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ON THE CLASS OF ORDER DUNFORD-PETTIS OPERATORS

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Abstract. We characterize Banach lattices $E$ and $F$ on which the adjoint of each operator from $E$ into $F$ which is order Dunford-Pettis and weak Dunford-Pettis, is Dunford-Pettis. More precisely, we show that if $E$ and $F$ are two Banach lattices then each order Dunford-Pettis and weak Dunford-Pettis operator $T$ from $E$ into $F$ has an adjoint Dunford-Pettis operator $T'$ from $F'$ into $E'$ if, and only if, the norm of $E'$ is order continuous or $F'$ has the Schur property. As a consequence we show that, if $E$ and $F$ are two Banach lattices such that $E$ or $F$ has the Dunford-Pettis property, then each order Dunford-Pettis operator $T$ from $E$ into $F$ has an adjoint $T': F' \to E'$ which is Dunford-Pettis if, and only if, the norm of $E'$ is order continuous or $F'$ has the Schur property.

Keywords: Dunford-Pettis operator, weak Dunford-Pettis operator, order Dunford-Pettis operator, order continuous norm, Schur property

MSC 2010: 46B40, 46B42, 47B60

1. INTRODUCTION

The problem discussed in the article [5] was to impose conditions on Banach lattices, $E$ and $F$, and the operator $T$ from $E$ to $F$ for its adjoint operator $T'$ to be weak Dunford-Pettis. In this paper, we continue our research on this way and give necessary and sufficient conditions on $E$, $F$ and $T$ to have a Dunford-Pettis adjoint operator $T'$. More precisely, we show that if $E$ and $F$ are two Banach lattices then each order Dunford-Pettis and weak Dunford-Pettis operator $T$ from $E$ into $F$ has an adjoint Dunford-Pettis operator $T'$ from $F'$ into $E'$ if, and only if, the norm of $E'$ is order continuous or $F'$ has the Schur property (Theorem 3.1). Our theorem, Theorem 3.1, appears to be a reformulation of Theorems 3.2 and 3.5 in [5] in the following sense. In the sufficient condition of Theorem 3.2 [5], the authors give the condition of AM-compactness property of spaces $E$ and $F$. However, under these
conditions, a positive weak Dunford-Pettis operator is an order and weak Dunford-Pettis operator. This shows that Theorem 3.2 [5] can be easily deduced from our Theorem 3.1 and the conditions that were sufficient are also necessary. Theorem 3.5 [1] which gives a necessary condition is also included in our theorem in the way that the conditions that were only necessary became also sufficient if the operator is supposed to be order Dunford-Pettis. Hence the importance of Theorem 3.1 given in this article.

2. Preliminaries and notation

In [2] K.T. Andrews said that a norm bounded subset \( A \) of a Banach space \( X \) is a Dunford-Pettis set whenever every weakly compact operator from \( X \) to an arbitrary Banach space carries \( A \) to a norm totally bounded set. Alternatively, a norm bounded subset \( A \) of a Banach lattice \( E \) is said to be a Dunford-Pettis set if every weakly null sequence \((f_n)\) of \( E \) converges uniformly to zero on the set \( A \), that is, \( \sup_{x \in A} |f_n(x)| \to 0 \) (see Theorem 5.98 of [1]). On the other hand, a Banach space \( X \) is said to have the Dunford-Pettis property if every weakly compact operator \( T \) defined on \( E \) and taking values in a Banach space \( F \) is Dunford-Pettis. For example, the Banach space \( \ell^\infty \) has the Dunford-Pettis property but the Banach space \( \ell^\infty(\ell^2) \) does not have the Dunford-Pettis property.

Based on the concept of Dunford-Pettis sets, the class of order Dunford-Pettis operators is defined in [4]. In fact, an operator \( T \) from a Banach lattice \( E \) into a Banach space \( X \) is said to be order Dunford-Pettis if it carries each order bounded subset of \( E \) into a Dunford-Pettis set of \( X \), i.e., if for each \( x \in E^+ \), the subset \( T([-x, x]) \) is Dunford-Pettis in \( X \).

Let \( X \) and \( Y \) be two Banach spaces. An operator \( T: X \to Y \) is called a Dunford-Pettis operator if \( T \) carries weakly convergent sequences to norm convergent sequences. (Equivalently, for each weakly null sequence \((x_n)\) we have \( \lim_{n \to \infty} \|T(x_n)\| = 0 \). Alternatively, an operator \( T: X \to Y \) is a Dunford-Pettis operator if and only if \( T \) carries relatively weakly compact sets to norm totally bounded sets.

On the other hand, unlike compact operators, there are operators \( T \) from a Banach space \( X \) into another \( Y \) whose dual operators \( T' \) from \( Y' \) into \( X' \) are not Dunford-Pettis. In fact, the dual operator of the identity operator of the Banach space \( \ell^1 \), which is the identity of the Banach space \( \ell^\infty \), is not Dunford-Pettis.

Recall from [1] that an operator \( T \) from a Banach space \( X \) into another \( Y \) is said to be weak Dunford-Pettis if \( y_n(T(x_n)) \) converges to 0 whenever \((x_n)\) converges weakly to 0 in \( X \) and \((y_n)\) converges weakly to 0 in \( Y \). Alternatively, \( T \) is weak...
Dunford-Pettis if the composed operator $S \circ T$ is Dunford-Pettis for each weakly compact operator $S$ from $Y$ into $G$, for an arbitrary Banach space $G$.

The latter class of operators was connected in Theorem 5.98 of [1] with the class of the Dunford-Pettis sets.

Let us recall that an operator $T$ from a Banach lattice $E$ into a Banach space $X$ is said to be AM-compact if it carries each order-bounded subset of $E$ onto a relatively compact subset of $X$. In [3], the Banach lattice $E$ is said to have the AM-compactness property if every weakly compact operator defined on $E$, and taking values in a Banach space $X$, is AM-compact. For example, the Banach lattice $L^2([0,1])$ does not have the AM-compactness property, but $\ell^1$ has the AM-compactness property.

It follows from Proposition 3.1 of [3] that a Banach lattice $E$ has the AM-compactness property if and only if for every weakly null sequence $(f_n)$ of $E$ we have $|f_n| \to 0$ for $\sigma(E',E)$.

On the other hand, it is well known that there exist weak Dunford-Pettis operators whose adjoints are not Dunford-Pettis. In fact, let us consider the Banach lattice $\ell^1$: its identity operator $\text{Id}_{\ell^1}: \ell^1 \to \ell^1$ is weak Dunford-Pettis while its dual operator $\text{Id}_{\ell^\infty}: \ell^\infty \to \ell^\infty$ is not Dunford-Pettis. Also, there exist order Dunford-Pettis operators whose adjoints are not Dunford-Pettis. In fact, as the Banach space $\ell^2$ has the AM-compactness property, the identity operator $\text{Id}_{\ell^2}$ is order Dunford-Pettis, but its dual operator, which is the identity operator of $\ell^2$, is not Dunford-Pettis (because the Banach space $\ell^2$ does not have the Schur property). However, we will prove that each operator is weak Dunford-Pettis and also order Dunford-Pettis if its adjoint is.

To state our results, we need to fix some notation and recall some definitions. A Banach lattice is a Banach space $(E, \| \cdot \|)$ such that $E$ is a vector lattice and its norm satisfies the following condition: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. A norm $\| \cdot \|$ of a Banach lattice $E$ is order continuous if for each generalized sequence $(x_\alpha)$ such that $x_\alpha \downarrow 0$ in $E$, $(x_\alpha)$ converges to 0 for the norm $\| \cdot \|$ where the notation $x_\alpha \downarrow 0$ means that $(x_\alpha)$ is decreasing, its infimum exists and $\inf(x_\alpha) = 0$. A vector lattice $E$ is Dedekind $\sigma$-complete if every majorized countable nonempty subset of $E$ has a supremum. A Banach lattice $E$ has the Schur property if each weakly null sequence in $E$ converges to zero in the norm. For example, the Banach lattice $\ell^1$ has the Schur property but the Banach lattice $L^1([0,1])$ does not have the Schur property. Note that if $E$ is a Banach lattice, its topological dual $E'$, endowed with the dual norm and the dual order, is also a Banach lattice.

We will use the term operator $T: E \to F$ between two Banach lattices to mean a bounded linear mapping. It is positive if $T(x) \geq 0$ in $F$ whenever $x \geq 0$ in $E$. The operator $T$ is regular if $T = T_1 - T_2$ where $T_1$ and $T_2$ are positive operators from $E$ into $F$. Note that each positive linear mapping on a Banach lattice is continuous.

If an operator $T: E \to F$ between two Banach lattices is positive, then its adjoint
\(T': F' \rightarrow E'\) is likewise positive, where \(T'\) is defined by \(T'(f)(x) = f(T(x))\) for each \(f \in F'\) and for each \(x \in E\).

For terminology concerning Banach lattice theory and positive operators we refer the reader to the excellent book of Aliprantis-Burkinshaw [1].

3. Main results

Let \(X\) and \(Y\) be two Banach spaces, and let \(E\) be a Banach lattice. We denote:
\[
\begin{align*}
\text{wDP}(X, Y), & \quad \text{the space of all weak Dunford-Pettis operators from } X \text{ into } Y, \\
oDP(E, Y), & \quad \text{the space of all order Dunford-Pettis operators from } E \text{ into } Y \text{ and } \\
\text{DP}(X, Y), & \quad \text{the space of all Dunford-Pettis operators from } X \text{ into } Y.
\end{align*}
\]

To give the proof of Proposition 3.1, we need the following lemma

Lemma 3.1. Let \(A\) be a bounded subset of a Banach space \(X\). If for each \(\varepsilon > 0\) there exists a Dunford-Pettis set \(A_\varepsilon\) in \(X\) such that \(A \subseteq A_\varepsilon + \varepsilon B_X\) (where \(B_X\) is the closed unit ball of \(X\)), then \(A\) is a Dunford-Pettis set.

Proof. Let \(Y\) be a Banach space and let \(T: X \rightarrow Y\) be a weakly compact operator. We have to prove that \(T(A)\) is relatively compact in \(Y\). Let \(\varepsilon > 0\), then by hypothesis there exists a Dunford-Pettis subset \(A_\varepsilon\) of \(X\) such that \(A \subseteq A_\varepsilon + \varepsilon B_X\), and then \(T(A) \subseteq T(A_\varepsilon) + \varepsilon\|T\| B_Y\). Now as \(A_\varepsilon\) is a Dunford-Pettis set, \(T(A_\varepsilon)\) is relatively compact in \(Y\) and hence by Theorem 3.1 of [1], \(T(A)\) is relatively compact in \(Y\). This shows that \(A\) is a Dunford-Pettis set. \(\Box\)

Proposition 3.1. Let \(E\) and \(F\) be two Banach lattices, and let \(X\) be a Banach space. Then
\[
\begin{align*}
(1) \quad & \text{oDP}(E, X) \text{ is a norm closed vector subspace of the space } L(E, X) \text{ of all operators from } E \text{ into } X, \\
(2) & \text{if } T: E \rightarrow F \text{ is an order Dunford-Pettis operator, then for each operator } S: F \rightarrow X, \text{ the composed operator } S \circ T \text{ is order Dunford-Pettis}, \\
(3) & \text{if } T: E \rightarrow F \text{ is an order bounded operator, then for each order Dunford-Pettis operator } S: F \rightarrow X, \text{ the composed operator } S \circ T \text{ is order Dunford-Pettis}.
\end{align*}
\]

Proof. (1) Clearly, \(oDP(E, X)\) is a vector subspace of \(L(E, X)\). To see that \(oDP(E, X)\) is also norm closed, let \(S\) be in the norm closure of \(oDP(E, X)\). To this end, let \(x\) be a nonzero in \(E^+\) and \(\varepsilon > 0\). Choose some \(T \in oDP(E, X)\) satisfying \(\|S - T\| \leq \varepsilon/\|x\|\), and observe that \(S([-x, x]) \subseteq T([-x, x]) + \varepsilon B_X\) holds. Since \(T\) is order Dunford-Pettis, \(T([-x, x])\) is a Dunford-Pettis set and hence by Lemma 3.1 \(S([-x, x])\) is a Dunford-Pettis set. This shows that \(S\) is order Dunford-Pettis.
(2) Let $T: E \to F$ be an order Dunford-Pettis operator. Then for each $x \in E^+$, $T([-x,x])$ is a Dunford-Pettis set in $F$ and hence $S(T[-x,x])$ is a Dunford-Pettis set in $X$. So, $S \circ T$ is order Dunford-Pettis.

(3) Let $T: E \to F$ be an order bounded operator. Then for each $x \in E^+$, $T([-x,x])$ is an order interval and since $S$ is order Dunford-Pettis, $S(T[-x,x])$ is a Dunford-Pettis set in $X$. Hence $S \circ T$ is order Dunford-Pettis. □

**Proposition 3.2.** Let $E$ be a Banach lattice and $X$ a Banach space. If the norm of $E$ is order continuous and $X$ has the Dunford-Pettis property then each operator $T$ from $E$ into $X$ is order Dunford-Pettis.

**Proof.** Since the norm of $E$ is order continuous, it follows from Theorem 2.4.3 of [7] that for each $x \in E^+$, the order interval $[-x,x]$ is weakly compact. If $T: E \to X$ is an operator, then $T([-x,x])$ is weakly compact in $X$.

On the other hand, since $X$ has the Dunford-Pettis property, the identity operator of $X$ is weak Dunford-Pettis and hence by Theorem 5.99 of [1], $T([-x,x])$ is a Dunford-Pettis set. This shows that $T$ is order Dunford-Pettis. □

The following proposition gives some characterizations of order Dunford-Pettis operators

**Proposition 3.3 ([4]).** Let $T$ be an operator from a Banach lattice $E$ into a Banach space $X$. Then the following assertions are equivalent:

1. $T$ is an order Dunford-Pettis operator,
2. for each weakly compact operator $S$ from $X$ into an arbitrary Banach space $Z$, the composed operator $S \circ T$ is AM-compact,
3. for each weakly null sequence $(f_n)$ in $X'$ we have $|T'(f_n)| \to 0$ for $\sigma(E',E)$.

There exist operators that are not order Dunford-Pettis. In fact, the identity operator of the Banach lattice $L^2([0,1])$ is not order Dunford-Pettis. The following result gives a characterization of a Banach lattice which has the AM-compactness property.

**Proposition 3.4.** Let $E$ be a Banach lattice. Then the following statements are equivalent:

1. each positive operator from $E$ into $E$ is order Dunford-Pettis,
2. the identity operator of $E$ is order Dunford-Pettis,
3. $E$ has the AM-compactness property.

**Proof.** (1) $\implies$ (2) Obvious.

(2) $\implies$ (3) Let $x \in E^+$ and let $T: E \to X$ be a weakly compact operator where $X$ is arbitrary Banach space.
Since the identity operator of $E$ is an order Dunford-Pettis, $[-x, x]$ is a Dunford-Pettis set in $E$ and hence $T([−x, x])$ is relatively compact. This shows that $T$ is AM-compact and hence $E$ has the AM-compactness property.

(3) $\implies$ (1) Let $T: E \to E$ be a positive operator and $S: E \to Z$ a weakly compact operator where $Z$ is an arbitrary Banach space. Since $E$ has the AM-compactness property, the operator $S$ is AM-compact and hence $S \circ T$ is AM-compact. Finally, it follows from Proposition 3.3 that $T$ is order Dunford-Pettis. □

**Proposition 3.5.** Let $T$ be an operator from a Banach lattice $E$ into a Banach space $F$. If $T' \in DP(F', E')$, then $T \in oDP(E, F)$.

**Proof.** Let $(f_n)$ be a sequence of $F'$ such that $f_n \to 0$ in the weak topology $\sigma(F', F'')$.

As the adjoint $T'$ is Dunford-Pettis from $F'$ into $E'$, we deduce that $T'(f_n) \to 0$ for the norm of $E'$ and hence $|T'(f_n)| \to 0$ for $\sigma(E', E)$. Finally, by Proposition 3.3, we deduce that $T$ is order Dunford-Pettis. □

**Proposition 3.6.** Let $T$ be an operator from a Banach lattice $E$ into a Banach space $F$. If $T' \in DP(F', E')$, then $T \in wDP(E, F)$.

**Proof.** Let $(x_n)$ (resp. $(f_n)$) be a sequence of $E$ (of $F'$) such that $x_n \to 0$ in the weak topology $\sigma(E, E')$ (f_n \to 0 in $\sigma(F', F'')$). We have to prove that $f_n(T(x_n)) \to 0$. As $(f_n)$ is a sequence of $F'$ such that $f_n \to 0$ in $\sigma(F', F'')$ and hence $T'$ is Dunford-Pettis then $T'(f_n) \to 0$ for the norm of $E'$.

On the other hand, since $x_n \to 0$ in the weak topology $\sigma(E, E')$ hence $(x_n)$ is norm bounded and by the inequality $|T'(f_n)(x_n)| \leq \|T'(f_n)||_{E'}$, we conclude that $T$ is weak Dunford-Pettis. □

**Theorem 3.1.** Let $E$ and $F$ be two Banach lattices. Then the following assertions are equivalent:

(1) each order Dunford-Pettis and weak Dunford-Pettis operator $T$ from $E$ into $F$ has an adjoint Dunford-Pettis operator $T'$ from $F'$ into $E'$,

(2) one of the following is valid:
   (a) the norm of $E'$ is order continuous,
   (b) $F'$ has the Schur property.

**Proof.** (1) $\implies$ (2) Assume that (2) is false, i.e., the norm of $E'$ is not order continuous and $F'$ does not have the Schur property. We will construct an operator $T: E \to F$ which is weak Dunford-Pettis and order Dunford-Pettis but its adjoint $T': F' \to E'$ is not Dunford-Pettis. Indeed, suppose that $E'$ does not have an order continuous norm. By Theorem 2.4.14 of [7] we may assume that $\ell^1$ is a closed
Dunford-Pettis property. Then the following assertions are equivalent: $x$ is order continuous, it follows from Corollary 2.9 of Dodds and Fremlin [6] that we have $T'$ is the weakly null sequence $(P_n)$ of $F^+$ with $\|y_n\| \leq 1$, and an $\varepsilon > 0$ such that $|f_n(y_n)| \geq \varepsilon$ for all $n$.

Now, we consider the operator $T = S \circ P: E \to \ell^1 \to F$ where $S$ is the operator defined by

$$S: \ell^1 \to F, (\alpha_n) \to \sum_n \alpha_n y_n.$$ 

Since $\ell^1$ has the Dunford-Pettis property, the operator $T$ is weak Dunford-Pettis. Also, $T$ is order Dunford-Pettis. In fact, since $\ell^1$ is discrete and its norm is order continuous, it is clear that $P([-x, x])$ is relatively compact in $\ell^1$. Then $T = S \circ P([-x, x])$ is relatively compact in $F$ and hence there is a Dunford-Pettis set in $F$ for each $x \in E_+$. Finally, we conclude that $T$ is order Dunford-Pettis.

But the adjoint $T': F' \to E'$ is not Dunford-Pettis. Indeed, the sequence $(f_n)$ is weakly null in $F'$. And as the operator $P: E \to \ell^1$ is surjective, there exists $\delta > 0$ such that $\delta \cdot B_{\ell^1} \subset P(B_E)$ where $B_H$ is the closed unit ball of $H = E$ or $\ell^1$. Hence

$$\|T'(f_n)\| = \sup_{x \in B_E}|T'(f_n)(x)| = \sup_{x \in B_E}|f_n(T(x))|$$

$$= \sup_{x \in B_E}|f_n \circ S(P(x))| \geq \delta \cdot |f_n \circ S((e_n))| \geq \delta \cdot |f_n(y_n)| > \delta \cdot \varepsilon$$

(where $(e_n)_{n=1}^\infty$ is the canonical basis of $\ell^1$). Then $\|T'(f_n)\| > \delta \cdot \varepsilon$ for all $n$, and we conclude that $T'$ is not Dunford-Pettis. This presents a contradiction.

$(2; a) \implies (1)$ Let $(f_n)$ be a disjoint sequence of $F'$ such that $(f_n) \to 0$ in $\sigma(F', F'')$. We have to prove that $(T'(f_n))$ converges to $0$ for the norm of $E'$. By using Corollary 2.7 of Dodds-Fremlin [6], it suffices to prove that $|T'(f_n)| \to 0$ in $\sigma(E', E)$ and $T'(f_n)(x_n) \to 0$ for every norm bounded disjoint sequence $(x_n) \in E_+$. In fact, as $(f_n)$ is a weakly null sequence in $F'$ and since $T$ is order Dunford-Pettis we have $|T'(f_n)| \to 0$ for $\sigma(E', E)$. On the other hand, since the norm of $E'$ is order continuous, it follows from Corollary 2.9 of Dodds and Fremlin [6] that $x_n \to 0$ in $\sigma(E, E')$. Hence, as $T$ is a weak Dunford-Pettis operator, we obtain

$T'(f_n)(x_n) = f_n(T(x_n)) \to 0$, and this proves that $T'$ is Dunford-Pettis.

$(2; b) \implies (1)$ Obvious.

**Corollary 3.1.** Let $E$ and $F$ be two Banach lattices such that $E$ or $F$ has the Dunford-Pettis property. Then the following assertions are equivalent:

1. Each order Dunford-Pettis operator $T$ from $E$ into $F$ has an adjoint Dunford-Pettis operator from $F'$ into $E'$,
(2) one of the following is valid:
(a) the norm of $E'$ is order continuous,
(b) $F'$ has the Schur property.

As consequences of Theorem 3.1 and Proposition 3.4, we obtain the following result:

**Corollary 3.2.** Let $E$ and $F$ be two Banach lattices such that $E$ has the AM-compactness property. Then the following assertions are equivalent:
(1) each weak Dunford-Pettis operator $T$ from $E$ into $F$ has an adjoint Dunford-Pettis operator from $F'$ into $E'$,
(2) one of the following is valid:
(a) the norm of $E'$ is order continuous,
(b) $F'$ has the Schur property.

As consequences of Theorem 3.1 and Proposition 3.2, we obtain the following result:

**Corollary 3.3.** Let $E$ and $F$ be two Banach lattices such that the norm of $E$ is order continuous and $F$ has the Dunford-Pettis property. Then the following assertions are equivalent:
(1) each operator $T$ from $E$ into $F$ has an adjoint which is Dunford-Pettis,
(2) one of the following is valid:
(a) the norm of $E'$ is order continuous,
(b) $F'$ has the Schur property.

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