta x, s) \\
&- 4v_{xxx}((k+\beta_1)\Delta x, s+\Delta t)) + 8v_{xxx}((k+\beta_2)\Delta x, s+\Delta t) \\
&- \gamma\varrho v_{xxx}(k\Delta x, s+\eta\Delta t))\}\,\mathrm{d}s \\
&- \frac{\Delta x}{2\times 3!}\sigma\gamma\varrho \int_{n\Delta t}^{(n+1)\Delta t} v_x(x,s)|_{x=x_k}\,\mathrm{d}W(s) \\
&+ \sigma\int_{n\Delta t}^{(n+1)\Delta t} (v(x,s)|_{x=x_k} - u_k^n)\,\mathrm{d}W(s).
\end{aligned}
$$

Applying $E|\cdot|^2$ to this equality we obtain

$$
\begin{aligned}
E\Big|\Big(1 - \frac{3\gamma\varrho}{2}\Big) r_k^{n+1} + 2\gamma\varrho r_{k-1}^{n+1} - \frac{\gamma\varrho}{2} r_{k-2}^{n+1}\Big|^2 &\leqslant 2E\Big| r_k^n + \frac{\gamma\varrho}{2}(u_{k-1}^n - 4u_k^n + 3u_{k+1}^n) \\
&+ \frac{\Delta x}{2\times 3!}\int_{n\Delta t}^{(n+1)\Delta t} \{((3v_{xxx}((k+\alpha_1)\Delta x, s) - v_{xxx}((k+\alpha_2)\Delta x, s) \\
&- 4v_{xxx}((k+\beta_1)\Delta x, s+\Delta t)) + 8v_{xxx}((k+\beta_2)\Delta x, s+\Delta t) \\
&- \gamma\varrho v_{xxx}(k\Delta x, s+\eta\Delta t))\}\,\mathrm{d}s \\
&- \frac{\Delta x}{2\times 3!}\sigma\gamma\varrho \int_{n\Delta t}^{(n+1)\Delta t} v_x(x,s)|_{x=x_k}\,\mathrm{d}W(s)\Big|^2 \\
&+ 2\sigma^2 E\Big|\int_{n\Delta t}^{(n+1)\Delta t} (v(x,s)|_{x=x_k} - u_k^n)\,\mathrm{d}W(s)\Big|^2,
\end{aligned}
$$

which can be written as

$$
\begin{aligned}
E\Big|\Big(1 - \frac{3\gamma\varrho}{2}\Big) r_k^{n+1} + 2\gamma\varrho r_{k-1}^{n+1} - \frac{\gamma\varrho}{2} r_{k-2}^{n+1}\Big|^2 &\leqslant 2E\Big| r_k^n + \frac{\gamma\varrho}{2}(r_{k-1}^n - 4r_k^n + 3r_{k+1}^n) \\
&+ \frac{\Delta x}{2\times 3!}\int_{n\Delta t}^{(n+1)\Delta t} \{((3v_{xxx}((k+\alpha_1)\Delta x, s) - v_{xxx}((k+\alpha_2)\Delta x, s) \\
&- 4v_{xxx}((k+\beta_1)\Delta x, s+\Delta t)) + 8v_{xxx}((k+\beta_2)\Delta x, s+\Delta t) \\
&- \gamma\varrho v_{xxx}(k\Delta x, s+\eta\Delta t))\}\,\mathrm{d}s \\
&- \frac{\Delta x}{2\times 3!}\sigma\gamma\varrho \int_{n\Delta t}^{(n+1)\Delta t} v_x(x,s)|_{x=x_k}\,\mathrm{d}W(s)\Big|^2 \\
&+ 4\sigma^2 \int_{n\Delta t}^{(n+1)\Delta t} E|(v(x,s)|_{x=x_k} - v_k^n)|^2\,\mathrm{d}s \\
&+ 4\sigma^2 \int_{n\Delta t}^{(n+1)\Delta t} E|(v_k^n - u_k^n)|^2\,\mathrm{d}s.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
E\Big|\Big(1-\frac{3\gamma\varrho}{2}\Big)r_k^{n+1}+2\gamma\varrho r_{k-1}^{n+1}-\frac{\gamma\varrho}{2}r_{k-2}^{n+1}\Big|^2 &\leqslant 4E\Big|(1-2\gamma\varrho)r_k^n+\frac{\gamma\varrho}{2}(r_{k-1}^n+3r_{k+1}^n)\Big|^2\\
&+8\Big(\frac{\Delta x}{2\times 3!}\Big)^2E\Big|\int_{n\Delta t}^{(n+1)\Delta t}\{((3v_{xxx}((k+\alpha_1)\Delta x,s)\\
&-v_{xxx}((k+\alpha_2)\Delta x,s)-4v_{xxx}((k+\beta_1)\Delta x,s+\Delta t))\\
&+8v_{xxx}((k+\beta_2)\Delta x,s+\Delta t)-\gamma\varrho v_{xxx}(k\Delta x,s+\eta\Delta t))\}\,\mathrm{d}s\Big|^2\\
&+8\Big(\frac{\Delta x}{2\times 3!}\Big)^2E\Big|\sigma\gamma\varrho\int_{n\Delta t}^{(n+1)\Delta t}v_x(x,s)|_{x=x_k}\,\mathrm{d}W(s)\Big|^2\\
&+4\sigma^2\int_{n\Delta t}^{(n+1)\Delta t}E|(v(x,s)|_{x=x_k}-v_k^n)|^2\,\mathrm{d}s\\
&+4\sigma^2\Delta tE|r_k^n|^2
\end{aligned}
$$

and consequently

$$
\begin{aligned}
&E\Big|\Big(1-\frac{3\gamma\varrho}{2}\Big)r_k^{n+1}+2\gamma\varrho r_{k-1}^{n+1}-\frac{\gamma\varrho}{2}r_{k-2}^{n+1}\Big|^2\\
&\qquad\leqslant 4\Big(|1-2\gamma\varrho|+\Big|\frac{\gamma\varrho}{2}\Big|+\Big|\frac{3\gamma\varrho}{2}\Big|+\sigma^2\Delta t\Big)\sup_k E|r_k^n|^2\\
&\qquad+8\Big(\frac{\Delta x}{2\times 3!}\Big)^2\sup_k E\Big|\int_{n\Delta t}^{(n+1)\Delta t}\{((3v_{xxx}((k+\alpha_1)\Delta x,s)\\
&\qquad-v_{xxx}((k+\alpha_2)\Delta x,s)-4v_{xxx}((k+\beta_1)\Delta x,s+\Delta t))\\
&\qquad+8v_{xxx}((k+\beta_2)\Delta x,s+\Delta t)-\gamma\varrho v_{xxx}(k\Delta x,s+\eta\Delta t))\}\,\mathrm{d}s\Big|^2\\
&\qquad+8\Big(\frac{\Delta x}{2\times 3!}\Big)^2(\sigma\gamma\varrho)^2\sup_k\int_{n\Delta t}^{(n+1)\Delta t}E|v_x(x,s)|_{x=x_k}|^2\,\mathrm{d}s\\
&\qquad+4\sigma^2\sup_k\int_{n\Delta t}^{(n+1)\Delta t}E|(v(x,s)|_{x=x_k}-v_k^n)|^2\,\mathrm{d}s.
\end{aligned}
$$

Assuming $\gamma\varrho\leqslant\frac{1}{2}$, introducing the notation $\varphi_{1k}=v_{xxx}((k+\alpha_1)\Delta x,s)<\infty$, $\varphi_{2k}=v_{xxx}((k+\alpha_2)\Delta x,s)<\infty$, $\varphi_{3k}=v_{xxx}((k+\beta_1)\Delta x,s+\Delta t)<\infty$, $\varphi_{4k}=v_{xxx}((k+\beta_2)\Delta x,s+\Delta t)<\infty$, $\varphi_{5k}=v_{xxx}(k\Delta x,s+\eta\Delta t)<\infty$, $\varphi_{6k}=v_x(x,s)<\infty$ and also considering

$$
\begin{aligned}
\int_{n\Delta t}^{(n+1)\Delta t}E|(v(x,s)|_{x=x_k}-v_k^n)|^2\,\mathrm{d}s&=E\int_{n\Delta t}^{(n+1)\Delta t}|(v(x,s)|_{x=x_k}-v_k^n)|^2\,\mathrm{d}s\\
&\leqslant\sup_{s\in[n\Delta t,(n+1)\Delta t]}|v(x,s)|_{x=x_k}-v(k\Delta x,n\Delta t)|^2\Delta t\leqslant\varphi'\Delta t,
\end{aligned}
$$

we get

$$\begin{aligned}
\sup_k E\Big|\Big(1-\frac{3\gamma\varrho}{2}\Big)r_k^{n+1}+2\gamma\varrho r_{k-1}^{n+1}-\frac{\gamma\varrho}{2}r_{k-2}^{n+1}\Big|^2& \\
\leqslant 4(1+\sigma^2\Delta t)\sup_k E|r_k^n|^2& \\
+8\Big(\frac{\Delta x}{2\times 3!}\Big)^2\sup_k E\Big|\int_{n\Delta t}^{(n+1)\Delta t}\{3\varphi_{1k}-\varphi_{2k}-4\varphi_{3k}+8\varphi_{4k}-\gamma\varrho\varphi_{5k}\}\,\mathrm{d}s\Big|^2& \\
+8\Big(\frac{\Delta x}{2\times 3!}\Big)^2(\sigma\gamma\varrho)^2\sup_k\int_{n\Delta t}^{(n+1)\Delta t}E|\varphi_{6k}|^2\,\mathrm{d}s+4\sigma^2\varphi'\Delta t.&
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\Big(\Big|1-\frac{3\gamma\varrho}{2}\Big|+|2\gamma\varrho|-\Big|\frac{\gamma\varrho}{2}\Big|\Big)\sup_k E|r_k^{n+1}|^2\leqslant 4(1+\sigma^2\Delta t)\sup_k E|r_k^n|^2& \\
+8\Big(\frac{\Delta x}{2\times 3!}\Big)^2\sup_k E|\Phi_1\Delta t|^2+8\Big(\frac{\Delta x}{2\times 3!}\Big)^2(\sigma\gamma\varrho)^2\sup_k E|\Phi_2|^2\Delta t+\Phi_3\Delta t,&
\end{aligned}$$

and

$$\sup_k E|r_k^{n+1}|^2\leqslant 4(1+\sigma^2\Delta t)\sup_k E|r_k^n|^2+\Phi\Delta t.$$

It follows that

$$\begin{aligned}
E\|r^{n+1}\|_\infty^2&\leqslant 4\{1+\sigma^2\Delta t\}E\|r^n\|_\infty^2+\Phi\Delta t\\
&\leqslant\Big\{1+\sigma^2\frac{t}{n+1}\Big\}^{n+1}\sum_{i=1}^n(4\Phi\Delta t)^i+\Phi\Delta t\\
&\leqslant \mathrm{e}^{\sigma^2 t}\sum_{i=1}^n(4\Phi\Delta t)^i+\Phi\Delta t.
\end{aligned}$$

When the time step Δt tends to zero, we have

$$\begin{aligned}
E\|r^{n+1}\|_\infty^2&\leqslant (n-1)\mathrm{e}^{\sigma^2 t}(4\Phi\Delta t)^2+4\mathrm{e}^{\sigma^2 t}\Phi\Delta t+\Phi\Delta t\\
&\leqslant t\mathrm{e}^{\sigma^2 t}(4\Phi)^2\Delta t+4\mathrm{e}^{\sigma^2 t}\Phi\Delta t+\Phi\Delta t\\
&=(t\mathrm{e}^{\sigma^2 t}(4\Phi)^2+4\mathrm{e}^{\sigma^2 t}\Phi+\Phi)\Delta t,
\end{aligned}$$

and consequently

$$E\|r^{n+1}\|_\infty^2\to 0.$$

□

R e m a r k 2. If $v \in H^1, H^3$, the stochastic R-L Liu scheme is convergent for the $\|.\|_\infty$-norm for $\gamma(\Delta t/\Delta x^2) =: \gamma\varrho \geqslant \frac{1}{2}$ for approximating the solution of the stochastic diffusion equation (2).

7.2. Convergence of stochastic Saul'yev/Robert and Weiss schemes for stochastic advection-diffusion equations.

Theorem 8. *Let $v \in H^1, H^2, H^3, H^4$. The stochastic L-R Saul'yev/Robert and Weiss scheme is convergent for the $\|.\|_\infty$-norm for $\nu\lambda/2 \leqslant \gamma\varrho \leqslant 1 + \nu\lambda/2$ for approximating the solution of the stochastic advection-diffusion equation* (1).

P r o o f. The L-R stochastic Saul'yev/Robert and Weiss scheme is given by

$$\Big(1 + \gamma\varrho + \frac{\nu\lambda}{2}\Big)u_k^{n+1} = \Big(1 - \gamma\varrho + \frac{\nu\lambda}{2}\Big)u_k^n + \Big(\gamma\varrho + \frac{\nu\lambda}{2}\Big)u_{k-1}^{n+1} + \Big(\gamma\varrho - \frac{\nu\lambda}{2}\Big)u_{k+1}^n + \sigma u_k^n \Delta W_n.$$

Considering the solution v_k^n, and using the Taylor expansion of $v_x(x,s)$ and $v_{xx}(x,s)$ with respect to the space expansion, we get

$$\begin{aligned}
v_k^{n+1} &= v_k^n - \nu \int_{n\Delta t}^{(n+1)\Delta t} v_x(x,s)|_{x=x_k}\,\mathrm{d}s \\
&\quad + \gamma \int_{n\Delta t}^{(n+1)\Delta t} v_{xx}(x,s)|_{x=x_k}\,\mathrm{d}s \\
&\quad + \sigma \int_{n\Delta t}^{(n+1)\Delta t} v(x,s)|_{x=x_k}\,\mathrm{d}W(s) \\
&= v_k^n - \nu \int_{n\Delta t}^{(n+1)\Delta t} \Big\{ \frac{v_k^{n+1} + v_{k+1}^n - v_{k-1}^{n+1} - v_k^n}{2\Delta x} \\
&\quad - \frac{\Delta x}{4}(v_{xx}((k+\alpha_1)\Delta x, s) - v_{xx}((k+\alpha_2)\Delta x, s+\Delta t))\Big\}\,\mathrm{d}s \\
&\quad + \gamma \int_{n\Delta t}^{(n+1)\Delta t} \Big\{ \frac{v_{k-1}^{n+1} - v_k^n + v_{k+1}^n - v_k^{n+1}}{\Delta x^2} \\
&\quad - \frac{\Delta x}{3!}(v_{xxx}((k+\beta_1)\Delta x, s) - v_{xxx}((k+\beta_2)\Delta x, s+\Delta t)) \\
&\quad - \frac{\Delta t}{\Delta x} v_{xt}(k\Delta x, s+\delta_1\Delta t) + \frac{\Delta t}{2} v_{xxt}(k\Delta x, s+\delta_2\Delta t)\Big\}\,\mathrm{d}s \\
&\quad + \sigma \int_{n\Delta t}^{(n+1)\Delta t} v(x,s)|_{x=x_k}\,\mathrm{d}W(s),
\end{aligned}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2, \delta_1, \delta_2 \in (0,1)$. Therefore, we have

$$
\begin{aligned}
v_k^{n+1} = v_k^n - \nu \int_{n\Delta t}^{(n+1)\Delta t} & \frac{v_k^{n+1} + v_{k+1}^n - v_{k-1}^{n+1} - v_k^n}{2\Delta x} \\
& - \frac{\Delta x}{4}(v_{xx}((k+\alpha_1)\Delta x, s) - v_{xx}((k+\alpha_2)\Delta x, s+\Delta t))\,\mathrm{d}s \\
& + \gamma \int_{n\Delta t}^{(n+1)\Delta t} \Big\{ \frac{v_{k-1}^{n+1} - v_k^n + v_{k+1}^n - v_k^{n+1}}{\Delta x^2} \\
& - \frac{\Delta x}{3!}(v_{xxx}((k+\beta_1)\Delta x, s) - v_{xxx}((k+\beta_2)\Delta x, s+\Delta t)) \\
& + \nu \frac{\Delta t}{\Delta x} v_{xx}(k\Delta x, s+\delta_1\Delta t) - \gamma \frac{\Delta t}{\Delta x} v_{xxx}(k\Delta x, s+\delta_1 \Delta t) \\
& - \nu \frac{\Delta t}{2} v_{xxx}(k\Delta x, s+\delta_2\Delta t) + \gamma \frac{\Delta t}{2} v_{xxxx}(k\Delta x, s+\delta_2\Delta t) \Big\}\,\mathrm{d}s \\
& - \gamma\sigma \frac{\Delta t}{\Delta x} \int_{n\Delta t}^{(n+1)\Delta t} v_x(x,s)|_{x=x_k}\,\mathrm{d}W(s) \\
& - \gamma\sigma \frac{\Delta t}{2} \int_{n\Delta t}^{(n+1)\Delta t} v_{xx}(x,s)|_{x=x_k}\,\mathrm{d}W(s) \\
& + \sigma \int_{n\Delta t}^{(n+1)\Delta t} v(x,s)|_{x=x_k}\,\mathrm{d}W(s).
\end{aligned}
$$

Assuming $r_k^n = v_k^n - u_k^n$, we get

$$
\begin{aligned}
r_k^{n+1} = r_k^n - \frac{\nu\lambda}{2}(r_k^{n+1} + r_{k+1}^n - r_{k-1}^{n+1} - r_k^n) + \gamma\varrho(r_{k-1}^{n+1} - r_k^n + r_{k+1}^n - r_k^{n+1}) \\
+ \nu \int_{n\Delta t}^{(n+1)\Delta t} \frac{\Delta x}{4}(v_{xx}((k+\alpha_1)\Delta x, s) - v_{xx}((k+\alpha_2)\Delta x, s+\Delta t))\,\mathrm{d}s \\
+ \int_{n\Delta t}^{(n+1)\Delta t} \Big\{ - \gamma \frac{\Delta x}{3!}(v_{xxx}((k+\beta_1)\Delta x, s) - v_{xxx}((k+\beta_2)\Delta x, s+\Delta t)) \\
+ \frac{\Delta t}{\Delta x} v_{xxx}(k\Delta x, s+\delta_1\Delta t) - \nu \frac{\Delta t}{2} v_{xxx}(k\Delta x, s+\delta_2\Delta t) \Big\}\,\mathrm{d}s \\
+ \nu \int_{n\Delta t}^{(n+1)\Delta t} \frac{\Delta t}{\Delta x} v_{xx}(k\Delta x, s+\delta_1\Delta t)\,\mathrm{d}s \\
+ \int_{n\Delta t}^{(n+1)\Delta t} \gamma \frac{\Delta t}{2} v_{xxxx}(k\Delta x, s+\delta_2\Delta t)\,\mathrm{d}s \\
- \gamma\sigma \frac{\Delta t}{\Delta x} \int_{n\Delta t}^{(n+1)\Delta t} v_x(x,s)|_{x=x_k}\,\mathrm{d}W(s) \\
- \gamma\sigma \frac{\Delta t}{2} \int_{n\Delta t}^{(n+1)\Delta t} v_{xx}(x,s)|_{x=x_k}\,\mathrm{d}W(s) \\
+ \sigma \int_{n\Delta t}^{(n+1)\Delta t} (v(x,s)|_{x=x_k} - u_k^n)\,\mathrm{d}W(s)
\end{aligned}
$$

where $\varrho = \Delta t/\Delta x^2$ and $\lambda = \Delta t/\Delta x$. Consequently,

$$\begin{aligned}
&\Big(1+\frac{\nu\lambda}{2}+\gamma\varrho\Big)r_k^{n+1} - \Big(\frac{\nu\lambda}{2}+\gamma\varrho\Big)r_{k-1}^{n+1} = \Big(1+\frac{\nu\lambda}{2}-\gamma\varrho\Big)r_k^n + \Big(\gamma\varrho - \frac{\nu\lambda}{2}\Big)r_{k+1}^n \\
&\quad + \nu\int_{n\Delta t}^{(n+1)\Delta t} \Big\{\frac{\Delta x}{4}(v_{xx}((k+\alpha_1)\Delta x, s) - v_{xx}((k+\alpha_2)\Delta x, s+\Delta t))\Big\}\,\mathrm{d}s \\
&\quad + \int_{n\Delta t}^{(n+1)\Delta t} \Big\{-\gamma\frac{\Delta x}{3!}(v_{xxx}((k+\beta_1)\Delta x, s) - v_{xxx}((k+\beta_2)\Delta x, s+\Delta t)) \\
&\quad + \frac{\Delta t}{\Delta x}v_{xxx}(k\Delta x, s+\delta_1\Delta t) - \nu\frac{\Delta t}{2}v_{xxx}(k\Delta x, s+\delta_2\Delta t)\Big\}\,\mathrm{d}s \\
&\quad + \nu\int_{n\Delta t}^{(n+1)\Delta t} \frac{\Delta t}{\Delta x}v_{xx}(k\Delta x, s+\delta_1\Delta t)\,\mathrm{d}s + \int_{n\Delta t}^{(n+1)\Delta t} \gamma\frac{\Delta t}{2}v_{xxxx}(k\Delta x, s+\delta_2\Delta t)\,\mathrm{d}s \\
&\quad - \gamma\sigma\frac{\Delta t}{\Delta x}\int_{n\Delta t}^{(n+1)\Delta t} v_x(x,s)|_{x=x_k}\,\mathrm{d}W(s) - \gamma\sigma\frac{\Delta t}{2}\int_{n\Delta t}^{(n+1)\Delta t} v_{xx}(x,s)|_{x=x_k}\,\mathrm{d}W(s) \\
&\quad + \sigma\int_{n\Delta t}^{(n+1)\Delta t} (v(x,s)|_{x=x_k} - u_k^n)\,\mathrm{d}W(s),
\end{aligned}$$

and applying $E|\cdot|^2$ to this equality, we obtain

$$\begin{aligned}
&E\Big|\Big(1+\frac{\nu\lambda}{2}+\gamma\varrho\Big)r_k^{n+1} - \Big(\frac{\nu\lambda}{2}+\gamma\varrho\Big)r_{k-1}^{n+1}\Big|^2 \leqslant 4E\Big|\Big(1+\frac{\nu\lambda}{2}-\gamma\varrho\Big)r_k^n + \Big(\gamma\varrho - \frac{\nu\lambda}{2}\Big)r_{k+1}^n\Big|^2 \\
&\quad + 8E\Big|\nu\int_{n\Delta t}^{(n+1)\Delta t} \Big\{\frac{\Delta x}{4}(v_{xx}((k+\alpha_1)\Delta x, s) - v_{xx}((k+\alpha_2)\Delta x, s+\Delta t))\Big\}\,\mathrm{d}s \\
&\quad + \int_{n\Delta t}^{(n+1)\Delta t} \Big\{-\gamma\frac{\Delta x}{3!}(v_{xxx}((k+\beta_1)\Delta x, s) - v_{xxx}((k+\beta_2)\Delta x, s+\Delta t)) \\
&\quad + \frac{\Delta t}{\Delta x}v_{xxx}(k\Delta x, s+\delta_1\Delta t) - \nu\frac{\Delta t}{2}v_{xxx}(k\Delta x, s+\delta_2\Delta t)\Big\}\,\mathrm{d}s \\
&\quad + \nu\int_{n\Delta t}^{(n+1)\Delta t} \frac{\Delta t}{\Delta x}v_{xx}(k\Delta x, s+\delta_1\Delta t)\,\mathrm{d}s \\
&\quad + \int_{n\Delta t}^{(n+1)\Delta t} \gamma\frac{\Delta t}{2}v_{xxxx}(k\Delta x, s+\delta_2\Delta t)\,\mathrm{d}s\Big|^2 \\
&\quad + 16(\gamma\sigma)^2 E\Big|\frac{\Delta t}{\Delta x}\int_{n\Delta t}^{(n+1)\Delta t} v_x(x,s)|_{x=x_k}\,\mathrm{d}W(s)\Big|^2 \\
&\quad + 16(\gamma\sigma)^2 E\Big|\frac{\Delta t}{2}\int_{n\Delta t}^{(n+1)\Delta t} v_{xx}(x,s)|_{x=x_k}\,\mathrm{d}W(s)\Big| \\
&\quad + 4\sigma^2\int_{n\Delta t}^{(n+1)\Delta t} E|v_k^n - u_k^n|^2\,\mathrm{d}s + 4\sigma^2\int_{n\Delta t}^{(n+1)\Delta t} E|v(x,s)|_{x=x_k} - v_k^n|^2.
\end{aligned}$$

Considering $\psi_1 = v_{xx}((k+\alpha_1)\Delta x, s) < \infty$, $\psi_2 = v_{xx}((k+\alpha_2)\Delta x, s+\Delta t) < \infty$, $\psi_3 = v_{xxx}((k+\beta_1)\Delta x, s) < \infty$, $\psi_4 = v_{xxx}((k+\beta_2)\Delta x, s+\Delta t) < \infty$, $\psi_5 = v_{xxx}(k\Delta x, s +$

$\delta_1\Delta t) < \infty$, $\psi_6 = v_{xxx}(k\Delta x, s+\delta_2\Delta t) < \infty$, $\psi_7 = v_{xx}(k\Delta x, s+\delta_1\Delta t) < \infty$, $\psi_8 = v_{xxxx}(k\Delta x, s+\delta_2\Delta t) < \infty$, $\psi_1' = v_x(x,s) < \infty$, $\psi_2' = v_{xx}(x,s) < \infty$, $\psi_1 \leqslant \infty$, we have

$$
\begin{aligned}
&E\Big|\Big(1+\frac{\nu\lambda}{2}+\gamma\varrho\Big)r_k^{n+1} - \Big(\frac{\nu\lambda}{2}+\gamma\varrho\Big)r_{k-1}^{n+1}\Big|^2 \\
&\quad \leqslant 4E\Big|\Big(1+\frac{\nu\lambda}{2}-\gamma\varrho\Big)r_k^n + \Big(\gamma\varrho-\frac{\nu\lambda}{2}\Big)r_{k+1}^n\Big|^2 \\
&\qquad + 8E\Big|\int_{n\Delta t}^{(n+1)\Delta t}\Big\{\nu\frac{\Delta x}{4}\psi_{1k} - \nu\psi_{2k} - \gamma\frac{\Delta x}{3!}(\psi_{3k}-\psi_{4k}) \\
&\qquad + \frac{\Delta t}{\Delta x}\psi_{5k} - \nu\frac{\Delta t}{2}\psi_{6k} + \nu\frac{\Delta t}{\Delta x}\psi_{7k} + \gamma\frac{\Delta t}{2}\psi_{8k}\Big\}\,\mathrm{d}s\Big|^2 \\
&\qquad + 16(\gamma\sigma)^2\int_{n\Delta t}^{(n+1)\Delta t}\Big\{\Big(\frac{\Delta t}{\Delta x}\Big)^2 E|\psi_{1k}'|^2 + \Big(\frac{\Delta t}{2}\Big)^2 E|\psi_{2k}'|^2\Big\}\,\mathrm{d}s \\
&\qquad + 4\sigma^2\int_{n\Delta t}^{(n+1)\Delta t} E|r_k^n|^2\,\mathrm{d}s + 4\sigma^2\int_{n\Delta t}^{(n+1)\Delta t} E|v(x,s)|_{x=x_k} - v_k^n|^2,
\end{aligned}
$$

and consequently, we get

$$
\begin{aligned}
&E\Big|\Big(1+\frac{\nu\lambda}{2}+\gamma\varrho\Big)r_k^{n+1} - \Big(\frac{\nu\lambda}{2}+\gamma\varrho\Big)r_{k-1}^{n+1}\Big|^2 \\
&\quad \leqslant 4E\Big|\Big(1+\frac{\nu\lambda}{2}-\gamma\varrho\Big)r_k^n + \Big(\gamma\varrho-\frac{\nu\lambda}{2}\Big)r_{k+1}^n\Big|^2 \\
&\qquad + 8E\Big|\int_{n\Delta t}^{(n+1)\Delta t}\Psi_1\,\mathrm{d}s\Big|^2 + 16(\gamma\sigma)^2\int_{n\Delta t}^{(n+1)\Delta t} E|\Psi_2|^2\,\mathrm{d}s \\
&\qquad + 4\sigma^2\int_{n\Delta t}^{(n+1)\Delta t} E|r_k^n|^2\,\mathrm{d}s + 4\sigma^2\int_{n\Delta t}^{(n+1)\Delta t} E|v(x,s)|_{x=x_k} - v_k^n|^2\,\mathrm{d}s.
\end{aligned}
$$

So it follows that

$$
\begin{aligned}
&E\Big|\Big(1+\frac{\nu\lambda}{2}+\gamma\varrho\Big)r_k^{n+1} - \Big(\frac{\nu\lambda}{2}+\gamma\varrho\Big)r_{k-1}^{n+1}\Big|^2 \\
&\quad \leqslant 4\Big(\Big(\Big|1+\frac{\nu\lambda}{2}-\gamma\varrho\Big| + \Big|\gamma\varrho-\frac{\nu\lambda}{2}\Big|\Big)^2 + \sigma^2\Delta t\Big)\sup_k E|r_k^n|^2 + \Psi\Delta t.
\end{aligned}
$$

Assuming $1+\nu\lambda/2-\gamma\varrho > 0$ and $\gamma\varrho-\nu\lambda/2 \geqslant 0$, we obtain

$$
\Big(\Big|1+\frac{\nu\lambda}{2}+\gamma\varrho\Big| - \Big|\frac{\nu\lambda}{2}+\gamma\varrho\Big|\Big)^2 \sup_k E|r_k^{n+1}|^2 \leqslant 4(1+\sigma^2\Delta t)\sup_k E|r_k^n|^2 + \Psi\Delta t,
$$

and considering $\gamma, \nu \geqslant 0$, we arrive at

$$
\sup_k E|r_k^{n+1}|^2 \leqslant 4(1+\sigma^2\Delta t)\sup_k E|r_k^n|^2 + \Psi\Delta t.
$$

So it follows that

$$E\|r^{n+1}\|_\infty^2 \to 0.$$

$\square$

R e m a r k 3. Let $v \in H^1, H^2, H^3, H^4$. The stochastic R-L Saul'yev/Robert and Weiss scheme is convergent for the $\|\cdot\|_\infty$-norm for $\nu\lambda/2 \leqslant \gamma\varrho \leqslant 1-\nu\lambda/2$ for approximating the solution of the stochastic advection-diffusion equation (1).

R e m a r k 4. Let $v \in H^1, H^2, H^3, H^4$. The stochastic R-L Saul'yev/Robert and Weiss scheme is convergent for the $\|\cdot\|_\infty$-norm for $\nu\lambda/2 \leqslant \gamma\varrho \leqslant 1-\nu\lambda/2$ for approximating the solution of the stochastic advection-diffusion equation (1).

R e m a r k 5. Let $v \in H^1, H^3$. The stochastic Saul'yev schemes are convergent for the $\|\cdot\|_\infty$-norm for $0 \leqslant \gamma\varrho \leqslant 1$ for approximating the solution of the stochastic diffusion equation (2).

8. Numerical results

Computational efficiency is another important factor in evaluating the superiority of numerical methods [5]. In this section, the performance of the presented numerical techniques described in the previous sections for solving the proposed SPDEs is considered and applied to some test problems. For computational purposes, it is useful to consider discreted Brownian motion where $W(t)$ is specified at discrete t values.

8.1. Example 1. We examine the performance of the proposed stochastic Saul'yev and Liu schemes for stochastic diffusion equation of the form

$$\frac{\partial u}{\partial t} - \gamma\frac{\partial^2 u}{\partial x^2} = \sigma\dot{W}(t) \tag{23}$$

subject to the initial condition

$$u(x,0) = \exp\Big(-\frac{(x-0.2)}{\gamma}\Big), \quad x \in [0,1],$$

and the boundary conditions

$$u(0,t) = \frac{1}{\sqrt{4t+1}}\exp\Big(-\frac{0.04}{\gamma(4t+1)}\Big),$$
$$u(1,t) = \frac{1}{\sqrt{4t+1}}\exp\Big(-\frac{0.64}{\gamma(4t+1)}\Big).$$

The problem has an exact expected solution given by

$$u(x,t) = \frac{1}{\sqrt{4t+1}} \exp\left(-\frac{x-0.2}{\gamma(4t+1)}\right).$$

The space domain of the interval $D = [0,1]$ is discretized into M uniform gridpoints. We carried out $10,000$ realizations for all tests, then we displayed the stochastic mean solutions along with some selected simulations. In this example, we used different values for the diffusion constant γ and the stochastic coefficient σ and the qualification of Saul'yev and Liu stochastic difference methods are investigated for various cases. In order to qualify the results for the stochastic diffusion equation (23), we plot in Figure 1 the stochastic solutions using both the Saul'yev and Liu schemes along with the deterministic numerical solution ($\sigma = 0$) on a mesh of 100 gridpoints, $\gamma = 0.005$, $\sigma = 5$ and $\Delta t = 0.005$.

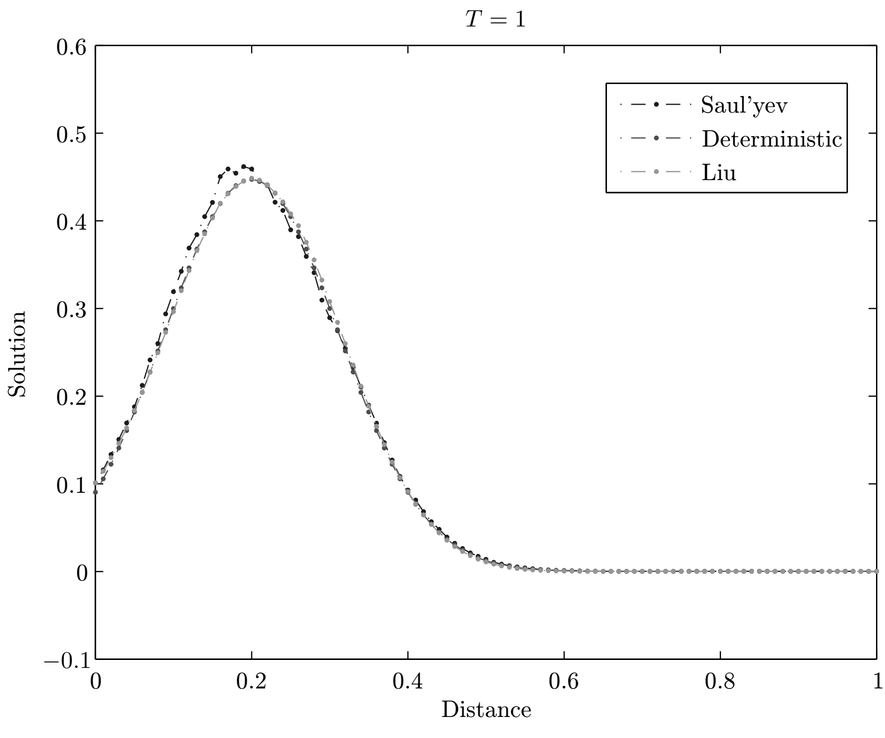

Figure 1. Mean and analytical solution of stochastic diffusion problem using Saul'yev and Liu methods with 100 mesh points.

In Table 1, some numerical results for solving the stochastic diffusion equation (23) using the unconditional stable alternating direction methods are presented. In all cases, time and space step sizes are considered to be $\Delta t = 0.01$ and $\Delta x = 0.01$, also the value of diffusion and stochastic constants are $\gamma = 0.005$ and $\sigma = 2.5$. As is shown in Table 1, the mean solutions of the considered SPDE are approximated by the L-R and R-L Saul'yev schemes and the averaged solution of these two methods for two opposite direction in each time step is presented. Similar computational results are given for stochastic Liu methods. Because of the significant property of stability of these stochastic explicit alternating direction methods, we have no limitation for considering the time and step sizes and the refinement of the computational domain does not impose any restriction on the stability conditions. So numerically explicit and unconditional stability of these stochastic alternating direction methods makes them applicable for approximating the solution of stochastic diffusion equations. The numerical solution of the stochastic diffusion equation (23) using the stochastic L-R Liu scheme is shown in Figure 2 on a 50×50 grid during the time interval $[0, 1]$ for $\gamma = 0.005$ and $\sigma = 10$.

Stochastic Scheme	$E(u(0.3,1))$	$E(u(0.4,1))$	$E(u(0.5,1))$	$E(u(0.6,1))$
L-R Saul'yev	0.31597	0.10496	0.01799	0.00170
R-L Saul'yev	0.33184	0.10505	0.01284	0.00054
Average of R-L and R-L Saul'yev	0.32424	0.10372	0.01527	0.00106
L-R Liu	0.32562	0.10672	0.01534	0.000895
R-L Liu	0.32460	0.10670	0.01534	0.00089
Average of R-L and R-L Liu	0.32460	0.10570	0.01493	0.00087

Table 1. Test of stochastic diffusion equation by the stochastic alternating direction methods.

For studying the performance of the proposed stochastic alternating direction methods, we use the exact expected solution to evaluate the expected error function at time t_n as

$$e_i^n = \langle u_i^n \rangle - u(x_i, t_n),$$

where $u(x_i, t_n)$ and $\langle u_i^n \rangle$ are respectively the exact and expected numerical solutions at the lattice points (x_i, t_n). So, the following spatial discrete error norms are defined:

$$\|e\|_{L_s^1} = \Delta x \sum_i |e_i|, \quad \|e\|_{L_s^2} = \Big(\Delta x \sum_i |e_i|^2\Big)^{1/2}.$$

Similarly, to investigate the time accuracy of the methods, we define the following temporal discrete error-norms:

$$\|e\|_{L_t^1} = \Delta t \sum_n \|e^n\|_{L_s^2}, \quad \|e\|_{L_t^2} = \Big(\Delta t \sum_n \|e^n\|_{L_s^2}^2\Big)^{1/2}.$$

where $\| \cdot \|_{L_s^p} = \| \cdot \|_{L^p(D)}$ and $| \cdot |_{L_t^p} = \| \cdot \|_{L^p([0,T))}$ denote the discrete L^p-norm in the space domain D and the time interval $[0,T)$, respectively.

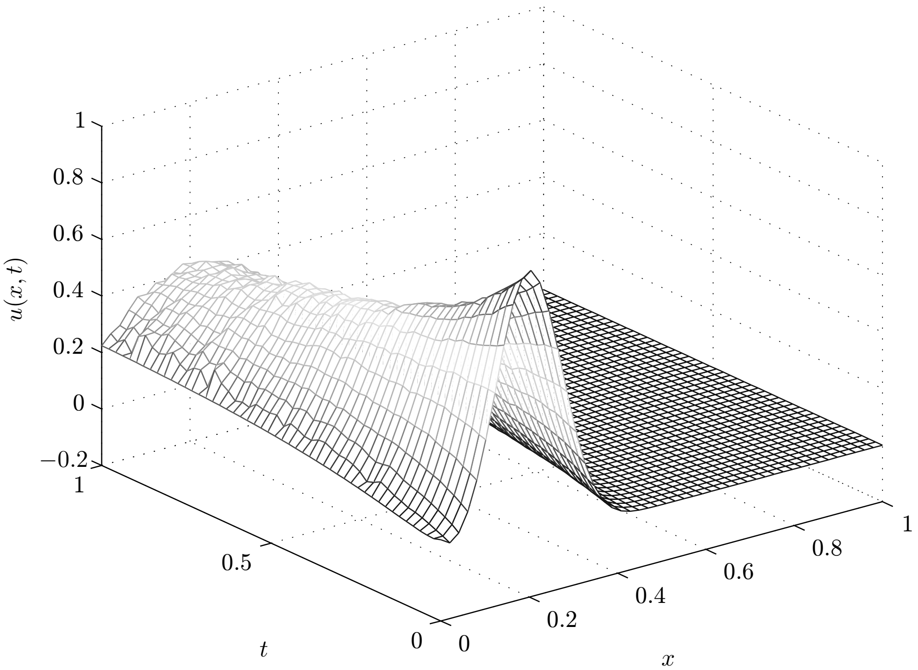

Figure 2. Mean solution of stochastic diffusion equation using L-R Liu method for $\gamma = 0.01$ and $\sigma = 10$.

In Table 2, we summarize the spatial and temporal errors for the diffusion equation (23) for the diffusion coefficient $\gamma = 0.01$, random coefficient $\sigma = 4.5$ and different values of M and N using the stochastic L-R Liu difference method. Further, we calculate the confidence interval with boundaries a and b to the level of 95 percent for the estimated $E(0.8)$.

M	N	$\| e \|_{L_s^1}$	$\| e \|_{L_s^2}$	$\| e \|_{L_t^1}$	$\| e \|_{L_t^2}$	$E(0.8)$	a	b
50	50	3.983E−03	7.962E−03	2.310E−02	3.913E−02	4.512E−01	4.449E−01	4.576E−01
100	200	3.979E−03	7.398E−03	1.467E−02	2.224E−02	4.528E−01	4.488E−01	4.569E−01
200	800	3.484E−03	7.326E−03	9.922E−03	1.333E−02	4.523E−01	4.496E−01	4.550E−01
400	3200	3.251E−03	7.145E−03	6.896E−03	8.445E−03	4.508E−01	4.448E−01	4.568E−01

Table 2. Spatial and temporal error norms and confidence intervals for stochastic diffusion L-R Liu method at time $t = 1$ with $\gamma = 0.01$ and $\sigma = 4.5$.

8.2. Example 2. In this example we investigate the efficiency of the stochastic Saul'yev/Robert and Weiss schemes for approximating the solution of the stochastic advection-diffusion equation of the form

$$\frac{\partial u}{\partial t}(x,t)+\nu\frac{\partial u}{\partial x}(x,t)=\gamma\frac{\partial^2 u}{\partial x^2}(x,t)+\sigma u(x,t)\dot{W}(t), \tag{24}$$

with the initial condition

$$u(x,0)=\exp\Big(-\frac{(x-0.5)^2}{\gamma}\Big),\quad x\in[0,1],$$

and boundary conditions

$$u(0,t)=\frac{1}{\sqrt{4t+1}}\exp\Big(-\frac{(-0.5-\nu t)^2}{\gamma(4t+1)}\Big),$$
$$u(1,t)=\frac{1}{\sqrt{4t+1}}\exp\Big(-\frac{(0.5-\nu t)^2}{\gamma(4t+1)}\Big).$$

It is easy to verify that in the absence of the noise term, the exact solution is

$$u(x,t)=\frac{1}{\sqrt{4t+1}}\exp\Big(-\frac{(x-0.5-\nu t)^2}{\gamma(4t+1)}\Big).$$

Numerical observations of the stability condition for the stochastic Saul'yev/Robert and Weiss schemes uphold the investigations of the previous sections. Applying the L-R stochastic Saul'yev/Robert and Weiss scheme to the stochastic initial value problem (24), the time and spatial step sizes do not impose any restrictions on the stability and convergence of the method.

Table 3 shows the spatial and temporal errors for the advection-diffusion equation (24) for $\gamma=0.005$, $\nu=0.5$ and $\sigma=2.5$ with several values of M and N using the L-R stochastic Saul'yev/Robert and Weiss difference method. The coefficient of the SPDE (24) and also the space and time step sizes are selected according to the convergence condition of the difference method. In addition, the confidence intervals with boundaries a and b to the level of 95 percent for the estimated $E(0.8)$ are given.

M	N	$\parallel e \parallel_{L^1_s}$	$\parallel e \parallel_{L^2_s}$	$\parallel e \parallel_{L^1_t}$	$\parallel e \parallel_{L^2_t}$	$E(0.8)$	a	b
25	50	2.109E-02	3.538E-02	1.403E-01	1.870E-01	5.135E-01	5.089E-01	5.181E-01
50	200	5.932E-03	1.135E-02	3.287E-03	1.348E-02	5.222E-01	5.136E-01	5.308E-01
100	800	1.822E-03	3.751E-03	3.026E-03	1.041E-02	5.312E-01	5.258E-01	5.367E-01
200	3200	8.480E-04	1.665E-03	2.961E-03	9.482E-03	5.380E-01	5.342E-01	5.418E-01

Table 3. Spatial and temporal error norms and confidence intervals for stochastic advection diffusion L-R Saul'yev/Robert and Weiss method at time $t=0.6$ with $\gamma=0.005$, $\nu=0.5$ and $\sigma=2.5$.

Figure 3 shows the approximations of the stochastic advection-diffusion equation using the L-R stochastic Saul'yev/Robert and Weiss scheme on a 100 by 100 grid with $\gamma=0.005$, $\nu=0.5$ and $\sigma=6$ during the time interval $[0,1]$.

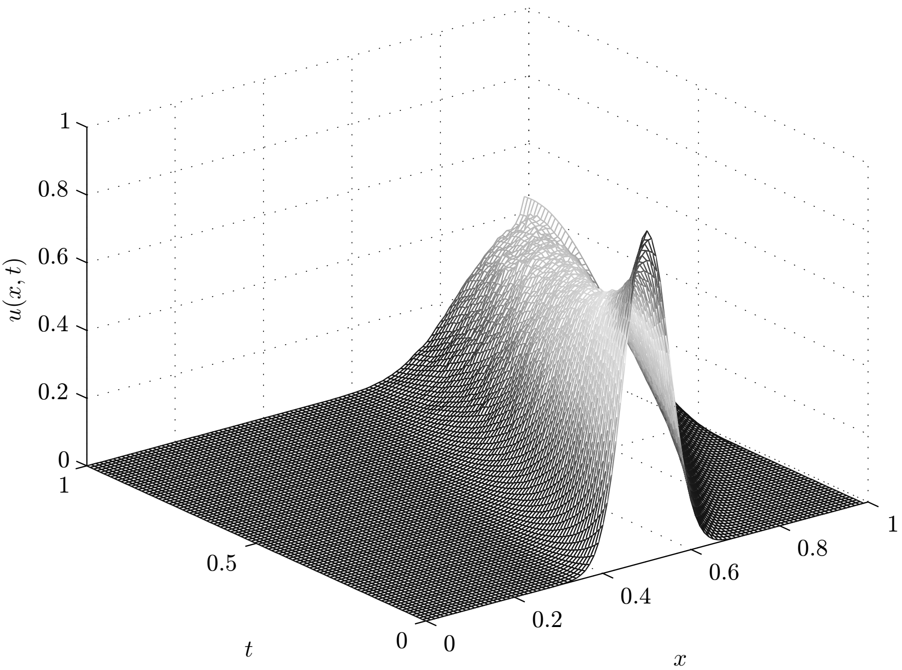

Figure 3. Mean solution of stochastic diffusion equation using L-R Saul'yev/Robert and Weiss method for $\gamma = 0.005$, $\nu = 0.5$ and $\sigma = 6$.

In Table 4, we show the dependence of the error on the number of the realizations for approximated solution of the SPDE (24) at the point $(0.6, 1)$ with $\Delta x = 0.01$ and $\Delta t = 0.005$.

number of realization	$E(0.6)$	a	b	$\| e \|_{L^1_s}$	$\| e \|_{L^1_s}$
10	2.665E-01	1.715E-01	3.614E-01	4.155E-02	6.437E-02
10^2	4.323E-01	3.602E-01	5.044E-01	7.954E-03	1.236E-02
10^3	4.369E-01	4.138E-01	4.599E-01	4.401E-03	6.422E-03
10^4	4.420E-01	4.346E-01	4.494E-01	3.447E-03	5.314E-03
10^5	4.429E-01	4.383E-01	4.476E-01	2.092E-03	2.896E-03
10^6	4.432E-01	4.425E-01	4.439E-01	1.998E-03	2.698E-03

Table 4. Confidence intervals for mean solution of SPDE (24) using L-R Saul'yev/Robert and Weiss method at time $T = 1$ with $\gamma = 0.01$, $\nu = 0.1$ and $\sigma = 3.5$.

Also, in Figure 4 the approximated solution of the stochastic advection-diffusion (24) using the L-R stochastic Saul'yev/Robert and Weiss scheme is represented for $\gamma = 0.01$, $\nu = 1.5$ and $\sigma = 8$ during the time interval $[0, 1]$. As is shown in Figure 5, applying the R-L stochastic Saul'yev/Robert and Weiss method to the stochastic advection-diffusion equation (24) with $\gamma\varrho = 4$, $\lambda\nu = 2$ and $\sigma = 1$, the computational results justify the theory of the stability conditions.

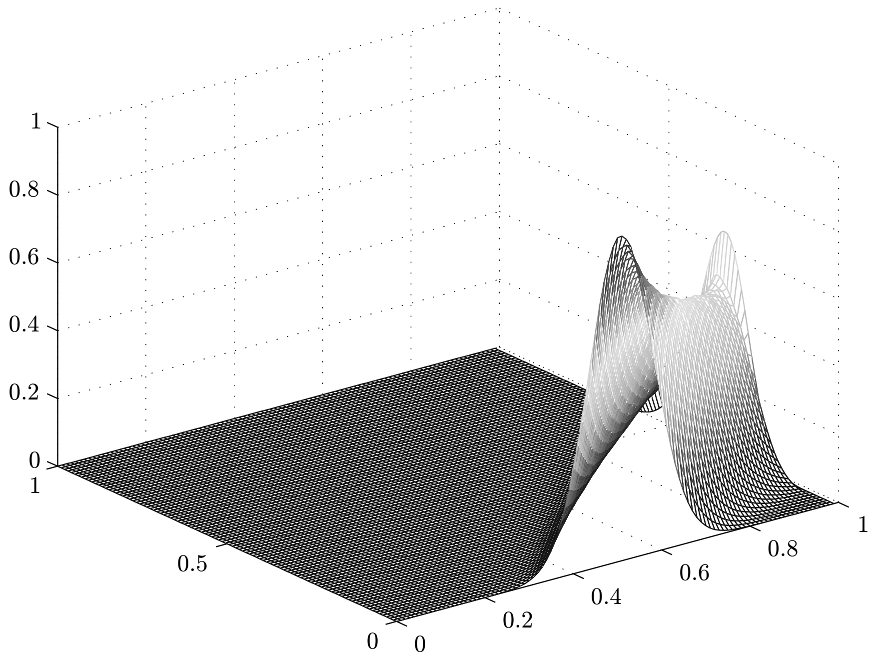

Figure 4. Mean solution of stochastic diffusion equation using L-R Saul'yev/Robert and Weiss method for $\gamma = 0.01$, $\nu = 1.5$ and $\sigma = 8$.

9. Conclusion

In this paper, numerical solutions of stochastic advection-diffusion and diffusion equations with real-valued Brownian motion are approximated using stochastic alternating direction methods. When applied to stochastic diffusion equations, the two Saul'yev and Liu SADE schemes retain their stability conditions with the calculation proceeding from left to right in the computational domain or in the opposite direction. The stability, consistency and convergence of stochastic Saul'yev/Robert and Weiss schemes are also investigated for solving stochastic advection-diffusion equations. Although the R-L Saul'yev/Robert and Weiss scheme is conditionally stable, the R-L Saul'yev/Robert and Weiss scheme is unconditionally stable for approximating the solution of the stochastic advection-diffusion equation. Since the proposed stochastic alternating direction methods are explicit in nature and because of the stability qualification of these stochastic difference schemes, they are computationally applicable in the comparison with the other unconditionally stable methods most of which are implicit and impose a large amount of computation to the algorithm, in particular for the stochastic case.

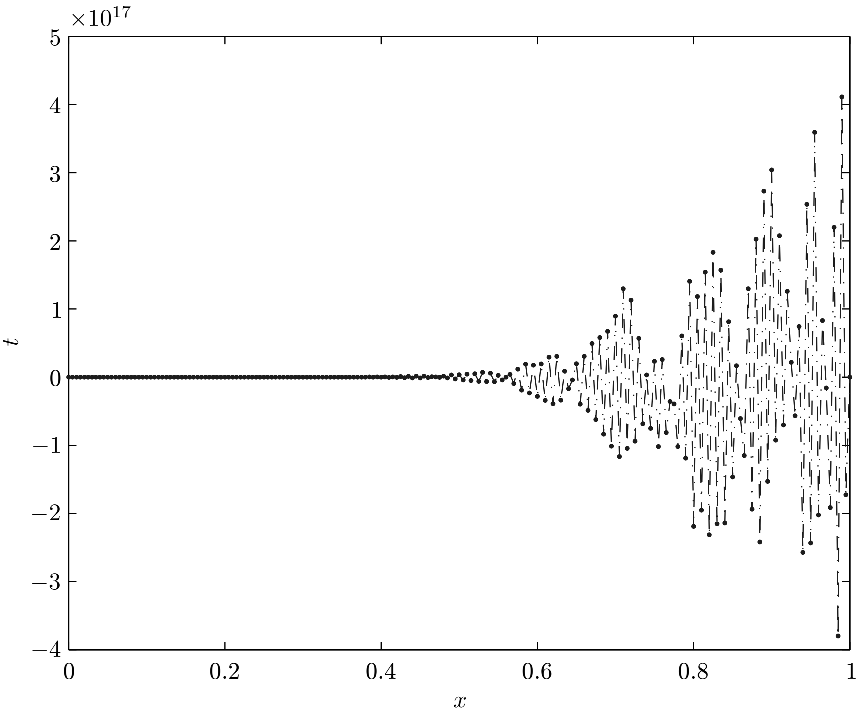

Figure 5. Representation of the unstability of the R-L Saul'yev/Robert and Weiss method for $\gamma\varrho = 4$, $\lambda\nu = 2$ and $\sigma = 1$.

In fact, when unconditional alternating direction methods are developed for the stochastic case with real-valued Brownian motions, the stochastic reformulated alternating direction methods are also unconditionally stable for approximating the stochastic partial differential equations. In numerical results, the performance of the stochastic alternating direction methods for stochastic advection-diffusion and diffusion equations is studied by computing discretized Brownian paths.

Another open question is how to extend such methods to SPDEs with space-time white noise process and demonstrate the stability and other main properties of the stochastic difference schemes.

References

[1] *E. J. Allen, S. J. Novosel, Z. Zhang*: Finite element and difference approximation of some linear stochastic partial differential equations. Stochastics Stochastics Rep. *64* (1998), 117–142.

[2] *W. F. Ames*: Numerical Methods for Partial Differential Equations. 3. ed. Computer Science and Scientific Computing. Academic Press, Boston, 1992.

[3] *L. J. Campbell, B. Yin*: On the stability of alternating-direction explicit methods for advection-diffusion equations. Numer. Methods Partial Differ. Equations *23* (2007), 1429–1444.

[4] *A. M. Davie, J. G. Gaines*: Convergence of numerical schemes for the solution of parabolic stochastic partial differential equations. Math. Comput. *70* (2001), 121–134.

[5] *D. J. Higham*: An algorithmic introduction to numerical simulation of stochastic differential equations. SIAM Rev. *43* (2001), 525–546.

[6] *P. E. Kloeden, E. Platen*: Numerical Solution of Stochastic Differential Equations. Applications of Mathematics 23. Springer, Berlin, 1992.

[7] *Y. Komori, T. Mitsui*: Stable ROW-type weak scheme for stochastic differential equations. Monte Carlo Methods Appl. *1* (1995), 279–300.

[8] *S. L. Liu*: Stable explicit difference approximations to parabolic partial differential equations. AIChE J. *15* (1969), 334–338.

[9] *S. McDonald*: Finite difference approximation for linear stochastic partial differential equation with method of lines. MPRA Paper No. 3983 (2006). `http://mpra.ub.uni-muenchen.de/3983`.

[10] *G. N. Milstein*: Numerical Integration of Stochastic Differential Equations. Transl. from the Russian. Mathematics and its Applications 313. Kluwer Academic Publishers, Dordrecht, 1994.

[11] *A. Rößler*: Stochastic Taylor expansions for the expectation of functionals of diffusion processes. Stochastic Anal. Appl. *22* (2004), 1553–1576.

[12] *A. Rößler, M. Seaïd, M. Zahri*: Method of lines for stochastic boundary-value problems with additive noise. Appl. Math. Comput. *199* (2008), 301–314.

[13] *C. Roth*: Difference methods for stochastic partial differential equations. Z. Angew. Math. Mech. *82* (2002), 821–830.

[14] *C. Roth*: Approximations of Solution of a First Order Stochastic Partial Differential Equation, Report. Institut Optimierung und Stochastik, Universität Halle-Wittenberg, Halle, 1989.

[15] *V. K. Saul'yev*: Integration of Equations of Parabolic Type by the Method of Nets. Translated by G. J. Tee. International Series of Monographs in Pure and Applied Mathematics Vol. 54 (K.L. Stewart, ed.). Pergamon Press, Oxford, 1964.

[16] *V. K. Saul'yev*: On a method of numerical integration of a diffusion equation. Dokl. Akad. Nauk SSSR *115* (1957), 1077–1080.

[17] *A. R. Soheili, M. B. Niasar, M. Arezoomandan*: Approximation of stochastic parabolic differential equations with two different finite difference schemes. Bull. Iran. Math. Soc. *37* (2011), 61–83.

[18] *J. C. Strikwerda*: Finite difference schemes and partial differential equations. 2nd ed. Society for Industrial and Applied Mathematics, Philadelphia, 2004.

[19] *J. W. Thomas*: Numerical Partial Differential Equations: Finite Difference Methods. Texts in Applied Mathematics 22. Springer, New York, 1995.

Authors' addresses: *Ali R. Soheili*, The Center of Excellence on Modeling and Control Systems, School of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Iran, e-mail: `soheili@ferdowsi.um.ac.ir`; *Mahdieh Arezoomandan*, Department of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran, e-mail: `arezoomandan@mail.usb.ac.ir`.