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ON THE FOURIER COSINE—KONTOROVICH-LEBEDEV  
GENERALIZED CONVOLUTION TRANSFORMS

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*Abstract.* We deal with several classes of integral transformations of the form

$$f(x) \rightarrow D \int_{\mathbb{R}_+^2} \frac{1}{u} (e^{-u \cosh(x+v)} + e^{-u \cosh(x-v)}) h(u) f(v) \, du \, dv,$$

where  $D$  is an operator. In case  $D$  is the identity operator, we obtain several operator properties on  $L_p(\mathbb{R}_+)$  with weights for a generalized operator related to the Fourier cosine and the Kontorovich-Lebedev integral transforms. For a class of differential operators of infinite order, we prove the unitary property of these transforms on  $L_2(\mathbb{R}_+)$  and define the inversion formula. Further, for an other class of differential operators of finite order, we apply these transformations to solve a class of integro-differential problems of generalized convolution type.

*Keywords:* convolution, Hölder inequality, Young's theorem, Watson's theorem, unitary, Fourier cosine, Kontorovich-Lebedev, transform, integro-differential equation

*MSC 2010:* 33C10, 44A35, 45E10, 45J05, 47A30, 47B15

## 1. INTRODUCTION

The Fourier cosine integral transform is of the form (see [10], [11])

$$(1.1) \quad (F_c f)(y) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos xy \, dx$$

for  $f \in L_1(\mathbb{R}_+)$ , and

$$(1.2) \quad (F_c f)(y) = \lim_{N \rightarrow \infty} \sqrt{\frac{2}{\pi}} \int_0^N f(x) \cos yx \, dx = \sqrt{\frac{2}{\pi}} \frac{d}{dx} \int_0^\infty f(x) \frac{\sin xy}{x} \, dx$$

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for  $f \in L_2(\mathbb{R}_+)$ ; here the limit is understood in  $L_2(\mathbb{R}_+)$  norm mean. These two definitions are equivalent if  $f \in L_1(\mathbb{R}_+) \cap L_2(\mathbb{R}_+)$ .

The Kontorovich-Lebedev integral transform was first investigated by M. J. Kontorovich and N. N. Lebedev in 1938–1939 and has the form (see [5], [6], [14])

$$(1.3) \quad K[f](y) = \int_0^\infty K_{ix}(y)f(x) dx,$$

which contains as the kernel the Macdonald function  $K_\nu(x)$  (see [1]) of the pure imaginary index  $\nu = iy$ . The function  $K_\nu(z)$  satisfies the differential equation

$$(1.4) \quad z^2 \frac{d^2 u}{dz^2} + z \frac{du}{dz} - (z^2 + \nu^2)u = 0.$$

The Macdonald function has the asymptotic behaviour (see [6])

$$(1.5) \quad K_\nu(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}[1 + O(1/z)], \quad z \rightarrow \infty,$$

and near the origin

$$(1.6) \quad z^\nu K_\nu(z) = 2^{\nu-1} \Gamma(\nu) + o(1), \quad z \rightarrow 0, \quad \nu \neq 0,$$

$$(1.7) \quad K_0(z) = -\log z + O(1), \quad z \rightarrow 0.$$

The following form for the Macdonald function is very useful (see [1], [6], [14]):

$$(1.8) \quad K_{iy}(x) = \int_0^\infty e^{-x \cosh u} \cos yu du, \quad x > 0.$$

The inverse Kontorovich-Lebedev transform (1.3) is of the form (see [5], [6])

$$(1.9) \quad f(x) = K^{-1}[g](x) = \frac{2}{\pi^2} x \sinh(\pi x) \int_0^\infty \frac{1}{y} K_{ix}(y)g(y) dy,$$

here,  $g(y) = K[f](y)$ .

Throughout this paper, we are interested in the Kontorovich-Lebedev transform (1.3). However, note that there is another version of the Kontorovich-Lebedev integral transform which is of the form (see [1], [6], [16])

$$(1.10) \quad g(y) = \tilde{K}[f](y) = \int_0^\infty K_{iy}(x)f(x) dx.$$

A generalized convolution for the Fourier cosine and the Kontorovich-Lebedev integral transforms has been studied in [12]:

$$(1.11) \quad (h \overset{\gamma}{*} f)(x) = \frac{1}{\pi^2} \int_0^\infty \int_0^\infty \frac{1}{u} [e^{-u \cosh(x+v)} + e^{-u \cosh(x-v)}] h(u)f(v) du dv, \quad x > 0.$$

The existence of the generalized convolution (1.11) for two functions in  $L_1(\mathbb{R}_+)$  with weight and its application to solving integral equations of generalized convolution type were studied in [12]. Namely, for  $h \in L_1(\mathbb{R}_+, 1/x)$ ,  $f \in L_1(\mathbb{R}_+, 1/\sinh x)$ , the following factorization equality holds (see [12]):

$$(1.12) \quad F_c(h \overset{\gamma}{*} f)(y) = \frac{1}{y \sinh \pi y} K^{-1}[h](y)(F_c f)(y), \quad \forall y > 0.$$

In any convolution  $(h * f)$  of two functions  $h$  and  $f$ , if we fix the function  $h$  and let  $f$  vary in a certain function space, then one can study convolution transforms of the type  $f \mapsto D(f * h)$ , where  $D$  is an operator. The most famous integral transforms constructed in this way are the Watson transforms that are related to the Mellin convolution and the Mellin transform (see [11])

$$f(x) \mapsto g(x) = \int_0^\infty k(xy)f(y) dy.$$

Recently, several authors have been interested in the convolution transforms of this type (see [3], [4], [13], [15]). In this paper, we are interested in the transform  $f \mapsto D(h \overset{\gamma}{*} f)$ , where  $(h \overset{\gamma}{*} f)$  is the generalized convolution (1.11). For the case  $D$  is the identity operator, in Section 2 we study several further operator properties in the Lebesgue spaces  $L_p(\mathbb{R}_+)$  with weight for the generalized convolution (1.11). In particular, Young's theorem and Young's inequality for this generalized convolution are obtained. In Section 3, for a class of differential operators  $D$  of infinite order, we obtain the necessary and sufficient condition such that the respective transforms are unitary on  $L_2(\mathbb{R}_+)$ , and define the inverse transforms. Finally, in Section 4, for an other class of differential operator  $D$  of finite order, we obtain the solution in closed form of a class of integro-differential equations.

## 2. GENERALIZED CONVOLUTION OPERATOR PROPERTIES

In this section, we will prove several norm properties of the generalized convolution (1.11). Throughout the paper, we are interested in the following two-parametric family of Lebesgue spaces.

**Definition 1** (see [16]). For  $\alpha \in \mathbb{R}$ ,  $0 < \beta \leq 1$ , we denote by  $L_p^{\alpha, \beta}(\mathbb{R}_+)$  the space of all functions  $f(x)$  defined in  $\mathbb{R}_+$  such that

$$(2.1) \quad \int_0^\infty |f(x)|^p K_0(\beta x) x^\alpha dx < \infty.$$

The norm of a function in this space is defined by

$$\|f\|_{L_p^{\alpha,\beta}(\mathbb{R}_+)} = \left( \int_0^\infty |f(x)|^p K_0(\beta x) x^\alpha dx \right)^{1/p}.$$

Using the asymptotics of the Macdonald function (1.5), (1.6), (1.7), formula (2.1) can be expressed in an equivalent form

$$\int_0^1 |f(x)|^p |\log x| x^\alpha dx + \int_1^\infty |f(x)|^p x^{\alpha-1/2} e^{-\beta x} dx < \infty.$$

The boundedness of the generalized convolution (1.11) on the spaces  $L_1(\mathbb{R}_+)$  is given by the following theorem; here we consider the function  $h \in L_1^{-1,\beta}(\mathbb{R}_+)$ .

**Theorem 2.1.** *Let  $h \in L_1^{-1,\beta}(\mathbb{R}_+)$  and  $g \in L_1(\mathbb{R}_+)$ ,  $0 < \beta \leq 1$ . Then the generalized convolution (1.11) exists for almost all  $x > 0$ , belongs to  $L_1(\mathbb{R}_+)$ , and the following estimation holds:*

$$(2.2) \quad \|h \overset{\gamma}{*} g\|_{L_1(\mathbb{R}_+)} \leq \frac{2}{\pi^2} \|h\|_{L_1^{-1,\beta}(\mathbb{R}_+)} \|g\|_{L_1(\mathbb{R}_+)}.$$

Moreover, the factorization property (1.12) holds true. Furthermore, if  $0 < \beta < 1$ , then the convolution (1.11) belongs to  $C_0(\mathbb{R}_+)$ , and the Parseval type equality takes place for all  $x > 0$ :

$$(2.3) \quad (h \overset{\gamma}{*} f)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{y \sinh \pi y} K^{-1}[h](y) (F_c f)(y) \cos xy dy.$$

*Proof.* Using formula (1.8) we obtain

$$(2.4) \quad \frac{1}{2} \int_0^\infty (e^{-u \cosh(x+v)} + e^{-u \cosh(x-v)}) dv = K_0(u).$$

Then

$$\|h \overset{\gamma}{*} f\|_{L_1(\mathbb{R}_+)} \leq \frac{2}{\pi^2} \int_0^\infty \int_0^\infty \frac{|h(u)|}{u} K_0(u) |f(v)| du dv = \frac{2}{\pi^2} \|h\|_{L_1^{-1,\beta}(\mathbb{R}_+, 1/x)} \cdot \|f\|_{L_1(\mathbb{R}_+)}.$$

We now prove the Parseval type equality. Using Fubini's theorem and the formula (2.16.48.19) in [9]

$$\int_0^\infty \cos by K_{iy}(u) dy = \frac{\pi}{2} e^{-u \cosh b},$$

we have

$$\begin{aligned}
 (h \overset{\gamma}{*} f)(x) &= \frac{1}{\pi^2} \int_0^\infty \int_0^\infty \frac{1}{u} [e^{-u \cosh(x+v)} + e^{-u \cosh(x-v)}] h(u) f(v) du dv \\
 &= \frac{1}{\pi^2} \int_0^\infty \int_0^\infty \int_0^\infty \frac{2}{\pi} \frac{1}{u} h(u) f(v) K_{iy}(u) (\cos(x+v)y + \cos(x-v)y) du dv dy \\
 &= \frac{4}{\pi^3} \int_0^\infty \int_0^\infty \int_0^\infty \frac{1}{u} h(u) f(v) K_{iy}(u) \cos xy \cos vy du dv dy \\
 &= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{y \sinh \pi y} \left( \frac{2}{\pi^2} y \sinh \pi y \int_0^\infty \frac{1}{u} K_{iy}(u) h(u) du \right) \\
 &\quad \times \left( \sqrt{\frac{2}{\pi}} \int_0^\infty f(v) \cos vy dv \right) \cos xy dy \\
 &= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{y \sinh \pi y} K^{-1}[h](y) (F_c f)(y) \cos xy dy.
 \end{aligned}$$

That gives the Parseval identity (2.3), and the proof of the theorem is complete.  $\square$

The next theorem draws a parallel with a result studied in [16], namely, the boundedness of the generalized convolution (1.11) on spaces  $L_r^{\alpha, \gamma}$ ,  $1 < r < \infty$ ,  $\alpha > -1$ ,  $0 < \gamma \leq 1$  is given.

**Theorem 2.2.** *Let  $1 < p < \infty$  be a real number and  $q$  its conjugate exponent, i.e.  $1/p + 1/q = 1$ . Then for any  $h \in L_p^{-p, \beta}(\mathbb{R}_+)$  and  $f \in L_q(\mathbb{R}_+)$ , the generalized convolution  $(h \overset{\gamma}{*} f)$  (1.11) is well-defined as a bounded continuous function on  $\mathbb{R}_+$ . Moreover,  $(h \overset{\gamma}{*} f)$  belongs to  $L_r^{\alpha, \gamma}(\mathbb{R}_+)$ ,  $1 \leq r < \infty$ ,  $\alpha > -1$ ,  $0 < \gamma \leq 1$ , and*

$$(2.5) \quad \|h \overset{\gamma}{*} f\|_{L_r^{\alpha, \gamma}(\mathbb{R}_+)} \leq C_{\alpha, \gamma}^{1/r} \|h\|_{L_p^{-p, \beta}(\mathbb{R}_+)} \|f\|_{L_q(\mathbb{R}_+)},$$

where  $C_{\alpha, \gamma} = (2^{r+\alpha-1}/\pi^{2r} \gamma^{\alpha+1}) \Gamma^2((\alpha+1)/2)$ .

**Proof.** Using the integral representation (2.4) for the function  $K_0(u)$ , the Hölder inequality, and the fact that  $e^{-u \cosh(x+v)} + e^{-u \cosh(x-v)} \leq 2e^{-u}$  for all positive  $u, x, v$ , we get

$$\begin{aligned}
 (2.6) \quad |(h \overset{\gamma}{*} f)(x)| &\leq \frac{1}{\pi^2} \int_0^\infty \int_0^\infty \left| \frac{h(u)}{u} \right| |f(v)| [e^{-u \cosh(x+v)} + e^{-u \cosh(x-v)}] du dv \\
 &\leq \frac{1}{\pi^2} \left( \int_0^\infty \int_0^\infty \left| \frac{h(u)}{u} \right|^p [e^{-u \cosh(x+v)} + e^{-u \cosh(x-v)}] du dv \right)^{1/p} \\
 &\quad \times \left( \int_0^\infty \int_0^\infty |f(v)|^q [e^{-u \cosh(x+v)} + e^{-u \cosh(x-v)}] du dv \right)^{1/q} \\
 &\leq \frac{2}{\pi^2} \left( \int_0^\infty \left| \frac{h(u)}{u} \right|^p K_0(u) du \right)^{1/p} \|f\|_{L_q(\mathbb{R}_+)}.
 \end{aligned}$$

Therefore, the generalized convolution is well-defined as a bounded operator and the estimation (2.6) holds. Moreover, in view of formula (2.16.2.2) in [9] we get

$$\begin{aligned} \|h \overset{\gamma}{*} f\|_{L_r^{\alpha, \gamma}(\mathbb{R}_+)} &\leq \frac{2}{\pi^2} \|h\|_{L_p^{-p, \beta}(\mathbb{R}_+)} \|f\|_{L_q(\mathbb{R}_+)} \left( \int_0^\infty x^\alpha K_0(\gamma x) dx \right)^{1/r} \\ &= \frac{2}{\pi^2} (2\gamma)^{-1/r} \left(\frac{\gamma}{2}\right)^{-\alpha/r} \Gamma^{2/r} \left(\frac{\alpha+1}{2}\right) \|h\|_{L_p^{-p, \beta}(\mathbb{R}_+)} \|f\|_{L_q(\mathbb{R}_+)}, \quad \alpha > -1. \end{aligned}$$

This yields (2.5) □

For the Fourier convolution (see [10])

$$(2.7) \quad (h \underset{F}{*} f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty h(x-y) f(y) dy,$$

Young's theorem and its corollary, the so-called Young inequality, are fundamental (see [2]). So, it is useful to study similar topics for convolutions and generalized convolutions for other integral transforms. Next, we will prove Young's type theorem for the generalized convolution (1.11).

**Theorem 2.3** (Young's Type Theorem). *Let  $p, q, r$  be real numbers in  $(1; \infty)$  such that  $1/p + 1/q + 1/r = 2$  and let  $f \in L_p^{-p, \beta}(\mathbb{R}_+)$ ,  $0 < \beta \leq 1$ ,  $g \in L_q(\mathbb{R}_+)$ ,  $h \in L_r(\mathbb{R}_+)$ . Then*

$$(2.8) \quad \left| \int_0^\infty (f \overset{\gamma}{*} g)(x) \cdot h(x) dx \right| \leq \frac{2^{(p-1)/p}}{\pi^2} \|f\|_{L_p^{-p, \beta}(\mathbb{R}_+)} \|g\|_{L_q(\mathbb{R}_+)} \|h\|_{L_r(\mathbb{R}_+)}.$$

*Proof.* Let  $p_1, q_1, r_1$  be the conjugate exponentials of  $p, q, r$ , respectively, it means

$$\frac{1}{p} + \frac{1}{p_1} = \frac{1}{q} + \frac{1}{q_1} = \frac{1}{r} + \frac{1}{r_1} = 1.$$

Then it is obvious that  $1/p_1 + 1/q_1 + 1/r_1 = 1$ . Put

$$\begin{aligned} F(x, u, v) &= |g(v)|^{q/p_1} |h(x)|^{r/p_1} [e^{-u \cosh(x+v)} + e^{-u \cosh(x-v)}]^{1/p_1}, \\ G(x, u, v) &= \left| \frac{f(u)}{u} \right|^{p/q_1} |h(x)|^{r/q_1} [e^{-u \cosh(x+v)} + e^{-u \cosh(x-v)}]^{1/q_1}, \\ H(x, u, v) &= \left| \frac{f(u)}{u} \right|^{p/r_1} |g(v)|^{q/r_1} [e^{-u \cosh(x+v)} + e^{-u \cosh(x-v)}]^{1/r_1}. \end{aligned}$$

We have

$$(2.9) \quad (F \cdot G \cdot H)(x, u, v) = \left| \frac{f(u)}{u} \right| |g(v)| |h(x)| [e^{-u \cosh(x+v)} + e^{-u \cosh(x-v)}].$$

On the other hand, in the space  $L_{p_1}(\mathbb{R}_+^3)$  we have

$$\begin{aligned}
 (2.10) \quad \|F\|_{L_{p_1}(\mathbb{R}_+^3)}^{p_1} &= \int_0^\infty \int_0^\infty \int_0^\infty |g(v)|^q |h(x)|^r [e^{-u \cosh(x+v)} + e^{-u \cosh(x-v)}] du dv dx \\
 &\leq 2 \int_0^\infty \int_0^\infty \int_0^\infty |g(v)|^q |h(x)|^r e^{-u} du dv dx \\
 &= 2 \|g\|_{L_q(\mathbb{R}_+)}^q \|h\|_{L_r(\mathbb{R}_+)}^r.
 \end{aligned}$$

Further, the fact that  $K_0(u) \leq K_0(\beta u)$  for  $0 < \beta \leq 1$  (see [16]) yields

$$\begin{aligned}
 (2.11) \quad \|G\|_{L_{q_1}(\mathbb{R}_+^3)}^{p_1} &= \int_0^\infty \int_0^\infty \int_0^\infty \left| \frac{f(u)}{u} \right|^p |h(x)|^r [e^{-u \cosh(x+v)} + e^{-u \cosh(x-v)}] du dv dx \\
 &\leq \int_0^\infty \int_0^\infty \int_0^\infty \left| \frac{f(u)}{u} \right|^p K_0(\beta u) |h(x)|^r du dv dx \\
 &= \|f\|_{L_p^{-p,\beta}(\mathbb{R}_+)}^p \|h\|_{L_r(\mathbb{R}_+)}^r,
 \end{aligned}$$

and similarly,

$$\begin{aligned}
 (2.12) \quad \|H\|_{L_{r_1}(\mathbb{R}_+^3)}^{r_1} &= \int_0^\infty \int_0^\infty \int_0^\infty \left| \frac{f(u)}{u} \right|^p |g(v)|^q [e^{-u \cosh(x+v)} + e^{-u \cosh(x-v)}] du dv dx \\
 &\leq \int_0^\infty \int_0^\infty \int_0^\infty \left| \frac{f(u)}{u} \right|^p K_0(\beta u) |g(v)|^q du dv dx \\
 &= \|f\|_{L_p^{-p,\beta}(\mathbb{R}_+)}^p \|g\|_{L_q(\mathbb{R}_+)}^q.
 \end{aligned}$$

From (2.10), (2.11) and (2.12) we have

$$\begin{aligned}
 (2.13) \quad \|F\|_{L_{p_1}(\mathbb{R}_+^3)} \|G\|_{L_{q_1}(\mathbb{R}_+^3)} \|H\|_{L_{r_1}(\mathbb{R}_+^3)} \\
 \leq 2^{(p-1)/p} \|f\|_{L_p^{-p,\beta}(\mathbb{R}_+)} \|g\|_{L_q(\mathbb{R}_+)} \|h\|_{L_r(\mathbb{R}_+)}.
 \end{aligned}$$

From (2.9) and (2.13), by three-function form of the Hölder inequality [2] we have

$$\begin{aligned}
 &\left| \int_0^\infty (f \overset{\gamma}{*} g)(x) \cdot h(x) dx \right| \\
 &\leq \frac{1}{\pi^2} \int_0^\infty \int_0^\infty \int_0^\infty \left| \frac{f(u)}{u} \right| |g(v)| |h(x)| [e^{-u \cosh(x+v)} + e^{-u \cosh(x-v)}] du dv dx \\
 &= \frac{1}{\pi^2} \int_0^\infty \int_0^\infty \int_0^\infty F(x, u, v) G(x, u, v) H(x, u, v) du dv dx \\
 &\leq \frac{1}{\pi^2} \|F\|_{L_{p_1}(\mathbb{R}_+^3)} \|G\|_{L_{q_1}(\mathbb{R}_+^3)} \|H\|_{L_{r_1}(\mathbb{R}_+^3)} \\
 &\leq \frac{2^{(p-1)/p}}{\pi^2} \|f\|_{L_p^{-p,\beta}(\mathbb{R}_+)} \|g\|_{L_q(\mathbb{R}_+)} \|h\|_{L_r(\mathbb{R}_+)}.
 \end{aligned}$$

The proof is complete. □



The following Young's type inequality is the direct corollary of the above theorem

**Corollary 2.1** (A Young's Type Inequality). *Let  $1 < p < \infty$ ,  $1 < q < \infty$ ,  $1 < r < \infty$  be such that  $1/p + 1/q = 1 + 1/r$  and let  $f \in L_p^{-p,\beta}(\mathbb{R}_+)$ ,  $0 < \beta \leq 1$ ,  $g \in L_q(\mathbb{R}_+)$ . Then the generalized convolution (1.11) is well-defined in  $L_r(\mathbb{R}_+)$ , moreover, the following inequality holds:*

$$(2.14) \quad \|f \overset{\gamma}{*} g\|_{L_r(\mathbb{R}_+)} \leq \frac{2^{(p-1)/p}}{\pi^2} \|f\|_{L_p^{-p,\beta}(\mathbb{R}_+)} \|g\|_{L_q(\mathbb{R}_+)}.$$

### 3. A WATSON TYPE THEOREM

An important part of the integral transforms theory is to study unitary transforms. In this section, for a class of differential operators of infinite order, we give a condition on the kernel  $h$  such that the convolution transformation (3.3) defines a unitary operator in  $L_2(\mathbb{R}_+)$ , and calculate the inverse transformation.

By an argument similar to that in the proof of Theorem 2.1, one can easily prove the following lemma.

**Lemma 3.1.** *Let  $h \in L_2^{-2,\beta}(\mathbb{R}_+)$ ,  $0 < \beta \leq 1$ , and  $f \in L_2(\mathbb{R}_+)$ . Then the generalized convolution (1.11) satisfies the factorization equality (1.12). Furthermore, the following generalized Parseval identity holds:*

$$(3.1) \quad (h \overset{\gamma}{*} f)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{y \sinh \pi y} K^{-1}[h](y) (F_c f)(y) \cos xy \, dy,$$

where the integral is understood in the  $L_2(\mathbb{R}_+)$  norm, if necessary.

**Theorem 3.1.** *Let  $h \in L_2^{-2,\beta}(\mathbb{R}_+)$ ,  $0 < \beta \leq 1$ . Then the condition*

$$(3.2) \quad |K^{-1}[h](\tau)| = \frac{1}{\cosh(\pi\tau)}$$

is necessary and sufficient for the transformation  $f \mapsto g$  given by formula

$$(3.3) \quad g(x) = \frac{d^2}{\pi^2 dx^2} \prod_{k=0}^{\infty} \left(1 - \frac{4 d^2}{k^2 dx^2}\right) \int_0^\infty \int_0^\infty \frac{1}{u} (e^{-u \cosh(x+v)} + e^{-u \cosh(x-v)}) h(u) f(v) \, du \, dv,$$

to be unitary on  $L_2(\mathbb{R}_+)$ . Moreover, the inverse transformation can be written in the symmetric form

(3.4)

$$f(x) = \lim_{N \rightarrow \infty} \frac{d^2}{\pi^2 dx^2} \prod_{k=0}^N \left(1 - \frac{4d^2}{k^2 dx^2}\right) \int_0^\infty \int_0^\infty \frac{1}{u} (e^{-u \cosh(x+v)} + e^{-u \cosh(x-v)}) \bar{h}(u) f(v) du dv.$$

Here, the limit is understood in the  $L_2(\mathbb{R}_+)$  norm.

*Proof. Sufficiency.* Suppose that the function  $h$  satisfies condition (3.2). Applying Lemma 3.1, it is easy to see that the generalized convolution transform (3.3) can be written in the form

(3.5)

$$g(x) = \sqrt{\frac{2}{\pi}} \lim_{N \rightarrow \infty} \frac{d^2}{dx^2} \prod_{k=0}^N \left(1 - \frac{4d^2}{k^2 dx^2}\right) \int_0^\infty \frac{1}{y \sinh \pi y} (K_{iy}[h])(F_c f)(y) \cos xy dy,$$

or equivalently,  $g(x) = \lim_{N \rightarrow \infty} g_N(x)$ , where

$$g_N(x) = \frac{d^2}{dx^2} \prod_{k=0}^N \left(1 - \frac{4d^2}{k^2 dx^2}\right) F_c \left( \frac{1}{y \sinh \pi y} (K_{iy}[h])(F_c f)(y) \right) (x).$$

It is well-known that  $h(y)$ ,  $yh(y)$ ,  $y^2h(y) \in L_2(\mathbb{R}_+)$  if and only if  $(Fh)(x)$ ,  $(d(Fh)(x)/dx)$ ,  $(d^2(Fh)(x)/dx^2) \in L_2(\mathbb{R}_+)$  (Theorem 68, page 92, [11]). Therefore,  $h(y)$ ,  $yh(y)$ ,  $y^2h(y), \dots, y^n h(y) \in L_2(\mathbb{R}_+)$  if and only if  $(Fh)(x)$ ,  $(d(Fh)(x)/dx)$ ,  $(d^2(Fh)(x)/dx^2), \dots, (d^n(Fh)(x)/dx^n) \in L_2(\mathbb{R}_+)$ . Moreover, for each positive integer  $n$  we have

$$\frac{d^{2n}}{dx^{2n}} (Fh)(x) = (-1)^n F(y^{2n}h(y))(x).$$

Therefore, if  $y^2 \prod_{k=0}^N (1 + 4y^2/k^2)h(y) \in L_2(\mathbb{R}_+)$  then the following formula holds:

$$(3.6) \quad \frac{d^2}{dx^2} \prod_{k=0}^N \left(1 - \frac{4d^2}{k^2 dx^2}\right) (F_c h)(x) = -F_c \left[ y^2 \prod_{k=0}^N \left(1 + \frac{4y^2}{k^2}\right) h(y) \right] (x).$$

From condition (3.2) and the infinite product form of  $\sinh z$  (see formula (4.5.68) in [1]) we have

$$\left| y^2 \prod_{k=0}^N (1 + (4y^2/k^2))(1/y \sinh \pi y) K^{-1}[h](y) \right| = 1 / \prod_{k=N+1}^\infty (1 + (4y^2/k^2)) < 1,$$

and hence it is bounded. Therefore

$$y^2 \prod_{k=0}^N \left(1 + \frac{4y^2}{k^2}\right) \frac{1}{y \sinh \pi y} K^{-1}[h](y)(F_c f)(y) \in L_2(\mathbb{R}_+),$$

and formula (3.6) yields

$$g_N(x) = F_c \left[ y^2 \prod_{k=0}^N \left(1 + \frac{4y^2}{k^2}\right) \frac{1}{y \sinh \pi y} K^{-1}[h](y)(F_c f)(y) \right] (x) \in L_2(\mathbb{R}_+).$$

This shows that  $g_N$  belongs to  $L_2(\mathbb{R}_+)$ . Applying the Fourier cosine transform to both sides of the above relation, we have

$$(F_c g_N)(y) = y^2 \prod_{k=0}^N \left(1 + \frac{4y^2}{k^2}\right) \frac{1}{y \sinh \pi y} K^{-1}[h](y)(F_c f)(y).$$

Besides, from the Parseval equality for the Fourier cosine transform  $\|F_c f\|_{L_2(\mathbb{R}_+)} = \|f\|_{L_2(\mathbb{R}_+)}$ , it follows that

$$\|F_c g_N - F_c g\|_{L_2(\mathbb{R}_+)} = \|g_N - g\|_{L_2(\mathbb{R}_+)} \rightarrow 0, \quad N \rightarrow \infty.$$

Therefore, using formula (4.5.68) in [1] we conclude that

$$\begin{aligned} (F_c g)(y) &= y^2 \prod_{k=0}^{\infty} \left(1 + \frac{4y^2}{k^2}\right) \frac{1}{y \sinh \pi y} K^{-1}[h](y)(F_c f)(y) \\ &= y \sinh 2\pi y \frac{1}{2y \sinh \pi y} K^{-1}[h](y)(F_c f)(y) \\ &= \cosh \pi y K^{-1}[h](y)(F_c f)(y). \end{aligned}$$

From condition (3.2), it is easy to see that  $|(F_c g)(y)| \equiv |(F_c f)(y)|$ , then  $\|f\|_{L_2(\mathbb{R}_+)} = \|g\|_{L_2(\mathbb{R}_+)}$ , which implies that the transform (3.3) is unitary. Again from condition (3.2) we obtain

$$\cosh \pi y K^{-1}[\bar{h}](y)(F_c g)(y) = (F_c f)(y).$$

Thus, in the same manner as above it corresponds to (3.4) and the inversion formula of the transform (3.3) follows.

*Necessity.* Suppose that transform (3.3) is unitary on  $L_2(\mathbb{R}_+)$  and the inversion formula is defined by (3.4). Then using the Parseval type identity (3.1), the Parseval identity for the Fourier cosine transform, and formula (4.5.68) in [1] we obtain

$$\|g\|_{L_2(\mathbb{R}_+)} = \|\cosh \pi y K^{-1}[h](y)(F_c f)(y)\|_{L_2(\mathbb{R}_+)} = \|F_c f\|_{L_2(\mathbb{R}_+)} = \|f\|_{L_2(\mathbb{R}_+)}.$$

The middle equality holds for all  $f \in L_2(\mathbb{R}_+)$  if and only if  $h$  satisfies the condition (3.2). This completes the proof of the theorem.  $\square$

#### 4. A CLASS OF INTEGRO-DIFFERENTIAL PROBLEMS

In spite of having many useful applications (see [7]), not many integro-differential equations can be solved in closed form. No application of convolution type transforms of solving integro-differential was presented in recent investigations [3], [4], [13], [15]. In this section, we apply a general class of Fourier cosine and Kontorovich-Lebedev generalized convolution transforms to solve a class of integro-differential problems, which seems to be difficult to solve in closed form by using other techniques. Namely, in case  $D = \left(\frac{d^2}{dx^2}\right) \prod_{k=1}^{n-1} \left(-\frac{d^2}{dx^2} + k^2\right)$ , the transform  $f \mapsto K_h(f) := D(h \overset{\gamma}{*} f)$  is of the form

$$(4.1) \quad (K_h f)(x) = \frac{1}{\pi^2} \frac{d^2}{dx^2} \prod_{k=1}^{n-1} \left(-\frac{d^2}{dx^2} + k^2\right) \int_0^\infty \int_0^\infty \frac{1}{u} [e^{-u \cosh(x+v)} + e^{-u \cosh(x-v)}] h(u) f(v) du dv.$$

We consider the integro-differential problem

$$(4.2) \quad \begin{aligned} f(x) + (K_h f)(x) &= g(x), \\ \frac{d^{2k-1}}{dx^{2k-1}} f(0) &= 0, \quad k = \overline{1, n}, \\ \lim_{x \rightarrow \infty} f^{(k)}(x) &= 0, \quad k = \overline{0, 2n-1}. \end{aligned}$$

Here,  $h, g$  are given functions in  $L_1(\mathbb{R}_+)$ , and  $f$  is the unknown function.

In order to give a solution of the above problem, note that, for  $h \in L_1(\mathbb{R}_+)$  such that  $h(0) = 0, \lim_{x \rightarrow \infty} h'(x) = 0$ , the Fourier sine and Fourier cosine transforms of  $h, h'$  exist. Furthermore,

$$(4.3) \quad \begin{aligned} (F_s h')(y) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty h'(x) \sin xy \, dx \\ &= \frac{1}{\sqrt{2\pi}} \left\{ h(x) \sin xy \Big|_0^\infty - y \int_0^\infty h(x) \cos xy \, dx \right\} \\ &= -y(F_c h)(y), \end{aligned}$$

and

$$(4.4) \quad (F_c h')(y) = \frac{1}{\sqrt{2\pi}} \int_0^\infty h'(x) \cos xy \, dx = y(F_s h)(y).$$

**Theorem 4.1.** *Suppose the following condition holds:*

$$(4.5) \quad 1 - \frac{(2n-1)!}{\sqrt{2\pi} \cdot 2^{2n-1}} F_c \left( h \overset{\gamma}{*} \frac{1}{\cosh^{2n} \tau/2} \right) (y) \neq 0, \quad \forall y > 0.$$

Then problem (4.2) has a unique solution in  $L_1(\mathbb{R}_+)$  whose closed form is

$$(4.6) \quad f(x) = g(x) + (g \underset{F_c}{*} l)(x),$$

where  $l \in L_1(\mathbb{R}_+)$  is defined by

$$(F_c l)(y) = \frac{((2n-1)!/\sqrt{2\pi} \cdot 2^{2n-1}) F_c(h \overset{\gamma}{*} \cosh^{-2n} \tau/2)(y)}{1 - ((2n-1)!/\sqrt{2\pi} \cdot 2^{2n-1}) F_c(h \overset{\gamma}{*} \cosh^{-2n} \tau/2)(y)}.$$

*Proof.* The equation (4.2) can be rewritten in the form

$$(4.7) \quad f(x) + \frac{d^2}{dx^2} \prod_{k=1}^{n-1} \left( -\frac{d^2}{dx^2} + k^2 \right) \{ (h \overset{\gamma}{*} f)(x) \} = g(x).$$

Applying the Fourier cosine transform to both sides of (4.7), by original conditions (4.2) and by virtue of the factorization equality (1.12) and formulas (4.3), (4.4) we obtain

$$(4.8) \quad (F_c f)(y) - y^2 \prod_{k=1}^{n-1} (y^2 + k^2) \cdot \frac{1}{y \sinh \pi y} K^{-1}[h](y) (F_c f)(y) = (F_c g)(y).$$

Using formula (see relation (1.9.3) in [5])

$$F_c \left( \frac{1}{\cosh^{2n} \tau/2} \right) (y) = \frac{\sqrt{2\pi} \cdot 2^{2n-1} y}{(2n-1)! \sinh \pi y} \prod_{k=1}^{n-1} (y^2 + k^2),$$

we have

$$(F_c f)(y) - \frac{(2n-1)!}{\sqrt{2\pi} \cdot 2^{2n-1}} F_c \left( \frac{1}{\cosh^{2n} \tau/2} \right) (y) K^{-1}[h](y) (F_c f)(y) = (F_c g)(y),$$

or equivalently,

$$(F_c f)(y) \left[ 1 - \frac{(2n-1)!}{\sqrt{2\pi} \cdot 2^{2n-1}} F_c \left( h(\tau) \overset{\gamma}{*} \frac{1}{\cosh^{2n} \tau/2} \right) (y) \right] = (F_c g)(y).$$

From condition (4.5) we get

$$(4.9) \quad (F_c f)(y) = \left( 1 + \frac{((2n-1)!/\sqrt{2\pi} \cdot 2^{2n-1})F_c(h \underset{F_c}{*} \cosh^{-2n} \tau/2)(y)}{1 - ((2n-1)!/\sqrt{2\pi} \cdot 2^{2n-1})F_c(h \underset{F_c}{*} \cosh^{-2n} \tau/2)(y)} \right) (F_c g)(y).$$

Recall that the Wiener-Levy theorem [8] states that if  $f$  is the Fourier transform of an  $L_1(\mathbb{R})$  function, and  $\varphi$  is analytic in a neighborhood of the origin that contains the domain  $\{f(y), \forall y \in \mathbb{R}\}$ , and  $\varphi(0) = 0$ , then  $\varphi(f)$  is also the Fourier transform of an  $L_1(\mathbb{R})$  function. For the Fourier cosine transform it means that if  $f$  is the Fourier cosine transform of an  $L_1(\mathbb{R}_+)$  function, and  $\varphi$  is analytic in a neighborhood of the origin that contains the domain  $\{f(y), \forall y \in \mathbb{R}_+\}$ , and  $\varphi(0) = 0$ , then  $\varphi(f)$  is also the Fourier cosine transform of an  $L_1(\mathbb{R}_+)$  function.

By the given condition (4.5) the function  $\varphi(z) = z/(1+z)$  satisfies the conditions of the Wiener-Levy theorem, and therefore, there exists a unique function  $l \in L_1(\mathbb{R}_+)$  such that

$$(F_c l)(y) = \frac{((2n-1)!/\sqrt{2\pi} \cdot 2^{2n-1})F_c(h \underset{F_c}{*} \cosh^{-2n} \tau/2)(y)}{1 - ((2n-1)!/\sqrt{2\pi} \cdot 2^{2n-1})F_c(h \underset{F_c}{*} \cosh^{-2n} \tau/2)(y)}.$$

Therefore the equation (4.9) becomes

$$(F_c f)(y) = (1 + (F_c l)(y))(F_c g) = F_c(g + g \underset{F_c}{*} l)(y),$$

which implies  $f(x) = g(x) + (g \underset{F_c}{*} l)(x)$ . The proof is complete.  $\square$

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