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AN ITERATIVE ALGORITHM FOR COMPUTING THE CYCLE MEAN OF A TOEPLITZ MATRIX IN SPECIAL FORM

Peter Szabó

The paper presents an iterative algorithm for computing the maximum cycle mean (or eigenvalue) of $n \times n$ triangular Toeplitz matrix in max-plus algebra. The problem is solved by an iterative algorithm which is applied to special cycles. These cycles of triangular Toeplitz matrices are characterized by sub-partitions of $n - 1$.

**Keywords:** max-plus algebra, eigenvalue, sub-partition of an integer, Toeplitz matrix

**Classification:** 90C27, 15B05, 15A80

1. INTRODUCTION

The class of Toeplitz matrices is much studied and still important within mathematics as well as in a wide range of applications (see [4, 6, 7]). Nevertheless, relatively little is known about their spectral properties. The aim of this work is to propose an efficient algorithm to find a real solution $\lambda$, $x_1, \ldots, x_n \in \mathbb{R}$ to the system of equations

$$\max\{t_{i-1} + x_1, t_{i-2} + x_2, \ldots, t_0 + x_i, x_{i+1}, \ldots, x_n\} = \lambda + x_i$$

for $i = 1, 2, \ldots, n$. It will be assumed that $t_i$, for $i = 0, 1, \ldots, n - 1$ are non-negative real values. The system of equations (1) can be written in the form

$$A \otimes x = \lambda \otimes x$$

where $A = (a_{kj})$ is a triangular Toeplitz matrix, $a_{kj} = t_{k-j}$ for $k \geq j$, $a_{kj} = 0$ for $k < j$ and $(\oplus, \otimes) = (\max, +)$ are operations of the max-plus algebra. For a general $n \times n$ real matrix $A = (a_{ij})$ there exist standard $O(n^3)$ algorithms (see [5]) to find $\lambda$, $x_1, \ldots, x_n$, solutions of the system

$$A \otimes x = \lambda \otimes x.$$  \hspace{1cm} (2)

The proposed iterative algorithm solves the problem (1) in time $O(n^3)$ and uses special, combinatorial properties of triangular Toeplitz matrices. The algorithm is applied to special cycles which are characterized by sub-partitions of $n - 1$. We show that using such cycles (sub-partitions), the values $\lambda$, $x_1, \ldots, x_n$ of system (1) can be computed.
2. COMPUTING THE EIGENVALUE IN MAX-PLUS ALGEBRA.

In general, max-plus algebra is understood as an algebraic structure \((\mathbb{R}, \max, +)\), where \(\mathbb{R}\) is the set of real numbers, \(\mathbb{R} = \mathbb{R} \cup \{-\infty\}\) and \(a \oplus b = \max\{a, b\}\), \(a \otimes b = a + b\) for all \(a, b \in \mathbb{R}\). Formally the operations \((\oplus, \otimes)\) can be extended to matrices and vectors in the same way as in linear algebra. The eigenvalue-eigenvector problem \([2]\) (shortly: eigenproblem) was one of the first problems studied in max-plus algebra. Here we only discuss the case when \(A\) does not contain \(-\infty\), where for every matrix there is exactly one eigenvalue.

We begin with the discussion of a special digraph \(D_A\) and the basic concept of the cycle mean. Let \(\mathbb{R}^{n \times n}\) denotes the set of real \(n \times n\) matrices. The associated digraph \(D_A = (V, E)\) of a real matrix \(A = (a_{ij}) \in \mathbb{R}^{n \times n}\) is defined as a complete weighted digraph with the node set \(V = N = \{1, \ldots, n\}\) and with the weights \(w(i, j) = a_{ji}\) for every \((i, j) \in E = N \times N\). The set \(E\) is called the edge set of \(D_A\) and \((i, j) \in E\) is called a directed edge. We say that the edge \((i, j) \in E\) is joining vertices \(i\) and \(j\). In general, the path \(p = \langle i_1, \ldots, i_k \rangle\) in a graph \(G = (V, E)\) is a sequence of vertices \(\{i_1, \ldots, i_k\} \subseteq V\) and edges \((i_{j-1}, i_j) \in E\) for \(j = 2, \ldots, k\). Vertex \(i_1\) is called the start vertex and vertex \(i_k\) the end vertex. The path \(s = \langle i_1, \ldots, i_l \rangle\) is a sub-path of \(p\) if \(1 \leq j \leq l \leq k\). The paths will also be marked as \(p = \langle p(1), p(2), \ldots, p(l + 1) \rangle\), where \(p(i)\) are vertices for \(i = 1, \ldots, l + 1\). If \(p\) contains no vertices and no edges then the path \(p\) is called empty. Let \(p = \langle i_1, \ldots, i_k \rangle\) be a path. The number \(k - 1\) is denoted as \(|p|\) and called the length of \(p\). The value \(w(p) = a_{i_1i_2} + \ldots + a_{i_{k-1}i_k}\) is termed the weight of \(p\). If start vertex and end vertex is the same \((i_1 = i_k)\) then path \(p\) is called a cycle. The cycle \(p\) is termed an elementary cycle if, moreover, \(i_j \neq i_l\) for \(j, l = 1, \ldots, k, j \neq l\). The cycle \(p\) is a loop if it contains only the vertex \(i_1\) and edge \((i_1, i_1)\). If \(\sigma\) is an elementary cycle then the value \(\frac{w(\sigma)}{|\sigma|}\) is called the cycle mean of \(\sigma\). A cycle with the maximum cycle mean is termed the critical cycle. The basic result of max-plus algebra \([2]\) states that the maximum cycle mean in \(D_A\) is equal to the unique eigenvalue of \(A\).

**Theorem 2.1.** For every matrix \(A = (a_{ij}) \in \mathbb{R}^{n \times n}\) there is a unique value of \(\lambda = \lambda(A)\) (called the eigenvalue of \(A\)) to which there is a vector \(x \in \mathbb{R}^n\) satisfying \([2]\). The unique eigenvalue is the maximum cycle mean in \(D_A\) that is

\[
\lambda(A) = \max_{\sigma} \frac{w(\sigma)}{|\sigma|}
\]

where \(\sigma = \langle i_1, \ldots, i_k \rangle\) denotes an elementary cycle in \(D_A\). The maximization is taken over elementary cycles of all lengths in \(D_A\), including loops.

In general, a matrix \(A \in \mathbb{R}^{n \times n}\) with \(-\infty\) has several eigenvalues and the value \(\lambda(A)\) from Theorem 2.1 is the greatest eigenvalue of \(A\). A summary of concepts, methods, applications and combinatorial character of max-plus algebra can be found in \([3]\) or \([4]\). One of the first publications to deal with max-plus algebra is \([9]\).

3. GRAPHS, CYCLES AND INTEGER PARTITIONS

The class of \(n \times n\) triangular Toeplitz matrices is defined as
Let $G$ cycle if it can be decomposed as

\[
T_n(t) = \begin{pmatrix}
t_0 & 0 & 0 & \ldots & 0 \\
t_1 & t_0 & 0 & \ldots & 0 \\
t_2 & t_1 & t_0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & 0 \\
t_{n-1} & \ldots & t_1 & t_0 & \ldots \\
\end{pmatrix}
\]

where $t = (t_0, t_1, \ldots, t_{n-1})^T$, $t_i \in \mathbb{R}_0^+$ for $i = 0, \ldots, n - 1$. With every matrix $A \in T_n(t)$, a directed acyclic graph (DAG) $G_t = (N, E_t)$ can be associated, where $N = \{1, \ldots, n\}$ are the vertices and $E_t = \{(i, j) | i < j; i, j = 1, \ldots, n\}$ are the edges of graph $G_t$ with weight function $w_G(i, j) = a_{ji} = t_{j-i}$ for all $(i, j) \in E_t$. If $D_A$ is the associated digraph of matrix $A$ then $G_t$ is a sub-graph of $D_A$. A characterization of cycles of triangular Toeplitz matrices is presented in [8]. We recall briefly the main results of this paper.

**Definition 3.1.** Let $A \in T_n(t)$. Cycle $c_p$ in $D_A = (N, E)$ is called a *triangular Toeplitz cycle* if it can be decomposed as $c_p = p \cup e$, where $p = \langle p(1), \ldots, p(l+1) \rangle$ is a path in $G_t$ and $e = \langle p(l+1), p(1) \rangle \in E$.

**Lemma 3.2.** Let $A \in T_n(t)$ then for every cycle $c'$ from $D_A$ there is a triangular Toeplitz cycle $c_p = p \cup e$ such that $w(p) = w(c_p)$ and $\frac{w(c)}{|c|} \geq \frac{w(c')}{|c'|}$.

Hence, it follows that it is sufficient to consider only the triangular Toeplitz cycles for the computation of the eigenvalue of $A \in T_n(t)$.

If $m = \sum_{k=1}^l i_k \leq n - 1$ and $l > 1$ then the sequence of positive integers $i_1, \ldots, i_l$ is termed a *sub-partition on the integer* $n - 1$ of size $l$. Also to be noted, that if $i_1, \ldots, i_l$ is a sub-partition on $n - 1$ then the order of the terms in the sum $\sum_{k=1}^l i_k$ is not significant. Let us assume that $A \in T_n(t)$ then we say that a path $p$ in $G_t$ is given by sub-partition $i_1, \ldots, i_l$ if (3) is fulfilled. We show that the paths in $G_t$ given by an arbitrary permutation of set $\{i_1, \ldots, i_l\}$ have the same weight. The next result of [8] describes the basic characteristics of paths in $G_t$.

**Lemma 3.3.** Let $A \in T_n(t)$. The sequence of positive integers $i_1, \ldots, i_l$ is a sub-partition on number $n - 1$ if and only if there is a path in graph $G_t$ such that

\[
p = \langle i_1 + 1, i_1 + i_2 + 1, \ldots, i_1 + \cdots + i_l + 1 \rangle = \langle p(1), p(2), \ldots, p(l+1) \rangle.
\]

**Lemma 3.4.** Let $A \in T_n(t)$, and $p = \langle p(1), \ldots, p(l+1) \rangle$ be a path in $G_t$ given by sub-partition $i_1, \ldots, i_l$. Let $\pi : \{i_1, \ldots, i_l\} \to \{i_1, \ldots, i_l\}$ be a permutation of the set $\{i_1, \ldots, i_l\}$ and the path $p_\pi$ be given by sub-partition $\pi(i_1), \ldots, \pi(i_l)$. Then $w(p) = w(p_\pi) = t_{i_1} + \cdots + t_{i_l}$ and $p(l + 1) = p_\pi(l + 1)$.

**Proof.** It follows from (3) that $p(1) = 1, p(j) = 1 + i_1 + \cdots + i_{j-1}$ for $j = 2, \ldots, l + 1$. Suppose that $A \in T_n(t)$ then the weight of edge $(p(j), p(j + 1))$ is equal to $w(p(j), p(j + 1)) = a_{p(j+1)p(j)} = t_{p(j+1)}p(j) = t_{p(j+1)-p(j)} = t_{i_j}$ for $j = 1, \ldots, l$. Therefore, the weight of path $p$ equals $w(p) = t_{i_1} + \cdots + t_{i_l}$ and the path $p_\pi$ given by sub-partition $\pi(i_1), \ldots, \pi(i_l)$ equals $w(p_\pi) = t_{\pi(i_1)} + \cdots + t_{\pi(i_l)}$. Thus, for each permutation
corresponds to a sub-partition 1, given by vector \( d \). It will be assumed that the triangular Toeplitz matrix 

\[
\begin{pmatrix}
\vdots \\
t_n & \cdots & t_2 & t_1 \\
\vdots 
\end{pmatrix}
\]

In this chapter we define a specific function. The features of function will serve to determine the wanted eigenvalue. It will be assumed that the triangular Toeplitz matrix 

\[
\begin{pmatrix}
\vdots \\
t_1 + t_2 + \cdots + t_i & \cdots & t_1 + t_2 & t_1 \\
\vdots 
\end{pmatrix}
\]

\[\pi : \{i_1, \ldots, i_t\} \to \{i_1, \ldots, i_t\} \text{ we have } w(p) = t_{i_1} + t_{i_2} + \cdots + t_{i_t} = t_{\pi(i_1)} + t_{\pi(i_2)} + \cdots + t_{\pi(i_t)} = w(p_\pi) \text{ and } p(l+1) = 1 + i_1 + \cdots + i_t = 1 + \pi(i_1) + \cdots + \pi(i_t) = p_\pi(l+1). \]

Figure 1 shows a graph \( G_t \), where \( t = (t_0, t_1, t_2, t_3, t_4) \), \( n - 1 = 4 \). The path \( p = (1, 2, 3, 5) \) in \( G_t \) corresponds to a sub-partition 1, 1, 2 of 4 and the path \( p_\pi = (1, 2, 4, 5) \) corresponds to a sub-partition 1, 2, 1 and vice versa. The weight of path \( p \) equals \( w(p) = t_1 + t_1 + t_2 = t_1 + t_2 + t_1 = w(p_\pi) \), \( l = 3 \) and \( p(4) = p_\pi(4) = 5 \).

4. AN ESTIMATION FUNCTION AND ITS FEATURES

In this chapter we define a specific function. The features of function will serve to determine the wanted eigenvalue. It will be assumed that the triangular Toeplitz matrix \( A \) given by vector \( t(z) = (z, t_1, \ldots, t_{n-1})^T \) where \( t_i \in \mathbb{R}^+_0 \) are fixed numbers for \( i = 1, \ldots, n-1 \) and \( z \in \mathbb{R}^+_0 \) is a variable. Note that it follows from the definition of graph \( G_t \) that \( G_{t(z)} = G_t \) for all \( z \in \mathbb{R}^+_0 \).

**Definition 4.1.** Let \( A \in T_n(t(z)) \) be a triangular Toeplitz square matrix given by the vector \( t(z) = (z, t_1, \ldots, t_{n-1}) \). The vector \( x(z) = (x_1(z), \ldots, x_n(z)) \) is called the sub-eigenvector of \( A \) corresponding to the value \( z \in \mathbb{R}^+_0 \) if it is defined by the formula:

\[
\begin{align*}
1. & \quad x_1(z) = 0 \\
2. & \quad x_i(z) = \max\{x_{i-1}(z), \max_{j=1,\ldots,i-1}\{t_{i-j} + x_j(z) - z\}\} \quad \text{for } i = 2, \ldots, n.
\end{align*}
\]

The sub-eigenvector \( x(z) \) may become an eigenvector of the matrix \( A \) due to the following Lemma.

**Lemma 4.2.** Let \( A \in T_n(t(z)) \), \( z \in \mathbb{R}^+_0 \) and \( x(z) \) be a sub-eigenvector of \( A \). Then \( A \otimes x(z) = z \otimes x(z) \) if and only if \( z \geq x_n(z) \).

**Proof.** Suppose that \( z \geq x_n(z) \). Let us denote \( [A \otimes x(z)]_i \) the \( i \)-th element of the vector \( [A \otimes x(z)] \). It follows from Definition 4.1 that \( 0 = x_1(z) \leq \cdots \leq x_n(z) \), therefore \( [A \otimes x(z)]_1 = \max\{z + x_1(z), x_2(z), \ldots, x_n(z)\} = \max\{z, x_n(z)\} = z \). For all \( i > 1 \) we have \( x_i(z) \geq \max_{j=1,\ldots,i-1}\{t_{i-j} + x_j(z)\} - z \) and by a simple computation \( [A \otimes x(z)]_i = \max\{t_{i-1} + x_1(z), t_{i-2} + x_2(z), \ldots, t_1 + x_{i-1}(z), z + x_i(z), x_{i+1}(z), \ldots, x_n(z)\} \).
\[ \text{max}\{x_{\ell}(z) + z, x_n(z)\} = x_{\ell}(z) + z \text{ is obtained. Hence, } A \otimes x(z) = z \otimes x(z). \] Let us assume that \( A \otimes x(z) = z \otimes x(z) \) and \( x(z) \) is a sub-eigenvector of \( A \). The relation \( z \geq x_n(z) \) is obtained after insertion of known data \( [A \otimes x(z)]_1 = \text{max}\{z + x_1(z), x_2(z), \ldots, x_n(z)\} = \text{max}\{z, x_n(z)\} = z. \)

**Lemma 4.3.** Let \( A \in T_n(t(z)) \), \( z \in \mathbb{R}^+_0 \) and \( x(z) \) be a sub-eigenvector of \( A \). Then \( x(z) = 0 \) if and only if \( z \geq \text{max}_{j=1,\ldots,n-1} t_j \).

**Proof.** Let \( A \in T_n(t(z)) \). Let us assume that \( z \geq \text{max}_{j=1,\ldots,n-1} t_j \). By a simple computation it follows that \( x_i(z) = 0 \) for all \( i = 1, \ldots, n \) (shortly: \( x(z) = 0 \)) and \( A \otimes x(z) = z \otimes x(z) \). In this case \( z \) is the eigenvalue and \( x(z) = 0 \) is the eigenvector. From the assumption \( x(z) = 0 \), it follows that \( z \geq \text{max}_{j=1,\ldots,n-1} t_j \).

Let \( A \in T_n(t(z)) \) be a triangular Toeplitz matrix where \( t(z) = (z, t_1, \ldots, t_{n-1}) \). In the next, it will be assumed that \( z < \text{max}_{j=1,\ldots,n-1} t_j \), i.e. \( x(z) \neq 0 \). Otherwise, according to Lemma 4.3 \( z = \lambda(A) \) and \( x(z) = 0 \). Let us focus on the real function \( y_A(z) = x_n(z) - z \).

**Definition 4.4.** Let \( x(z) = (x_1(z), \ldots, x_n(z)) \) be a sub-eigenvector of a matrix \( A \in T_n(t(z)) \). The expression

\[ y_A(z) = x_n(z) - z \]

is termed an estimation function of eigenvalue \( \lambda(A) \).

**Theorem 4.5.** Let \( x(z) = (x_1(z), \ldots, x_n(z)) \) be a sub-eigenvector of a matrix \( A \in T_n(t(z)) \). For each \( z \in \langle 0, \text{max}_{j=1,\ldots,n-1} t_j \rangle \) there is a path \( p \) in \( G_t \) such that

\[ y_A(z) = x_n(z) - z = w(p) - (|p| + 1)z \]

and \( n \) is the end vertex of \( p \).

**Proof.** Let \( z \) be an arbitrary element of the interval \( \langle 0, \text{max}_{j=1,\ldots,n-1} t_j \rangle \) and \( x(z) \) be a sub-eigenvector of \( A \). We shall show first that there is a path \( p \) in graph \( G_t \) such as

\[ y_A(z) = x_n(z) - z = w(p) - (|p| + 1)z. \]  

\[ \text{(4)} \]

From the assumption \( z \in \langle 0, \text{max}_{j=1,\ldots,n-1} t_j \rangle \) and from Lemma 4.3 it follows that the sub-eigenvector \( x(z) \neq 0 \) and \( x_n(z) > 0 \). It follows from the definition of \( x(z) \) that the vector components are non-decreasing, non-negative and \( x_n(z) \geq t_{n-k} + x_k(z) - z \) for all \( k = 1, \ldots, n-1 \).

We will first prove that the set \( M_n(z) = \{l; x_n(z) = t_{n-l} + x_l(z) - z\} \) is non empty. If we assume that \( x_n(z) > t_{n-k} + x_k(z) - z \) for all \( k = 1, \ldots, n-1 \) then \( x_n(z) = x_{n-1}(z) \) by Definition 4.1. The condition \( x_n(z) > 0 \) implies that there is an index \( j \) such that \( x_n(z) = x_{n-1}(z) = \cdots = x_{n-j}(z) \) and \( x_{n-j}(z) = t_{n-j-l} + x_l(z) - z > 0 \) for some \( l \), moreover \( n - j - l \geq 1 \). Therefore, we obtain \( x_n(z) = x_{n-j}(z) = t_{n-j-l} + x_l(z) - z \leq t_{n-(j+l)} + x_{j+l}(z) - z \), where \( j + l \leq n - 1 \), which is a contradiction.
An iterative algorithm

Let \( l_1 \in M_n(z) \) be an arbitrary index and let \( p \) be an empty path in \( G_t \). We add vertices \( l_1, n \) and the edge \((l_1, n)\) to the path \( p \). The value \( y_A(z) \) can be written as follows: 
\[
y_A(z) = x_n(z) - z = t_{n-l_1} + x_l(z) - 2z.
\]
If \( x_{l_1}(z) = 0 \) then \( y_A(z) = t_{n-l_1} - 2z = w(p) - (|p| + 1)z \). If \( x_{l_1}(z) > 0 \) then \( M_1(z) = \{ z; x_l(z) = t_{n-l_1} + x_l(z) - z \} \) is non-empty. Let \( l_2 \in M_1(z) \) be an arbitrary index \((l_2 < l_1)\). We add the vertex \( l_2 \) and the edge \((l_2, l_1)\) to the path \( p \). If \( x_{l_2}(z) = 0 \) then \( y_A(z) = t_{n-l_1} + t_{l_1-l_2} - 3z = w(p) - (|p| + 1)z \).

\[\text{while } x_{l_k}(z) > 0 \text{ this procedure is repeated. If the condition } x_{l_j}(z) = 0 \text{ is met, the procedure is finished. Such a component } x_{l_j}(z) \text{ exists because } x_1(z) = 0 \text{ and } x_1(z) \leq \ldots \leq x_n(z). \]

Note, if for \( z \in (0, \max_{j=1,\ldots,n-1} t_j) \) there is a path \( p \) from \( G_t \) such that 
\[
y_A(z) = x_n(z) - z = w(p) - (|p| + 1)z,
\]
so there exists such a path \( p^* \) of minimum length, i.e. 
\[
|p^*| = \min \{ |p|; y_A(z) = x_n(z) - z = w(p) - (|p| + 1)z \}.
\]

We show how to construct such a path in time \( O(n^2) \). Each element \( l_1 \in M_n(z) \) from the proof of Theorem 4.5 defines a class of paths in \( G_t \). This class of paths is characterized by integers \( n-l_1, l_1-l_2, \ldots, l_{j-1}-l_j \) or by directed edges with weights \( t_{n-l_1}, t_{l_1-l_2}, \ldots, t_{l_{j-1}-l_j} \), which define the path \( p_{l_1} = (l_j, \ldots, l_1, n) \). We denote \( M_i(z) = \min \{ z; x_l(z) = t_{l_{i-1}+l_1(z)-z} \} \) for \( i = 1, \ldots, n \) and we define \( M_j(z) = 0 \) when \( M_j(z) = \emptyset \) for some \( j \). The \( l_i \) values are computed as \( l_i = M_{l_{i-1}}(z) \) for \( i = 1, \ldots, j \). The complexity of the computation of the integers \( n-l_1, l_1-l_2, l_2-l_3, \ldots, l_{j-1}-l_j \) or path \( p_{l_1} \) is \( O(n) \). The computation and the assignment of a path \( p^*_i \) is performed for each element \( i \in M_n(z) \). Now just assign \( |p^*| = \min \{ |p^*_i|; y_A(z) = x_n(z) - z = w(p^*_i) - (|p^*_i| + 1)z \} \).

The overall complexity of the procedure is \( O(n^2) \), because \( |M_n(z)| \leq n \). We will refer to the procedure of creation the path \( p^* \) as a path assignment procedure. So the next claim is proved.

**Lemma 4.6.** For each \( z \in (0, \max_{j=1,\ldots,n-1} t_j) \) the path assignment procedure finds all paths \( p \) in \( G_t \) such that 
\[
y_A(z) = w(p) - (|p| + 1)z \text{ in time } O(n^2).
\]

Now, we can define an equivalence relation of paths in \( G_t \). Two paths \( p_1, p_2 \) are said to be equivalent if and only if \( w(p_1) = w(p_2) \) and \( |p_1| = |p_2| \). If a path \( p \) belongs to the same class of equivalence then this class is marked as \( [p] \).

**Theorem 4.7.** Let \( x(z) = (x_1(z), \ldots, x_n(z)) \) be a sub-eigenvector of a matrix \( A \in T_n(t(z)) \). The function \( y_A(z) = x_n(z) - z \) is decreasing and piecewise linear on interval \( (0, \max_{j=1,\ldots,n-1} t_j) \) with integer slopes and moreover \( y_A(z^*) = 0 \) if only if \( z^* = \lambda(A) \).

**Proof.** Let \( z \) be an arbitrary element of interval \( (0, \max_{j=1,\ldots,n-1} t_j) \). From Theorem 4.5 it follows that there is a path in \( G_t \) such that \( y_A(z) = x_n(z) - z = w(p) - (|p| + 1)z \).

If there is only one equivalence class \([p^*]\) such that \( y_A(z) = x_n(z) - z = w(p^*) - (|p^*| + 1)z \) (in other words, if \([z, y_A(z)]\) is not an intersection point of two lines) then there is a small neighbourhood \((z_1, z_2)\) around \( z \) where \( y_A(z) \) is linear (with negative slope) and decreasing. Assume now that \( y_A(z) = w(p_1) - (|p_1| + 1)z = w(p_2) - (|p_2| + 1)z \) and \( |p_1| < |p_2| \). Therefore, there are two paths \( p^* \) and \( \tilde{p}^* \) such that \( y_A(z) = w(p^*) - (|p^*| + 1)z \).
\((|p^*| + 1)z = w(\overline{p^*}) - (|p^*| + 1)z\) and \(p^*\) has a minimum and \(\overline{p^*}\) a maximum length of such paths, hence \(|p^*| < |\overline{p^*}|\). For this reason, there is a small interval \((z_1, z)\) where \(y_A(z) = w(\overline{p^*}) - (|p^*| + 1)z\) and a small interval \((z, z_2)\) where \(y_A(z) = w(p^*) - (|p^*| + 1)z\). Function \(y_A(z)\) on intervals \((z_1, z)\) and \((z, z_2)\) is linear and decreasing, therefore \(y_A(z)\) is a piecewise linear and decreasing on interval \((0, \max_{j=1,\ldots,n-1}t_j)\).

Now we prove the second part of the theorem. If the condition \(y_A(\overline{z}) = 0\) is met then \(\overline{z} = \lambda(A)\) with regard to Lemma 4.2. Now we suppose that \(\overline{z} < \max_{j=1,\ldots,n-1}t_j\) and \(\overline{z} = \lambda(A)\). It is necessary to prove that \(y_A(\overline{z}) = x_n(\overline{z}) - \overline{z} = 0\). The condition \(y_A(\overline{z}) = x_n(\overline{z}) - \overline{z} > 0\) implies that \(\overline{z} \neq \lambda(A)\) by Lemma 4.2. Assume that \(y_A(\overline{z}) = x_n(\overline{z}) - \overline{z} < 0\). From Lemma 4.2 it follows that for any non-critical cycle \(c\) of \(D_A\) the inequality \(y_A(\frac{w(c)}{|c|}) > 0\) is fulfilled. The function \(y_A(z)\) is piecewise linear on the interval \((\frac{w(c)}{|c|}, \overline{z}) \subseteq (0, \max_{j=1,\ldots,n-1}t_j)\). Therefore \(y_A(z)\) is also a continuous function. Hence, there exists \(z' \in (\frac{w(c)}{|c|}, \overline{z})\) such as \(y_A(z') = x_n(z') - z' = 0\). The already proved sufficient condition implies that \(z' = \lambda(A)\). From Theorem 2.1 it follows that \(\lambda(A) = \overline{z}\) is a unique eigenvalue, but \(\lambda(A) = z' \neq \overline{z}\), which contradicts with condition \(y_A(\overline{z}) < 0\).

5. AN ITERATIVE ALGORITHM

We propose a simple iterative algorithm to obtain the eigenvalue \(\lambda(A)\) based on Theorem 4.7.

![Fig. 2. An iterative step of the algorithm.](image)

The Figure 2 shows an iterative step of the algorithm, where \(z_i, z_{i+1}\) are estimates of the eigenvalue \(\lambda(A)\). The algorithm solves problem (1) in \(O(n^3)\) steps. Each iterative step has a complexity \(O(n^2)\) (paths \(p_i\) with minimum slope are created by path assignment procedure, see Lemma 4.6). The number of iterative steps does not exceed \(n\), the maximum possible slope of function \(y_A(z)\). The number of iterative steps depends on the initial estimate \(z_0\), but on the general complexity of the iterative method it has no effect.
Algorithm 1 An iterative algorithm

{Input: \( A \in T_n(t) \), where \( t = (t_0, t_1, \ldots, t_{n-1})^T \), \( t_j \in \mathbb{R}_0^+ \) for \( j = 0 \ldots, n - 1 \).}

\( i = 0; \ z_0 = t_0; \)

if \( y_A(z_0) \leq 0 \) then

\( \{ z_0 = t_0 \) is the eigenvalue, \( x(z_0) \) is an eigenvector of matrix \( A \) and the loop \((1, 1)\) is a critical cycle.\}

end if

while \( y_A(z_i) > 0 \) do

\( i = i + 1; \)

\( z_i = \frac{w(p_{i-1})}{|p_{i-1}|+1}; \)

end while

\( \{ \text{If } y_A(z_i) = w(p_i) - (|p_i|+1)z_i > 0 \) then \( i = i+1 \) and \( z_i = \frac{w(p_{i-1})}{|p_{i-1}|+1} \) is the next estimate of \( \lambda(A) \). If \( y_A(z_i) = w(p_i) - (|p_i|+1)z_i = 0 \) then \( z_i \) is the eigenvalue of \( A \), \( x(z_i) \) is an eigenvector (see Theorem 4.7) and \( c_{p_i} = p_i \cup e \) is a critical cycle. The value of \( w(p_i) \) can be expressed as \( t_{i_1} + \cdots + t_{i_l} \) and the indices \( i_1, \ldots, i_l \) define a sub-partition of \( n-1 \).} \}

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