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FINDING TARGET UNITS IN FDH MODEL BY LEAST-DISTANCE MEASURE MODEL

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Recently, some authors used the Least-Distance Measure model in order to obtain the shortest distance between the evaluated Decision Making Unit (DMU) and the strongly efficient production frontier. But, their model is not applicable for situation in which the production possibility set satisfies free disposability property. In this paper, we propose a new approach to this end in FDH model which improves the application potential of the Least-Distance Measure and overcomes the mentioned shortcoming. The applicability of the proposed method is illustrated with two numerical examples and proves to be persuasive and acceptable to real-world problem.

Keywords: data envelopment analysis, least distance, FDH, target unit

Classification: 90C05, 90BXX

1. INTRODUCTION

Data Envelopment Analysis (DEA), first proposed by Charnes et al. [4] is a non-parametric approach to evaluate the performance or efficiency of various organizations in public and private sectors with multiple inputs and multiple outputs. Based on information about existing data on the performance of the units and some preliminary assumptions, DEA forms an empirical efficient frontier. If a DMU lies on the frontier, it is referred to as an efficient unit, otherwise inefficient. DEA also provides efficiency scores and reference set for inefficient DMU. The efficiency scores are used in practical applications as performance indicators of the DMUs. The reference set for inefficient units consists of efficient units and determines a virtual unit on the efficient surface. The virtual unit can be regarded as a target unit for the inefficient unit. The target unit is found in DEA by projecting an inefficient DMU radially to the efficient surface that usually imposes some restrictions on the direction of improvement.

It needs to point out that the selection of the reference set, which is used for obtaining the target unit, is crucial for evaluating the potential performance of the DMU, as well as for providing information on how to improve its performance. If the reference set is not selected appropriately, then target unit obtained, as well as the associated level of efficiency, might give misleading indications about how to improve the efficiency. González and Álvarez [13] suggested that the reference set should be located in the efficient subset of the isoquant, and have the shortest distance from the inefficient subset. Similarly, Bogetoft and Houggard [2]. also make the point that the closest DMU should be chosen as the reference point, and that the efficiency measure is only designed to provide a simple representation of closeness.

The most common approach of taking into account the decision maker's (DM's) preference information to derive most effective targets is through the use of multi-objective programming. The solution of a multi-objective optimization problem is dependent upon the decision makers preferences, which could be represented by a utility function that aggregates all objective functions into a scalar criterion. In most decision situations, a global utility function is not known explicitly and only local information about the utility function could be elicited. This leads to interactive procedures facilitating trade-off analysis. One of the earliest attempts to integrate multi-objective procedures and DEA techniques was made by Golany [12], who suggests an interactive multi-objective linear programming (MOLP) procedure for estimating a target set of output levels given the available input levels of a DMU. Thanassoulis and Dyson [18] proposed a weights-based general preference structure model in which the DM selects a subset of inputs and outputs whose targets should be preferentially improved and specifies weights that reflect the relative importance of such improvements. Post and Spronk [17] combined the use of DEA and interactive multiple goal programming where preference information are incorporated interactively with the DM by adjusting the upper and lower feasible boundaries of the input and output levels. Li and Reeves [14] proposed a multiple criteria data envelopment analysis model which can be used to improve discriminating power of classical DEA method and also effectively yield more reasonable input and output weights without a priori information about the weights. The proposed model involved broader definitions of relative efficiency than the classical one introduced by Charnes et al. [4]. More specifically, in that model, several different efficiency measures defined under the same constraints. Then, efficiencies evaluated under the framework of MOLP. Estellita Lins et al. [10] proposed a multi-objective ratio optimization (MORO) to generate efficient operation points from which the DM may a posteriori choose the one of her preference and, also, an interactive method for multi-objective target optimization. Bogetoft and Nielsen [3] proposed interactive benchmarking using a directional distance function approach where the direction vector components can be directly given as relative weights of inputs and outputs or, by subtraction, as aspiration, goal or reference target levels. Yang et al. [20] investigated equivalence models and interactive tradeoff analysis procedures in MOLP, such that DEA-oriented performance assessment and target setting can be integrated in a way that the decision makers' preferences can be taken into account in an interactive fashion. In a similar vein, Ebrahimnejad and Lotfi [10] established an equivalence model between the general combined-oriented CCR model and MOLP and also using ZiontsWallenius's method to integrate combined-oriented CCR performance assessment and target setting such that the DMs preference can be taken into account in an interactive fashion. Even though the most satisfactory reference set, based on the subjective criterion of the manager, may be chosen by iterative process, the manager's judgment requires qualitative information beyond the traditional scope of DEA. Alternatively, Coelli [6] suggests multi-stage DEA as a tool for finding the nearest efficient point, based upon the idea that the sum of slacks should be minimized rather than maximized. Unfortunately, multi-stage DEA is a kind of orientation model, so that its benchmarking information is obtained in such a way as to hold inputs (outputs) constant, while determining how much of an improvement in the output (input) dimensions is necessary in order for the system to become efficient [11]. Moreover, multi-stage DEA focuses only on finding representative benchmark information, and it cannot provide a measure of efficiency. Cheryche and Puyenbroeck [5] also point out that, in some cases, the projection point of multi-stage DEA may not be the most representative one. Recently, Baek and Lee [1] proposed the Least-Distance Measure (LDM) for obtaining the most relevant and easily attainable target unit, while at the same time, providing a well defined efficiency measure. But, their model in not applicable for situation in which the production possibility set has not the convexity property. In this paper, we generalize their model to non-convexity technology which improves the application potential of the LDM.

The remainder of this paper is organized as follow. Section 2 reviews the current methods for finding target unit with convex technology. Section 3 introduces our proposed method to obtain target unit based on the Least-Distance Measure model in non-convex space. In Section 4 we illustrate our method with two numerical examples and discuss about the advantages of the proposed method. Section 5 gives some conclusions.

2. TARGET UNIT OF PREVIOUS DEA MODEL WITH CONVEX TECHNOLOGY

In this section, we review that how the target unit has been deal with in extant DEA models based on convex technology. This review would motivate the introduction of a generalized LDM which gives a target unit for DEA model with non-convex technology.

Consider the production possibility set (PPS) under the assumption convex technology where x and y are the m-dimensional input and s-dimensional output vectors, respectively, $X = (x_j) \in R^{m+n}$ and $Y = (y_j) \in R^{s+n} (j = 1, 2, \dots, n)$ of n DMUs constitute the given data set and e is an n-dimensional row vector with all elements being equal to 1 and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ is the intensity vector that denotes the fractions the observations represent in the projection point, in order to eliminate inefficiency:

$$\text{PPS} = \{(x, y) | x \geq X\lambda, y \leq Y\lambda, e\lambda = 1, \lambda \geq 0\}.$$

Now consider the following input-oriented BCC model in which θ is a scalar and $\text{DMU}_o(x_o, y_o)$ is the DMU under evaluation:

$$\begin{aligned} & BCC_I = \min \theta \\ \text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta x_{io}, \quad i = 1, \dots, m, \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro}, \quad r = 1, \dots, s, \\ & \sum_{j=1}^{n} \lambda_j = 1, \\ & \lambda_j \geq 0, \quad j = 1, \dots, n. \end{aligned} \tag{1}$$

In this case,$(\theta^* x_o, y_o)$ is the target unit of $\text{DMU}_o(x_o, y_o)$ and can be obtained for eliminating inefficiency, through proportionate input contraction, while keeping the output fixed at current level. In a similar way, based on the following output-oriented BCC model the target unit $(x_o, \theta^* y_o)$ can be obtained through proportionate output expansion, while keeping the input fixed at current level:

$$\begin{aligned} & BCC_O = \max \theta \\ \text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{io}, \quad i = 1, \ldots, m, \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \geq \theta y_{ro}, \quad r = 1, \ldots, s, \\ & \sum_{j=1}^{n} \lambda_j = 1, \\ & \lambda_j \geq 0, \quad j = 1, \ldots, n. \end{aligned} \tag{2}$$

Tone [19] proposed a slack-based measure (SBM) of efficiency in DEA. In an effort to estimate the efficiency of $\text{DMU}_o(x_o, y_o)$, he formulated the following fractional program known as SBM model:

$$\begin{aligned} & \text{SBM}_O = \min \quad \rho = \frac{1-(\frac{1}{m})\sum_{i=1}^{m} \frac{s_i^-}{x_{io}}}{1+(\frac{1}{s})\sum_{r=1}^{s} \frac{s_r^+}{y_{ro}}} \\ \text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{io}, \quad i = 1, \ldots, m, \\ & \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{ro}, \quad r = 1, \ldots, s, \\ & \sum_{j=1}^{n} \lambda_j = 1, \\ & s_i^- \geq 0, \quad i = 1, \ldots, m, \\ & s_r^+ \geq 0, \quad r = 1, \ldots, s, \\ & \lambda_j \geq 0, \quad j = 1, \ldots, n. \end{aligned}$$

In this case, $(\lambda^* x + s^{-*}, \lambda^* y - s^{+*})$ is the target unit of $\text{DMU}_o(x_o, y_o)$. It needs to point out in models (1) and (2) the manager is restricted to making changes in only of either the input or output dimensions. Thus to have an ability to move along all dimensions, Cooper et al. [7] proposed the following Ranged Adjusted Measures (RAM) based on the additive DEA model, which maximizes the sum of slacks:

$$\begin{aligned} & RAM = \min 1 - \frac{1}{m+s}\left(\sum_{i=1}^{m} \frac{s_{io}^-}{R_i^-} + \sum_{r=1}^{s} \frac{s_{ro}^+}{R_i^+}\right) \\ \text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{io}, \quad i = 1, \ldots, m, \\ & \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{ro}, \quad r = 1, \ldots, s, \\ & \sum_{j=1}^{n} \lambda_j = 1, \\ & s_i^- \geq 0, \quad i = 1, \ldots, m, \\ & s_r^+ \geq 0, \quad r = 1, \ldots, s, \\ & \lambda_j \geq 0, \quad j = 1, \ldots, n, \end{aligned} \tag{3}$$

where $R_i^- = \max_j\{x_{ij}\} - \min_j\{x_{ij}\}$, $R_r^+ = \max_j\{y_{rj}\} - \min_j\{y_{rj}\}$, $s_j^- = (s_{ij}^-) \in R^m$ and $s_j^+ = (s_{rj}^+) \in R^s$.

In this model the unit $(x_o - s_o^{-*}, y_o + s_o^{+*})$ is considered as a target unit which is the farthest point from the evaluated DMU toward the production frontier.

Frei and Harker [11] used the multiplier form of the additive DEA model to find the closest efficient point on the supporting hyperplane. The DMU being evaluated is projected not on the efficient frontier, but on the supporting hyperplane. Portela et al. [16] used the Pareto-efficient facet instead of the supporting hyperplane to obtain the closest benchmark. However, their non-linear model, which is in the form of a multiplication, cannot be solved analytically.

Recently, Baek and Lee [1] introduced a weighted Least-Distance as an efficiency measure in DEA. They discussed that their new approach provides both closest target units and a well-defined efficiency measure. In order to prove that the proposed efficiency measure is well defined, Baek and Lee checked the four properties: (a) efficiency value is in the range of 0 to 1, (b) unit invariance of the efficiency measure, (c) strong monotonicity and (d) translation invariance. However, Pastor and Aparicio [15] by means of counterexample showed that the third property, strong monotonicity, is not satisfied.

Definition 2.1. (Baek and Lee [1]) The set of observations satisfying the Pareto efficiency conditions and their convex combinations is defined as a strongly efficient set, E, such that

$$E = \Big\{(x, y)|\max(e^t s^+ + e^t s^-) = 0, \\ \text{s.t.} \quad (s^+, s^-) = (x - X\lambda,\, Y\lambda - y),\; e^t\lambda = 1,\; \lambda \geq 0\Big\}, \tag{4}$$

where $e^t = (1, \ldots, 1)$, $e^t s^+ = \sum_{r=1}^{s} s_r^+$, $e^t s^- = \sum_{i=1}^{m} s_i^-$.

The objective function of the LDM converts the distance between the evaluated $\text{DMU}_o(x_o, y_o)$, and the strongly efficient set, E, into an efficiency measure, and can be described as follows.

$$\theta = \max \left[1 - \frac{1}{m+s} \left(\sum_{i=1}^{m} (\frac{x_i - x_i^o}{R_i^-})^2 + \sum_{r=1}^{s} (\frac{y_r - y_r^o}{R_r^+})^2 \right)^{\frac{1}{2}} \right] \quad \text{s.t.} \quad (x, y) \in E. \tag{5}$$

Baek and Lee [1] used a relevant algorithm to solve above problem. Nevertheless their algorithm does not work when the possibility production set is not convex.

From the preceding brief review, it is evident that a new model is required to provide a new efficiency measure in DEA model with non-convex technology, while at the same time allowing the generation of reasonable target unit. To do this, we try to find a target unit in FDH Model in the next section.

3. LEAST-DISTANCE MEASURE IN NON-CONVEX SPACE

In this section, we generalize the LDM in free disposal hull (FDH) model which assumes a non-convex possible production set.

3.1. Free Disposal Hull (FDH) model

An interesting model which has received a considerable amount of research attention is the FDH (Free Disposal Hull) model as first formulated by Deprins, et al. [8] in Belgium. The basic motivation is to ensure that efficiency evaluations are effected from only actually observed performances.

The PPS of FDH can be described as follows:

$$\mathrm{PPS}_{\mathrm{FDH}} = \bigcup_{j=1}^{n} \{(X,Y)|X \geq x_j, Y \leq y_j\}.$$

The FDH input radial efficiency of $\mathrm{DMU}_o(x_o, y_o)$ is obtained by solving the following mixed integer linear program:

$$\begin{aligned} & \min \theta \\ \text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta x_{io}, \quad i = 1, \ldots, m, \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro}, \quad r = 1, \ldots, s, \\ & \sum_{j=1}^{n} \lambda_j = 1, \\ & \lambda_j \in \{0, 1\}. \end{aligned} \tag{6}$$

3.2. The geometry interpretation of the LDM in FDH model

Consider the non-convex PPS and FDH model. Since in the FDH model, the convex property has been removed from the PPS, thus it is impossible to use combinations of m+s components of efficient set E for finding the distance between each inefficient DMU and the plane generated by each combination. This approach is applicable for convex PPS as it is done by Baek and Lee [1]. As before said, the aim of Least-Distance Measure model is to find the shortest path from each inefficient unit to efficiency frontier which requires to the defining hyperplanes of PPS. To do this in FDH model, first the inefficient $\mathrm{DMU}_o(x_o, y_o)$ is projected on efficient frontier. By considering the property of additive FDH frontier, it is apparent that the projection point is located only on extreme points. Now to obtain the shortest distance between the inefficient $\mathrm{DMU}_o(x_o, y_o)$and efficiency frontier, we first compute the distance between inefficient $\mathrm{DMU}_o(x_o, y_o)$ and each of defining hyperplanes corresponding to the reference unit (projection point) and then consider the obtained lease measure as the least distance.

We first illustrate this approach geometry. Consider Figure 1. First, the projection point of inefficient $\mathrm{DMU}_o(x_o, y_o)$ is located on DMU_c which consists of two defining hyperplanes H_1 and H_2. We note the FDH model identifies the $\mathrm{DMU}_c(x_c, y_c)$ as the target unit of inefficient unit $\mathrm{DMU}_o(x_o, y_o)$ that is not easily attainable for this unit. Thus our proposed algorithm tries to find another efficient DMU on efficiency frontier as the target of unit $\mathrm{DMU}_o(x_o, y_o)$ that has the least distance to this DMU (unit D in Figure 1). To do this, it should be calculated the distance of inefficient $\mathrm{DMU}_o(x_o, y_o)$ to each of hyperplanes H_1 and H_2. We note that similar to convex PPS (as discussed in [1]), it is maybe that the least distance from each inefficient unit is located on the inefficient region of PPS (s_1^-). Therefore using only the reference point of each DMU and their corresponding hyperplanes is not sufficient to obtain the least distance on the efficient frontier. In fact, we need a region that the defining hyperplanes form the efficiency frontier of PPS. Then we should find the least distance to efficiency frontier in that region.

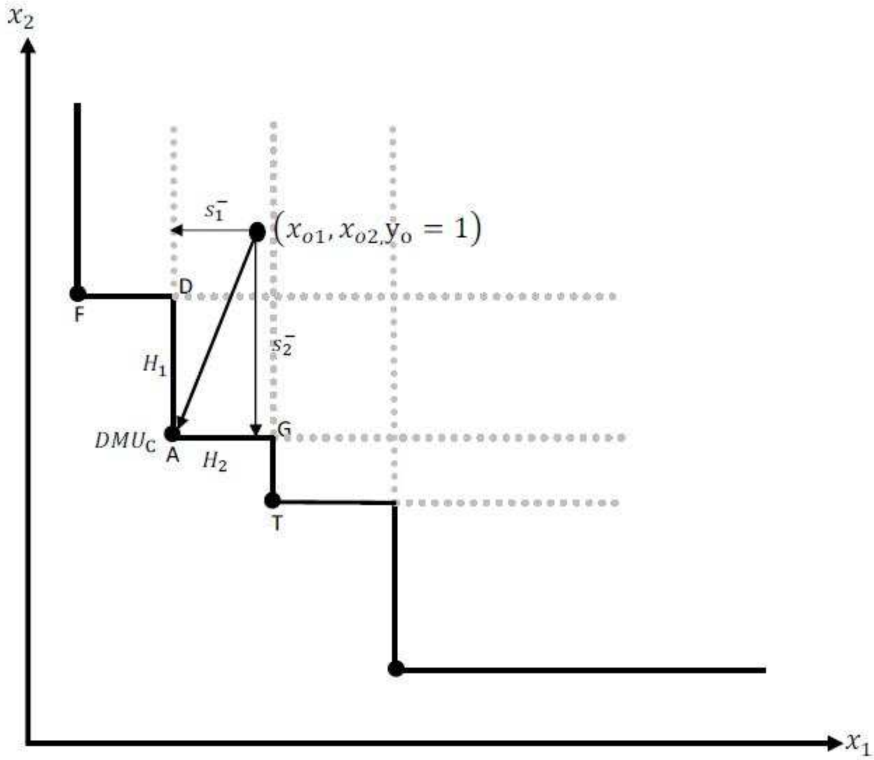

Fig. 1. The project of DMU_o on non-convex frontier.

According to Figure 1, the distance between inefficient $\mathrm{DMU}_o(x_o, y_o)$ and each of two segments H_1 and H_2 is calculated first and then the shortest path on efficient frontier (H_1 in this figure) is considered as a benchmark.

In fact it should move from the reference unit DMU_c along with obtained defining hyperplanes such that the distance becomes minimized. This is done by movement as $(x + d_1)$ or $(y - d_2)$ which d shows the amount of movement such that it remains on the efficient frontier. We note that all of points on these segments (AD and AG) are only dominated by the reference unit $\mathrm{DMU}_c(x_c, y_c)$ except of the terminal points (D and G) on these segments which are dominated by other extreme points (F and T). Based on concept of dominance, the change value of d is determined according to a special procedure such that two following properties are satisfied:

(i) The reference unit $\mathrm{DMU}_c(x_c, y_c)$ dominate $(x + d_1, y - d_2)$, i.e. $x_c \leq x + d_1$, $y_c \geq y - d_2$.

(ii) There is no extreme unit (except one) that dominates $(x + d_1, y - d_2)$, i.e. $x_k \geq x + d_1$, $y_k \leq y - d_2$, $\mathrm{DMU}_k(x_k, y_k) \in E$, $k \neq c$.

However, it is impossible to determine the priority of DMUs. Thus, we test the second property for those DMUs that have more inputs and less outputs of the reference unit DMU_c. Also, since at each step it moves only on one hyperplane corresponding to the reference unit, the concept of dominance is done for all indexes opposite to index of defining hyperplane. Now by use of region that the defining hyperplanes forms the PPS, it is possible to determine the nearest projection of inefficient unit on efficient frontier based on Euclidean norm. In the next subsection, we give an illustrative algorithm to do these steps.

3.3. The algorithm of Least-Distance Measure in non-convex space

Assume that we have a set of n DMUs with m inputs and s outputs, $\{(X_j, Y_j) = (x_{1j}, \ldots, x_{mj}, y_{1j}, \ldots, y_{rj})\}$, where all inputs and outputs are positive.

Step 1. Solve the following additive FDH model for each DMU and categorize each DMU as either Pareto efficient or inefficient. Pareto efficient DMUs (x^E, y^E) will be those that have zero as the optimal value of the additive FDH model and consist of the set E such that:

$$
\begin{aligned}
& E = \{(x, y) | \max \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ = 0\} \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{io}, \quad i = 1, \ldots, m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{ro}, \quad r = 1, \ldots, s, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& s_i^- \geq 0, \quad i = 1, \ldots, m, \\
& s_r^+ \geq 0, \quad r = 1, \ldots, s, \\
& \lambda_j \geq 0, \quad j = 1, \ldots, n.
\end{aligned}
\tag{7}
$$

Suppose $|E| = L$. To obtain the target unit of each Pareto inefficient DMU do the following steps:

Step 2. Suppose the projection point of inefficient unit $\mathrm{DMU}_o(x_o, y_o)$ be $\mathrm{DMU}_c(x_c, y_c)$ in which $\mathrm{DMU}_c \in E$.

Step 3. Find the passing hyperplanes of $\mathrm{DMU}_c(x_c, y_c)$ which consist of the defining hyperplanes of PPS as follows:

The passing hyperplanes of DMU_c: $\begin{cases} x_i = x_{ic}, & i = 1, \cdots, m, \\ y_r = y_{rc}, & r = 1, \cdots, s. \end{cases}$

Now, we find the distance between DMU_o and each of these defining hyperplanes. This depends on the kind of defining hyperplanes and explores below:

If the hyperplane is as $x_k = x_{kc}$. It should move in other dimensions of input and the all dimensions of output in such way it remains on the frontier while at a same time gives the least distance to frontier based on Euclidean norm. It is done by solving the following model:

$$
\begin{aligned}
\min \quad & \sum_{i=1, i\neq k}^{m} (x_{io} - (x_i + d_{1i}))^2 + \sum_{r=1}^{s} (y_{ro} - (y_r - d_{2r}))^2 + (x_k - x_{ko})^2 \\
\text{s.t.} \quad & \sum_{j:(x_j,y_j)\in E} \lambda_j x_{ij} + \lambda_{L+1}(x_i + d_{1i}) = (x_i + d_{1i}), \quad i = 1,\dots,m,\, i \neq k, \\
& \sum_{j:(x_j,y_j)\in E} \lambda_j y_{rj} + \lambda_{L+1}(y_r - d_{2r}) = (y_r - d_{2r}), \quad r = 1,\dots,s, \\
& \sum_{j:(x_j,y_j)\in E} \lambda_j x_{kj} + \lambda_{L+1} x_k = x_k, \\
& x_k = x_{kc}, \\
& x_i + d_{1i} \geq x_{ic}, \quad i = 1,\dots,m, i \neq k, \\
& y_r - d_{2r} \leq y_{rc}, \quad r = 1,\dots,s, \\
& x_i + d_{1i} \leq x_{it}, \quad i = 1,\dots,m, i \neq k, \\
& y_r - d_{2r} \geq y_{rt}, \quad r = 1,\dots,s, \\
& \lambda_j \in \{0,1\}, \quad j = 1,\dots,n, \\
& x_{ij} \geq 0,\ y_{rj} \geq 0, \quad j = 1,\dots,n,\ i = 1,\dots,m,\ r = 1,\dots,s,
\end{aligned}
\tag{8}
$$

where

$$
x_{it} = \{x_{ij} | x_{ij} > x_{ic}, j = 1,\dots,n\}, \text{ and } \{y_{rt} = y_{rj} | y_{rj} < y_{rc}, j = 1,\dots,n\}.
$$

We note that in model (8) three first constraints guaranty that the movement only is done on the efficient frontier. The first constraint confirms that the movement is done in direction of all inputs except of x_k and the second constraint confirms that it is done in direction of all outputs. The forth constraint shows that the referenced point is on the hyperplane $x_k = x_{kc}$. The fifth and sixth constraints guaranty that the reference unit, $\text{DMU}_c(x_c, y_c)$, dominates $(x + d_1, y - d_2)$. Moreover, the seventh and eighth constraints guaranty that no extreme unit can dominate $(x + d_1, y - d_2)$. In fact, these constraints guaranty that the two properties given in the previous subsection are satisfied.

If the hyperplane is as $y_k = y_{kc}$. It should move in direction of other outputs and in direction of all inputs in such way it remains on the frontier while at a same time gives the least distance to frontier. It is done by solving the following model:

$$\begin{aligned}
&\min \sum_{i=1}^{m}(x_{io}-(x_i+d_{1i}))^2+\sum_{r=1,r\neq k}^{s}(y_{ro}-(y_r-d_{2r}))^2+(y_k-y_{ko})^2\\
\text{s.t.}\quad &\sum_{j:(x_j,y_j)\in E}\lambda_j x_{ij}+\lambda_{L+1}(x_i+d_{1i})=x_i+d_{1i},\quad i=1,\ldots,m,\\
&\sum_{j:(x_j,y_j)\in E}\lambda_j y_{rj}+\lambda_{L+1}(y_r-d_{2r})=y_r-d_{2r},\quad r=1,\ldots,s,\ r\neq k,\\
&\sum_{j:(x_j,y_j)\in E}\lambda_j y_{kj}+\lambda_{L+1}y_k=y_k,\\
&y_k=y_{kc},\\
&x_i+d_{1i}\geq x_{ic},\quad i=1,\ldots,m\\
&y_r-d_{2r}\leq y_{rc},\quad r=1,\ldots,s,r\neq k\\
&x_i+d_{1i}\leq x_{it},\quad i=1,\ldots,m\\
&y_r-d_{2r}\geq y_{rt},\quad r=1,\ldots,s,\ r\neq k\\
&\lambda_j\in\{0,1\},\quad j=1,\ldots,n,\\
&x_{ij}\geq 0,\ y_{rj}\geq 0,\quad j=1,\ldots,n,\ i=1,\ldots,m,\ r=1,\ldots,s,
\end{aligned}\tag{9}$$

where

$$x_{it}=\{x_{ij}|x_{ij}>x_{ic},\,j=1,\ldots,n\}\quad\text{and}\quad y_{rt}=\{y_{rj}|y_{rj}<y_{rc},\,j=1,\ldots,n\}.$$

It should be noted that there is a similar discussion for the constraints of model (9).

Step 4. The solutions received in models (8) and (9) are ordered increasing according to the objective value of model (8) or (9). The first DMU evaluated as being additive efficient is defined as (x^*, y^*) and then is the nearest projection point on efficient frontier to inefficient $\text{DMU}_o(x_o, y_o)$ which is considered as the target unit of inefficient unit under consideration. In this case the efficiency measure of the LDM is obtained as follows:

$$\theta=\max\left[1-\frac{1}{m+s}\left\{\sum_{i=1}^{m}(\frac{x_i^*-x_{io}}{R_i^-})^2+\sum_{r=1}^{s}(\frac{y_r^*-y_{ro}}{R_r^+})^2\right\}^{\frac{1}{2}}\right].\tag{10}$$

Remark 1. If in formula (10) $R_r^+=0$ (in the case of there is only one output), we should modify the formula (10) as follows:

$$\theta=\max\left[1-\frac{1}{m+s}\left\{\sum_{i=1}^{m}(\frac{x_i^*-x_{io}}{R_i^-})^2\right\}^{\frac{1}{2}}\right].\tag{11}$$

It is evident that there is a similar discussion when $R_i^-=0$.

These steps are summarized in Figure 2.

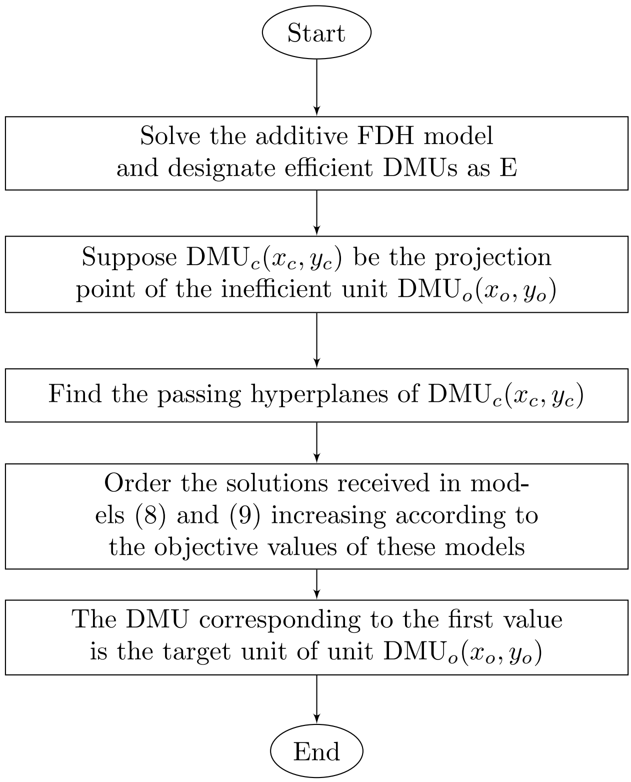

Fig. 2. Flow chart of the Least-distance Measure in non-convex space.

4. NUMERICAL EXAMPLE

In this section, we provide two numerical examples to illustrate the meaning of the Least-Distance Measure. In this first example, each unit uses two inputs to produce one output. In this example, we assume that all output have the same value equal to 1. The reason of using such data is that we be able to illustrate the steps of the proposed algorithm geometry. However, in the second example, we evaluate fifteen units consists of three inputs and three outputs and obtain a target unit for each inefficient unit based on our proposed algorithm. Then, we compare the obtained target unit with that the FDH model gives. Our results show that our approach provides a better target unit than the FDH model for each inefficient unit. In fact the distance between inefficient unit and target unit based on our algorithm is less than the distance between inefficient unit and target unit based on our FDH model. Thus, if inefficient unit uses the target unit of the proposed algorithm rather than FDH model, it can reach to efficient frontier in an easier manner.

Example 4.1. The data of 6 units is used as the first numerical example and consists of two inputs and one input, as shown in Table 1.

	A	B	C	D	E	F
x_1	2	3	4	6	8	5
x_2	8	6	3	2	1	8
y_1	1	1	1	1	1	1

Tab. 1. The data of Example 4.1.

In addition, the corresponding PPS is shown in Figure 3.

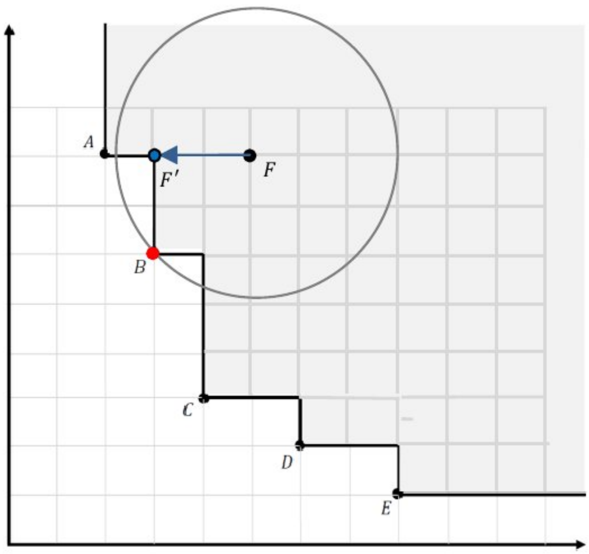

Fig. 3. The FDH frontier in Example 4.1.

By following the algorithm of the Least-distance measures, units A, B, C, D and E are evaluated as Pareto efficient units and unit F as inefficient unit, in the first step. In the second step, unit A is considered as the projection of unit F on efficient frontier. In the third step, three hyperplanes pass through the unit A, namely $x_1 = 2$, $x_2 = 8$ and $y_1 = 1$. Solving model (8) for $x_1 = 2$ gives an optimal solution as (2, 8, 1) with objective value 9. Also solving model (8) for $x_2 = 8$ gives an optimal solution as (3, 8, 1) with objective value 4. Finally, solving model (9) for $y_1 = 1$ gives an optimal solution as (2, 8, 1) with objective value 9. Therefore, in the forth step, by sorting the objective value in an increasing order and choosing the optimal solution corresponding to first value, we obtain the point $\acute{F} = (3, 8, 1)$ as the target unit for inefficient unit F. We note that in this example $m = 2, s = 1, R_1^- = 6, R_2^- = 7$ and $R_1^+ = 0$. Thus, in this case substituting (3, 8, 1) in the formula (11) (given in Remark 1) yields the LDM equal to 0.89 as follows:

$$1 - \frac{1}{3}\left\{\left(\frac{3-5}{6}\right)^2 + \left(\frac{8-8}{7}\right)^2\right\}^{\frac{1}{2}} = 0.89.$$

It is worth to note that the unit $\acute{\mathrm{F}} = (3, 8, 1)$ is a better target in compare of unit A=(2, 8, 1) for the inefficient unit F=(5, 8, 1). As we see all these DMUs are same in the second input and in the unique output. They differ only in the first input. It is clear that the unit F can decrease its first input from 5 to 3 in an easier way rather than 2 and so unit $\acute{\mathrm{F}}$ (which has the less distance to F) will be better target than of unit A.

Example 4.2. This example is about the evaluation of 15 units that use three inputs to produce three outputs. Table 2 shows the data of inputs and outputs of these units. In addition, the last column in this table shows the optimal objective values of model (7) that use to recognize the Pareto-efficient units and inefficient units.

DMU	x_1	x_2	x_3	y_1	y_2	y_3	Optimal value of model (7)
1	26	20	10	15.5	8	26	0
2	20	15	7	17.2	5	25	0
3	22	10	9.5	14.3	8	23	0
4	15	12	8.4	14	5	20	0
5	30	22	10	12	3	20	33.8
6	35	15	11	16.3	4	22	23.9
7	35	25	12	12	2	18	46.8
8	34	24	12	12	2	19	43.8
9	20	16	9.5	14.3	10	28	0
10	22	17	10	13.5	8	26	8.3
11	24	19	8	15	9	29	0
12	18	20	12	16	8	23	0
13	28	20	10.1	14.2	6	20	24.7
14	30	12	9	14	5	20	15.6
15	25	15	10	17	3	29	0

Tab. 2. The data of Example 4.2.

Step 1: By solving the additive FDH model (7) for each DMU, we categorize them as either Pareto efficient or inefficient. In addition, this model provides the projection of each inefficient unit on strong frontier as its target unit. The optimal value of model (7) shows that units $5-8$, 10, 13 and 14 are inefficient units and others are Pareto-efficient units giving the set E as $E = \{1, 2, 3, 4, 9, 11, 12, 15\}$.

Inefficient Unit	5	6	7	8	10	13	14
Target Unit	9	2	9	9	9	9	4

Tab. 3. The projection of inefficient units.

Step 2: Table 3 shows the target unit of each inefficient unit based on FDH model (7).

Now, we should run the steps 3 and 4 for each inefficient unit.

Step 3 for unit 5: As Table 3 shows, unit 9 is the projection point of inefficient unit 5. Also, the passing hyperplane of unit 9 are as $x_1 = 20,\ x_2 = 16,\ x_3 = 9.5,\ y_1 = 14.3,\ y_2 = 10,\ y_3 = 28$. Now to find the distance between unit 5 and each of these hyperplanes, models (7) or (8) is solved. The results are given in Table 4 and Table 5.

Hyperplane	$x_1 = 20$	$x_2 = 16$	$x_3 = 9.5$
Objective value of model	185	176	**149**
Target unit	(20,19,10,14,9,26)	(22,16,10,14,9,26)	(22,19,10,14,9,26)

Tab. 4. Target unit based on model (8).

Hyperplane	$y_1 = 14.3$	$y_2 = 10$	$y_3 = 28$
Objective value of model	177	**162**	150.29
Target unit	(22,19,10,14,9,28)	(22,19,10,14,10,26)	(22,19,10,14.3,9,26)

Tab. 5. Target unit based on model (9).

Step 4 for unit 5: From Tables 4 and 5, hyperplane $x_3 = 9.5$ has the least distance to unit 5 and so the corresponding target unit, i. e. (22, 19, 10, 14, 9, 26) is considered as the target unit of unit 5. In addition, based on formula (10), the efficiency measure of the LDM is equal to $\theta = 0.96887$. Now, we are in a position to compare two obtained target units based on model (7) and our proposed algorithm for inefficient unit 5 with input data as (32, 22, 10) and output data as (12, 3, 20). The FDH model gives unit 9 with input data (20, 16, 9.5) and output data (14.3, 10, 28) as the target of unit 5, while our proposed algorithm gives a target unit with input data (22, 19, 10) and output data (14, 9, 26). Noting to input values of these units, the unit 5 to reach the inputs of unit 9 should decrease 12, 6 and 0.5 units respectively, while to reach the our proposed target unit needs to decrease its inputs 10, 3 and 0 units, respectively. Thus, our proposed target unit will be better than target unit of FDH model for unit 5 from inputs point of view. Moreover, by comparing the outputs of these units, we find that if

Inefficient units	Target unit	Least distance	Measure of LDM
6	(22,15,8,16.3,4,23)	179	0.81012
7	(22,19,10,14,9,26)	327.29	0.85179
8	(22,19,10,14.3,9,26)	115.29	0.75364
10	(22,17,10,14.3,9,26)	1.64	0.96887
13	(22,19,10,14.3,9,26)	82.02	0.87828
14	(18,12,9,14,5,20)	144	0.94

Tab. 6. Target units of inefficient units based on proposed algorithm.

unit 5 considers unit 9 as its target unit, it needs to increase the outputs as 2.3, 7 and 8 units respectively, while if it accepts our target unit, it should to increase its outputs as 2, 6 and 6 units, respectively. Thus, in this case unit 5 can reach to our proposed target unit easier than target unit 9 from output point of view. In sum, our proposed target unit is easily attainable for inefficient unit 5. In addition, target units of other inefficient units based on our proposed algorithm are given in Table 6. A similar discussion shows that the target units based on our proposed algorithm are better that based on FDH model (7) for all inefficient units.

5. CONCLUSION

Data envelopment analysis provides a target unit for each inefficient DMU that usually is the farthest projection point on efficiency frontier from the inefficient unit under evaluation. Thus, it is difficult to the inefficient unit to improve its data according to this target unit to be efficient. Therefore, some researches concentrate themselves on how to obtain a target unit for each inefficient unit being easily attainable. Baek and Lee [1] based on concept of the efficiency measure of LDM proposed the most relevant target unit on the efficiency frontier which has the shortest distance from the evaluated inefficient unit. Although, this approach gives a reasonable target unit and the inefficient unit can change its data easily to be efficient, but it fails when the PPS is non-convex. Noting that, finding an easily attainable target unit is much more complicated in the case of the non-convex returns to scale, we proposed a new algorithm that is efficient in such situations.

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