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TRAJECTORIES, FIRST RETURN LIMITING NOTIONS AND RINGS OF H-CONNECTED AND ITERATIVELY H-CONNECTED FUNCTIONS

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Abstract. In the paper the existing results concerning a special kind of trajectories and the theory of first return continuous functions connected with them are used to examine some algebraic properties of classes of functions. To that end we define a new class of functions (denoted $Conn^*$) contained between the families (widely described in literature) of Darboux Baire 1 functions (DB_1) and connectivity functions ($Conn$). The solutions to our problems are based, among other, on the suitable construction of the ring, which turned out to be in some senses an "optimal construction". These considerations concern mainly real functions defined on $[0,1]$ but in the last chapter we also extend them to the case of real valued iteratively H-connected functions defined on topological spaces.

Keywords: trajectory; first return continuity; H-connected function; ring of functions; D-ring; iteratively H-connected function

MSC 2010: 26A15, 54H20, 26A21, 54C40, 54C30

1. Introduction

Until recently one dimensional dynamical systems were analyzed almost exclusively with respect to continuous functions and thinner classes of functions. However, some considerations provoked a necessity to examine discontinuous derivatives. Therefore many papers regarding dynamic of discontinuous functions (e.g. [1], [6], [14], [22], [23]) have appeared recently. The interesting fact about the quoted papers is that all of them are connected with Darboux like functions. Simultaneously, many papers regarding "first return" notions based on a special kind of trajectories appeared at the end of 20. century and at the beginning of the current century (e.g. [7], [8], [9], [10], [19], [21]). The connection of these trajectories with transitive continuous functions has been described among others in [9].

Although this paper does not refer to dynamical systems directly, we use tools connected with some trajectories to consider problems of existence of "optimal constructions" of rings of functions. Several previous papers published by the second author concern dynamical systems generated by Darboux like functions and consequently our considerations are also connected with such transformations.

Algebraic structures play a special role among many problems currently being examined with respect to various classes of functions. This is indicated by the fact that in some cases some equivalents to these structures are created being, on the one hand, the extensions of their concept and, on the other, distant from their prototypes in abstract algebra. For example, a notion of pseudogroup was developed by É. Cartan in the early 1900s, which, after some modification, is now used, among other things, in research dealing with dynamical systems (e.g. in [2]).

The research related to algebraic operations and algebraic structures has been carried on also with respect to Darboux-like functions. The results published till now are the starting point of considerations presented in this paper.

In Section 3 we introduce a subfamily, $Conn^*$, of the connectivity functions family, and we present its primary properties. The direct motivation to distinguish this class of functions were the results connected with the theory of dynamical systems included in papers [6] and [23] and the desire to obtain a wide class of functions with connected graph, for which there is a possibility to construct rings of functions belonging to this class. The direction of the research was also connected with the paper [21]. Further considerations presented in the following parts of the paper address the class $Conn^*$.

In Section 4 we examine the possibility of constructing rings (more precisely, complete rings) of functions belonging to the family introduced in Section 3. In the proof of Theorem 4.1 the construction of a complete ring containing a fixed function $f \in Conn^*$ and included in $Conn^*$, is presented. Furthermore, this construction gives an "optimal ring". The consequence of "the optimality of the construction" is Theorem 4.3 saying (under some additional assumptions) that if two functions belong to a common additive group of Darboux functions, then they also belong to a common ring which may be constructed by the method presented in the proof of Theorem 4.1.

Section 5 contains results regarding rings of iteratively H-connected functions, i.e. functions being of the form $f \circ g$ where $g\colon [0,1] \to \mathbb{R}$ is an H-connected function and $f\colon X \to [0,1]$ is a continuous function defined on a topological space. The starting point of the considerations in this part of the paper are the results presented in [15], which concludes that there exist topological spaces such that there is no ring of Darboux functions defined on them. This raises questions concerning assumptions that would cause existence of the respective rings.

The paper is closed with an open problem that is, in some senses, a continuation of Theorem 5.3.

2. Preliminaries

We will use mostly standard notation and definitions ([5], [10], [11], [15], [20], [21]). In particular, by the letters $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{N}$ we denote the sets of all real numbers, rational numbers and positive integers, respectively.

The closure, interior and boundary of a set A are denoted by $\mathrm{Cl}(A)$, $\mathrm{Int}(A)$ and $\mathrm{Fr}(A)$, respectively.

The symbol $C(f)$ ($D(f)$) stands for the set of all continuity (discontinuity) points of a function f, and $Z(f) = f^{-1}(0)$. By $\Gamma(f)$ and $f \restriction A$ we denote the graph of a function f and the restriction of a function f to a set A, respectively.

For a set $H \subset [0,1]$ open in $[0,1]$, the symbol $\mathrm{Cp}(H)$ denotes the set of all components of H.

To denote a family of all Darboux functions $f\colon [a,b] \to \mathbb{R}$ we use the symbol $\mathrm{DB}x([a,b])$ while the symbol $\mathrm{DB}_1([a,b])$ stands for the family of all Darboux Baire 1 functions $f\colon [a,b] \to \mathbb{R}$.

According to the commonly used definition, a real function f defined on a topological space X is called a Darboux function if the image $f(C)$ is a connected set, for each connected set $C \subset X$.

We say that a function $f\colon [0,1] \to \mathbb{R}$ does not intersect the axis OX if $f(x) \geqslant 0$ for $x \in [0,1]$ or $f(x) \leqslant 0$ for $x \in [0,1]$.

A function $f\colon [0,1] \to \mathbb{R}$ belongs to the class B_1^{**} if $D(f) = \emptyset$ or $f \restriction D(f)$ is continuous.

Let f, g be real functions defined on $[0,1]$. By ϱ we denote the metric of uniform convergence defined as $\varrho(f,g) = \min(1, \sup\limits_{x\in[0,1]} |f(x) - g(x)|)$.

Let $\Re$ be a ring of fuctions defined on a topological space X. Let us introduce the following notation:

$$D(\Re) = \bigcup_{h\in\Re} D(h).$$

A ring $\Re$ of functions defined on a topological space X is called a D-ring if $D(\Re) \neq \emptyset$. A D-ring $\Re$ of functions defined on a topological space X is called a prime ring if $D(f) \subset Z(f)$ for each $f \in \Re$.

We call the set $H \subset [0,1]$ the od-set if it is open and dense in $[0,1]$.

Let H be an arbitrary od-set. By an H-trajectory we mean any sequence $\bar{q} = \{q_n\}_{n\in\mathbb{N}}$ of distinct points such that $\{q_n\colon\ n = 1, 2, \ldots\}$ is a dense subset of H.

Let $\bar{q} = \{q_n\}_{n\in\mathbb{N}}$ be a fixed H-trajectory. For any interval $(c,d) \subset [0,1]$, by $r_{\bar{q}}((c,d))$ we will denote the first element of the sequence $\bar{q}$ belonging to (c,d) (i.e. $r_{\bar{q}}((c,d)) = q_{n_0}$ if and only if $n_0 = \min\{n\colon\ q_n \in (c,d)\}$).

For $x \in (0,1]$ the left first return path to x based on the H-trajectory $\bar{q}$, $P_l(x,\bar{q}) = \{t_k\}_{k\in\mathbb{N}}$, is defined recursively via the following formulas:

$$\begin{cases} t_1 = r_{\bar{q}}((0,x)), \\ t_{k+1} = r_{\bar{q}}((t_k,x)), \quad \text{for } k = 1,2,\ldots. \end{cases}$$

For $x \in [0,1)$ the right first return path to x based on the H-trajectory $\bar{q}$, $P_r(x,\bar{q}) = \{s_k\}_{k\in\mathbb{N}}$, is defined recursively via the following formulas:

$$\begin{cases} s_1 = r_{\bar{q}}((x,1)), \\ s_{k+1} = r_{\bar{q}}((x,s_k)), \quad \text{for } k = 1,2,\ldots. \end{cases}$$

We say that a function $f\colon\ [0,1] \to \mathbb{R}$ is first return continuous from the left (right) at a point $x \in (0,1]$ ($x \in [0,1)$) with respect to the H-trajectory $\bar{q}$, if

$$\lim_{\substack{t\to x\\ t\in P_l(x,\bar{q})}} f(t) = f(x) \quad \Bigl(\lim_{\substack{t\to x\\ t\in P_r(x,\bar{q})}} f(t) = f(x)\Bigr).$$

If f is first return continuous from the left and from the right at $x \in (0,1)$ with respect to the H-trajectory $\bar{q}$, then we say that it is first return continuous at x with respect to $\bar{q}$. Moreover, we will say that f is first return continuous at 0 (at 1) with respect to $\bar{q}$ if it is first return continuous from the right (from the left) at 0 (at 1) with respect to $\bar{q}$.

Let H be a fixed od-set and $\bar{q}$ a fixed H-trajectory. We say that a function $f\colon\ [0,1] \to \mathbb{R}$ is $(H,\bar{q})$-first return continuous if it is first return continuous with respect to $\bar{q}$ at each point $x \in H$ and f is first return continuous from the left with respect to $\bar{q}$ at the right end of any component of H and f is first return continuous from the right with respect to $\bar{q}$ at the left end of any component of H. The set of all $(H,\bar{q})$-first return continuous functions will be denoted by $\mathrm{FRC}(H,\bar{q})$.

Two known facts concerning the classical notion of first return continuity will be very useful in the next sections. Now we will formulate them in the terminology used in this paper.

Lemma 2.1 ([9])**.** *A function $f\colon\ [0,1] \to \mathbb{R}$ is Darboux Baire 1 if and only if there exists a $[0,1]$-trajectory $\bar{q}$ such that $f \in \mathrm{FRC}([0,1],\bar{q})$.*

Lemma 2.2 ([7]). *Let $D \subset [0,1]$ be countable and dense in $[0,1]$. If a function $f\colon [0,1] \to \mathbb{R}$ is Darboux Baire 1 with $\Gamma(f \restriction D)$ dense in $\Gamma(f)$, then there exists an ordering $\bar{d} = \{d_n\}_{n\in\mathbb{N}}$ of D such that $f \in \mathrm{FRC}([0,1], \bar{d})$.*

If $H \subset [0,1]$ is an od-set, $I \in \mathrm{Cp}(H)$, a and b are the left and the right endpoint of I respectively, $\bar{q}$ is an H-trajectory and $t\colon [0,1] \to \mathbb{R}$ is defined as follows: $t(x) = f(x)$ for $x \in (a,b)$; $t(x) = f(a)$ for $x \in [0,a]$; $t(x) = f(b)$ for $x \in [b,1]$, then $t \in \mathrm{FRC}([0,1], \bar{q})$. By Lemma 2.1 we have:

Proposition 2.3. *Let $H \subset [0,1]$ be an od-set and $\bar{q}$ an H-trajectory. If $f \in \mathrm{FRC}(H, \bar{q})$, then for any component $I \in \mathrm{Cp}(H)$, $f \restriction \mathrm{Cl}(I)$ is a Darboux Baire 1 function.*

Let H be a fixed od-set and $\bar{q} = \{q_n\}_{n\in N}$ a fixed H-trajectory. We say that a function $f\colon [0,1] \to \mathbb{R}$ has a $D(H, \bar{q})$ property at a point $x \in [0,1]$ if for any $\varepsilon > 0$ there exists $\delta \in (0, \varepsilon)$ such that for any component I of the set H the following condition is fulfilled:

$$(I \cap (x-\delta, x+\delta) \neq \emptyset)$$
$$\Rightarrow (f(\{q_n\colon n = 1,2,\ldots\} \cap I \cap (x-\delta, x+\delta)) \cap (f(x)-\varepsilon, f(x)+\varepsilon) \neq \emptyset).$$

One can easily observe:

Proposition 2.4. *Let H be a fixed od-set and $\bar{q}$ a fixed H-trajectory. If $f \in \mathrm{FRC}(H, \bar{q})$, then f has a $D(H, \bar{q})$ property at each point of H.*

We say that a function $f\colon [0,1] \to \mathbb{R}$ is H-connected with respect to H-trajectory $\bar{q}$ if $f \in \mathrm{FRC}(H, \bar{q})$ and f has a $D(H, \bar{q})$ property at each point $x \in [0,1] \setminus H$.

3. The family *Conn**

Let us introduce the following notation:

The symbol *Conn** will denote the family of all functions $f\colon [0,1] \to \mathbb{R}$ such that there exist an od-set H_f and an H_f-trajectory $\bar{q}$ such that f is H_f-connected with respect to $\bar{q}$.

Functions with connected graph play a special role in generalizations of considerations dealing with dynamical systems in the case of discontinuous functions (e.g. [6], [23]). Using the term "H-connected function" suggests that functions with connected graph are the subject of our considerations. The next theorem will show that each function belonging to the class *Conn** has indeed a connected graph. In the proof of this theorem we will use the following lemma.

Lemma 3.1 ([24]). *Let K be a closed and nowhere dense subset of the interval $[0,1]$ such that $K = [0,1] \setminus \bigcup_{i\in\mathbb{N}} (a_i, b_i)$, where all the intervals (a_i, b_i) are pairwise disjoint. Let $K_0 = [0,1] \setminus \bigcup_{i\in\mathbb{N}} [a_i, b_i]$ and assume that a function $f\colon [0,1] \to \mathbb{R}$ fulfils the following conditions:*

(i) *f has a connected graph on each interval $[a_i, b_i]$;*

(ii) *for every point $x \in K_0$ and subsequence $\{a_{p_n}\}_{n\in\mathbb{N}}$ converging to x there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \in [a_{p_n}, b_{p_n}]$ (for $n \in \mathbb{N}$) and $\lim_{n\to\infty} (x_n, f(x_n)) = (x, f(x))$.*

Then the function f has a connected graph.

Theorem 3.2. *Let $H \subset [0,1]$ be an od-set and $f\colon [0,1] \to \mathbb{R}$ an H-connected function with respect to an H-trajectory $\bar{q} = \{q_n\}_{n\in\mathbb{N}}$. Then the function f has a connected graph.*

P r o o f. If the set H consists of a finite number of components, then (using Lemma 2.1) it is easy to show that $f \in \mathrm{DB}_1([0,1])$, so f has a connected graph.

Let us assume now that H has an infinite number of components. Then $H = \bigcup_{i=1}^{\infty} I_i$, where I_i are open subintervals of $(0,1)$ or they are either of the form $[0,c)$ or $(d,1]$ for some $c, d \in (0,1)$. For $i \in \mathbb{N}$ let us denote by a_i the left endpoint of the interval I_i and by b_i the right endpoint of I_i. Put $K = [0,1] \setminus \bigcup_{i\in\mathbb{N}} (a_i, b_i)$. Of course K is a closed and nowhere dense subset of $[0,1]$. Let $K_0 = [0,1] \setminus \bigcup_{i\in\mathbb{N}} [a_i, b_i]$. Since $f \in \mathrm{FRC}(H, \bar{q})$, so by Proposition 2.3, for each $i \in \mathbb{N}$, $f \restriction [a_i, b_i] \in \mathrm{DB}_1([a_i, b_i])$, so it has a connected graph. Thus the condition (i) of Lemma 3.1 is fulfilled.

Now we will show that the condition (ii) of this lemma is also fulfilled.

Let $x_0 \in K_0$ and let $\{a_{p_n}\}_{n\in\mathbb{N}}$ be a subsequence of $\{a_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} a_{p_n} = x_0$.

Fix a decreasing and converging to zero sequence of positive numbers $\{\varepsilon_i\}_{i\in\mathbb{N}}$. For simplicity of notation, let us assume that $\varepsilon_1 < \frac{1}{2}$ when $x_0 = 0$ or $x_0 = 1$ and, $\varepsilon_1 < \frac{1}{2}\min(x_0, 1 - x_0)$ when $x_0 \neq 0$ and $x_0 \neq 1$.

For each $i \in \mathbb{N}$ choose $\delta_i \in (0, \varepsilon_i)$ such that for each $j \in \mathbb{N}$, if $I_j \cap (x_0 - \delta_i, x_0 + \delta_i) \neq \emptyset$, then

$$f(\{q_k\colon\ k = 1, 2, \ldots\} \cap I_j \cap (x_0 - \delta_i, x_0 + \delta_i)) \cap (f(x_0) - \varepsilon_i, f(x_0) + \varepsilon_i) \neq \emptyset. \tag{3.1}$$

For each $l \in \mathbb{N}$, fix $n_l \in \mathbb{N}$ such that

$$(a_{p_n}, b_{p_n}) \cap (x_0 - \delta_l, x_0 + \delta_l) \neq \emptyset, \quad \text{for } n \geqslant n_l. \tag{3.2}$$

We may require the sequence $\{n_l\}_{n\in\mathbb{N}}$ to be increasing.

We shall define a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \in [a_{p_n}, b_{p_n}]$, for $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty}(x_n, f(x_n)) = (x_0, f(x_0))$.

Let $n \in \mathbb{N}$. One of the following cases is possible:

(1) $n < n_1$. Then put $x_n = a_{p_n}$.

(2) There exists $j_0 \in \mathbb{N}$ such that $n \in [n_{j_0}, n_{j_0+1})$. By (3.2), we have $(a_{p_n}, b_{p_n}) \cap (x_0 - \delta_{j_0}, x_0 + \delta_{j_0}) \neq \emptyset$. Using the implication (3.1), choose an element q_{k_n} of the H-trajectory $\bar{q}$ such that $q_{k_n} \in (a_{p_n}, b_{p_n}) \cap (x_0 - \delta_{j_0}, x_0 + \delta_{j_0})$ and $f(q_{k_n}) \in (f(x_0) - \varepsilon_{j_0}, f(x_0) + \varepsilon_{j_0})$. Put $x_n = q_{k_n}$.

It is easy to notice that $\lim\limits_{n\to\infty}(x_n, f(x_n)) = (x_0, f(x_0))$. Thus the condition (ii) of Lemma 3.1 is fulfilled, and this completes the proof. $\square$

It is easy to notice that the following inclusion is true:

$$\mathrm{DB}_1([0,1]) \subset Conn^*.$$

What is more, it occurs that the set $\mathrm{DB}_1([0,1])$ is porous in the space $Conn^*$ with the metric of uniform convergence. The proof of this fact will be preceded with some construction and two technical lemmas.

Let H be an od-set, $\bar{q} = \{q_n\}_{n\in\mathbb{N}}$ an H-trajectory and let $\breve{H}$ be an open and dense subset of H. Then we can construct an $\breve{H}$-trajectory generated by the H-trajectory $\bar{q}$. To this end we will define (by induction) an increasing sequence $\{k_n\}_{n\in\mathbb{N}}$ of positive integers in the following way. Let $k_1 = \min\{k\colon\ q_k \in \breve{H}\}$. Assume that the elements $k_1, k_2, \ldots, k_n$ of this sequence have been already defined, where n is some positive integer. Then we put $k_{n+1} = \min\{k > k_n\colon\ q_k \in \breve{H}\}$. Continuing this process we obtain an infinite sequence $\bar{q}^* = \{q_{k_n}\}_{n\in\mathbb{N}}$. It is easy to notice that $\bar{q}^*$ consists of these and only these elements of $\bar{q}$ which belong to $\breve{H}$. Then the sequence $\bar{q}^*$ is an $\breve{H}$-trajectory. We will call it an $\breve{H}$-trajectory generated by the H-trajectory $\bar{q}$.

Lemma 3.3. *Let $\breve{H} \subset H$ be two od-sets in $[0,1]$, $\bar{q} = \{q_n\}_{n\in\mathbb{N}}$ an H-trajectory, $I \in \mathrm{Cp}(\breve{H})$ and let $\bar{q}^* = \{q_{k_n}\}_{n\in\mathbb{N}}$ be an $\breve{H}$-trajectory generated by the H-trajectory $\bar{q}$. Then:*

(i) *for every $x \in [a,b)$ (where a, b denote the endpoints of the interval I) and for each $j \in \{0,1,2,\ldots\}$ we have $s_{m_0+j} = w_{l_0+j}$, where $\{s_m\}_{m\in\mathbb{N}} = P_r(x,\bar{q})$, $\{w_l\}_{l\in\mathbb{N}} = P_r(x,\bar{q}^*)$, $m_0 = \min\{m\colon\ s_m \in (x,b)\}$, $l_0 = \min\{l\colon\ w_l \in (x,b)\}$;*

(ii) *for every $x \in (a,b]$ and for each $j \in \{0,1,2,\ldots\}$ we have $z_{i_0+j} = u_{r_0+j}$, where $\{z_i\}_{i\in\mathbb{N}} = P_l(x,\bar{q})$, $\{u_r\}_{r\in\mathbb{N}} = P_l(x,\bar{q}^*)$, $i_0 = \min\{i\colon\ z_i \in (a,x)\}$, $r_0 = \min\{r\colon\ u_r \in (a,x)\}$.*

P r o o f. We will prove (i). Let $x \in [a,b)$. It is easy to show

$$r_{\bar{q}^*}((x,c)) = r_{\bar{q}}((x,c)) \quad \text{for any } c \in (x,b]. \tag{3.3}$$

First we will show that $s_{m_0} = w_{l_0}$.

By the definition of l_0 and by the form of the sequence $\{w_l\}_{l\in\mathbb{N}}$, we can conclude that $w_{l_0} = r_{\bar{q}^*}((x, w_{l_0-1}))$ and $b \leqslant w_{l_0-1}$ (if $l_0 = 1$ we can put $w_{l_0-1} = 1$ and then $r_{\bar{q}^*}((x,b)) = r_{\bar{q}^*}((x,1)))$. Since $w_{l_0} \in (x,b) \subset (x, w_{l_0-1})$, we have $r_{\bar{q}^*}((x, w_{l_0-1})) = r_{\bar{q}^*}((x,b))$. Consequently,

$$w_{l_0} = r_{\bar{q}^*}((x,b)). \tag{3.4}$$

An exactly similar reasoning shows that

$$s_{m_0} = r_{\bar{q}}((x,b)). \tag{3.5}$$

By (3.3)–(3.5), we have $s_{m_0} = r_{\bar{q}}((x,b)) = r_{\bar{q}^*}((x,b)) = w_{l_0}$.

Assume now that $s_{m_0+j} = w_{l_0+j}$ for some integer $j \geqslant 0$. We shall show that $s_{m_0+j+1} = w_{l_0+j+1}$.

By the inductive assumption we have $(x, s_{m_0+j}) = (x, w_{l_0+j}) \subset (x,b)$. Thus and by (3.3) we obtain

$$s_{m_0+j+1} = r_{\bar{q}}((x, s_{m_0+j})) = r_{\bar{q}}((x, w_{l_0+j})) = w_{l_0+j+1},$$

which completes the proof of (i).

The proof of (ii) is analogous. $\square$

Lemma 3.4. *Let $f\colon [0,1] \to \mathbb{R}$, $H \subset [0,1]$ be an od-set, $\bar{q}$ an H-trajectory and $x_0 \in [0,1]$. Assume that for every $\varepsilon > 0$ there exists $\delta \in (0,\varepsilon)$ such that for every component I of H having nonempty intersection with the interval $(x_0-\delta, x_0+\delta)$, there exists a point $x \in C(f) \cap I \cap (x_0-\delta, x_0+\delta)$ with the property $f(x) \in (f(x_0)-\varepsilon, f(x_0)+\varepsilon)$. Then the function f has a $D(H,\bar{q})$ property at the point x_0.*

Proof. Let $\varepsilon > 0$ and let $\delta \in (0,\varepsilon)$ be the number chosen according to the assumptions of the lemma. Let us consider a component I of the set H such that $I \cap (x_0-\delta, x_0+\delta) \neq \emptyset$. Then there exists a point $x_1 \in C(f) \cap I \cap (x_0-\delta, x_0+\delta)$ such that $f(x_1) \in (f(x_0)-\varepsilon, f(x_0)+\varepsilon)$. One can find $\sigma > 0$ such that $(x_1-\sigma, x_1+\sigma) \cap I \subset (x_0-\delta, x_0+\delta) \cap I$ and

$$f((x_1-\sigma, x_1+\sigma) \cap I) \subset (f(x_0)-\varepsilon, f(x_0)+\varepsilon). \tag{3.6}$$

Of course $\operatorname{Int}((x_1-\sigma, x_1+\sigma) \cap I) \neq \emptyset$, so there exists a positive integer k_0 such that $q_{k_0} \in (x_1-\sigma, x_1+\sigma) \cap I \subset (x_0-\delta, x_0+\delta) \cap I$. Moreover, by (3.6) we have $f(q_{k_0}) \in (f(x_0)-\varepsilon, f(x_0)+\varepsilon)$.

Thus the function f has the $D(H,\bar{q})$ property at x_0. $\square$

Theorem 3.5. *The set* $\mathrm{DB}_1([0,1])$ *is porous in the space* $(Conn^*, \varrho)$.

P r o o f. Let $f \in Conn^*$ and $\varepsilon > 0$. There exist an od-set $H \subset [0,1]$ and an H-trajectory $\bar{q}$ such that f is H-connected with respect to $\bar{q}$. Let $I^* \in \mathrm{Cp}(H)$. Denote by a^* and b^* the left and the right endpoint of I^*, respectively. By Proposition 2.3, we have $f \restriction \mathrm{Cl}(I^*) \in \mathrm{DB}_1(\mathrm{Cl}(I^*))$, so the function f has a continuity point $x_0 \in (a^*, b^*)$. We can choose a number $\delta > 0$ such that $[x_0-\delta, x_0+\delta] \subset (a^*, b^*)$ and

$$f([x_0-\delta, x_0+\delta]) \subset \Big(f(x_0) - \frac{\varepsilon}{3}, f(x_0) + \frac{\varepsilon}{3}\Big). \tag{3.7}$$

Now define a function f_1.

On the interval $[x_0-\delta, x_0+\delta]$ let us consider a Cantor-like set $\breve{C}$ containing $x_0-\delta$ and $x_0+\delta$ and denote by $\breve{C}^*$ the set of ends of all components of the set $[x_0-\delta, x_0+\delta] \setminus \breve{C}$.

Let $\breve{H} = H \setminus \breve{C}$. Obviously $\breve{H}$ is an open and dense subset of H, so we can consider an $\breve{H}$-trajectory $\bar{q}^* = \{q_{k_n}\}_{n\in\mathbb{N}}$ generated by the H-trajectory $\bar{q}$.

Let (a,b) be a component of the set $[x_0-\delta, x_0+\delta] \setminus \breve{C}$ and let $P_r(a, \bar{q}^*) = \{t_k(a)\}_{k\in\mathbb{N}}$ be the right first return path at a with respect to the $\breve{H}$-trajectory $\bar{q}^*$ and $k_0 = \min\{k\colon t_k(a) \in (a, (b-a)/2)\}$. For $i \geqslant k_0$, let x_1^i, x_2^i be some points fulfilling inequalities $t_{i+1}(a) < x_1^i < x_2^i < t_i(a)$. Similarly, let $P_l(b, \bar{q}^*) = \{s_l(b)\}_{l\in\mathbb{N}}$ be the left first return path at b with respect to the $\breve{H}$-trajectory $\bar{q}^*$ and $l_0 = \min\{l\colon s_l(b) \in ((b-a)/2, b)\}$. For $j \geqslant l_0$, let y_1^j, y_2^j be some points fulfilling inequalities $s_j(b) < y_1^j < y_2^j < s_{j+1}(b)$. On the interval $[a,b]$ define the function f_1 as follows: $f_1(x) = f(x_0)$ for $x \in \{a,b\} \cup \{t_i(a)\colon i \geqslant k_0\} \cup \{s_j(b)\colon j \geqslant l_0\} \cup (t_{k_0}(a), s_{l_0}(b))$; $f_1(x) = f(x_0) - \varepsilon/3$ for $x \in \{x_1^i\colon i \geqslant k_0\} \cup \{y_1^j\colon j \geqslant l_0\}$; linear on the intervals $[t_{i+1}(a), x_1^i]$, $[x_1^i, x_2^i]$, $[x_2^i, t_i(a)]$ $(i \geqslant k_0)$, $[s_j(b), y_1^j]$, $[y_1^j, y_2^j]$, $[y_2^j, s_{j+1}(b)]$ $(j \geqslant l_0)$.

In an analogous way we define the function f_1 on each component of the set $[x_0-\delta, x_0+\delta] \setminus \breve{C}$.

Consider a function $g\colon [0,1] \to \mathbb{R}$ defined as follows: $g(x) = f_1(x)$ for $x \in (x_0-\delta, x_0+\delta) \setminus (\breve{C} \setminus \breve{C}^*)$; $g(x) = f(x_0) + \varepsilon/3$ for $x \in \breve{C} \setminus (\breve{C}^* \cup \{x_0-\delta, x_0+\delta\})$; $g(x) = f(x)$ for $x \in [0, x_0-\delta] \cup [x_0+\delta, 1]$.

We will show that $g \in \mathrm{FRC}(\breve{H}, \bar{q}^*)$.

First we will prove that the function g is first return continuous from the right with respect to $\bar{q}^*$ at each point of the set $\breve{H}$ and at the left endpoint of each component of the set $\breve{H}$.

Let $x^* \in \breve{H}$ or let x^* be the left endpoint of some component of $\breve{H}$. The proof conveniently splits into the following cases:

1.1. $x^* \in [x_0+\delta, 1)$. Then it is sufficient to notice that $P_r(x^*, \bar{q}^*) = P_r(x^*, \bar{q})$.

1.2. $x^* \in (x_0-\delta, x_0+\delta)$. The property to be proved is evidently true if $x^* \in \breve{H}$. So let us consider the case when x^* is the left endpoint of some component of the set $\breve{H}$. Denote $P_r(x^*, \bar{q}^*) = \{t_i(x^*)\}_{i\in\mathbb{N}}$. There exists $j_0 \in \mathbb{N}$ such that for every $i \geqslant j_0$ we have $g(t_i(x^*)) = f(x_0) = g(x^*)$. It means that

$$\lim_{\substack{t\to x^* \\ t\in P_r(x^*,\bar{q}^*)}} g(t) = g(x^*),$$

so g is first return continuous from the right at x^* with respect to $\bar{q}^*$.

1.3. $x^* \in [0, x_0 - \delta)$. Let $J \in \mathrm{Cp}(H)$ be such that $x^* \in \mathrm{Cl}(J)$. Let us denote by a_J and b_J the left and the right endpoint of J, respectively. Consider first return paths $P_r(x^*, \bar{q}) = \{s_m\}_{m\in\mathbb{N}}$ and $P_r(x^*, \bar{q}^*) = \{w_p\}_{p\in\mathbb{N}}$. Put $m_0 = \min\{m\colon s_m \in (x^*, b_J)\}$ and $p_0 = \min\{p\colon w_p \in (x^*, b_J)\}$. By Lemma 3.3 we have $s_{m_0+i} = w_{p_0+i}$ for $i \in \{1, 2, \ldots\}$. So the conclusion that g is first return continuous from the right with respect to $\bar{q}^*$ at x^* is immediate.

An exactly similar reasoning shows that the function g is first return continuous from the left with respect to $\bar{q}^*$ at each point of the set $\breve{H}$ and at the right end of each component of the set $\breve{H}$. We next show that g has the $D(\breve{H}, \bar{q}^*)$ property at each point of the set $[0,1] \setminus \breve{H}$.

Let $z \in [0,1] \setminus \breve{H}$ and $\varepsilon_0 > 0$.

2.1. Assume $z \in [0, x_0 - \delta) \setminus \breve{H}$. Let $\varepsilon_1 \in (0, \min\{\varepsilon_0, x_0 - \delta - z\})$. There exists $\delta_1 \in (0, \varepsilon_1)$ such that for each $I \in \mathrm{Cp}(H)$ the following condition is fulfilled if $(z - \delta_1, z + \delta_1) \cap I \neq \emptyset$ then

$$f(\{q_k\colon k = 1, 2, \ldots\} \cap I \cap (z - \delta_1, z + \delta_1)) \cap (f(z) - \varepsilon_1, f(z) + \varepsilon_1) \neq \emptyset. \tag{3.8}$$

Of course $z + \delta_1 < x_0 - \delta$ and $\delta_1 \in (0, \varepsilon_0)$. Let $\breve{I}_1 \in \mathrm{Cp}(\breve{H})$ be such that $\breve{I}_1 \cap (z - \delta_1, z + \delta_1) \neq \emptyset$. Then the following cases are possible:

a) $\breve{I}_1 \in \mathrm{Cp}(H)$ or
b) $\breve{I}_1 = (a^*, x_0 - \delta) \subset I^* \in \mathrm{Cp}(H)$.

In the case a) the condition (3.8), and in the case b) the condition (3.8) and the equality $\breve{I}_1 \cap (z - \delta_1, z + \delta_1) = I^* \cap (z - \delta_1, z + \delta_1)$, imply the existence of $n^1 \in \mathbb{N}$ such that $q_{n^1} \in \breve{I}_1 \cap (z - \delta_1, z + \delta_1)$ and $g(q_{n^1}) = f(q_{n^1}) \in (f(z) - \varepsilon_1, f(z) + \varepsilon_1) \subset (g(z) - \varepsilon_0, g(z) + \varepsilon_0)$. Of course there exists also n_1 such that $q^*_{k_{n_1}} = q_{n^1}$. Thus g has the $D(\breve{H}, \bar{q}^*)$ property at z.

2.2. An exactly similar reasoning applies to the case $z \in (x_0 + \delta, 1] \setminus \breve{H}$.

2.3. Assume now $z \in \breve{C} \setminus (\breve{C}^* \cup \{x_0 - \delta, x_0 + \delta\})$. Let $\delta_2 \in (0, \varepsilon_0)$ be sufficiently small for the inclusion $(z - \delta_2, z + \delta_2) \subset (x_0 - \delta, x_0 + \delta)$ to be true. Let $\breve{I}_2 \in \mathrm{Cp}(\breve{H})$ and $\breve{I}_2 \cap (z - \delta_2, z + \delta_2) \neq \emptyset$. Of course $\breve{I}_2 = (a_0, b_0)$ for some $a_0, b_0 \in \breve{C}^*$. We have $a_0 > z$ or $b_0 < z$. Without loss of generality, we may assume that $a_0 > z$. From the properties of g we can deduce that there exists a point $y^* \in (a_0, \min\{z + \delta_2, b_0\})$

such that $g(y^*) = f(x_0) + \varepsilon/3 = g(z)$. Of course $y^* \in C_g \cap \breve{I}_2 \cap (z - \delta_2, z + \delta_2)$ and $g(y^*) \in (g(z) - \varepsilon_0, g(z) + \varepsilon_0)$. By Lemma 3.4, the function g has the $D(\breve{H}, \bar{q}^*)$ property at z.

2.4. Now let $z \in \breve{C}^*$. Then z is the left or the right end of some component (a_1, b_1) of $\breve{H}$. Assume that $z = a_1$ (the case $z = b_1$ is analogous). Let

$$\delta_3 \in (0, \min\{\varepsilon_0, b_1 - z, z - x_0 + \delta\}). \tag{3.9}$$

Moreover, let $\breve{I}_3 = (a_3, b_3) \in \mathrm{Cp}(\breve{H})$ and $(a_3, b_3) \cap (z - \delta_3, z + \delta_3) \neq \emptyset$. Then, by (3.9), there are two possibilities:

2.4.a) $(a_3, b_3) = (a_1, b_1)$ or

2.4.b) $x_0 - \delta < a_3$ and $z - \delta_3 < b_3 < z$.

In both the cases there exists a point $y^{**} \in (a_3, b_3) \cap (z - \delta_3, z + \delta_3)$ such that $g(y^{**}) = f(x_0) = g(z)$. Of course $y^{**} \in C_g \cap \breve{I}_3 \cap (z - \delta_3, z + \delta_3)$ and $g(y^{**}) \in (g(z) - \varepsilon_0, g(z) + \varepsilon_0)$. By Lemma 3.4, the function g has the $D(\breve{H}, \bar{q}^*)$ property at z.

2.5. Now let $z \in \{x_0 - \delta, x_0 + \delta\}$. Assume that $z = x_0 - \delta$ (the case $z = x_0 + \delta$ is analogous). The equality $g(z) = f(x_0 - \delta)$ and the condition (3.7) imply $g(z) \in (f(x_0) - \varepsilon/3, f(x_0) + \varepsilon/3)$. Let $\delta_4 \in (0, \min\{\varepsilon_0, 2\delta, x_0 - \delta - a^*\})$, $\breve{I}_4 \in \mathrm{Cp}(\breve{H})$ and $\breve{I}_4 \cap (z - \delta_4, z + \delta_4) \neq \emptyset$. Then we can consider two cases:

2.5.a) $\breve{I}_4 \subset (x_0 - \delta, x_0 + \delta)$ or

2.5.b) $\breve{I}_4 = (a^*, x_0 - \delta)$ (or $\breve{I}_4 = [a^*, x_0 - \delta)$, if $a^* = 0$ and $0 \in H$).

In the case 2.5.a) one can apply a reasoning exactly similar to that in 2.3 and 2.4. In the case 2.5.b) it is sufficient to notice that $g(x) = f(x)$ for $x \in [0, x_0 - \delta]$ and if $q_n \in [0, x_0 - \delta)$, then $q_n \in \{q_k^*\colon\ k = 1, 2 \ldots\}$ for $n \in \mathbb{N}$.

We have completed the proof that $g \in Conn^*$.

We will show now that $B(g, \varepsilon/9) \subset B(f, \varepsilon)$.

Let $\xi \in B(g, \varepsilon/9)$. We have

$$\varrho(f, \xi) \leqslant \varrho(f, g) + \varrho(g, \xi) < \frac{2\varepsilon}{3} + \frac{\varepsilon}{9} < \varepsilon.$$

Since $\xi \in B(g, \varepsilon/9)$ was chosen arbitrarily, $B(g, \varepsilon/9) \subset B(f, \varepsilon)$.

The task is now to show that $B(g, \varepsilon/9) \cap \mathrm{DB}_1([0, 1]) = \emptyset$.

Let $\tau \in B(g, \varepsilon/9)$. It is easy to notice that $\tau(x) \in (f(x_0) - \varepsilon/9, f(x_0) + \varepsilon/9)$ for $x \in \breve{C}^*$ and $\tau(x) \in (f(x_0) + 2\varepsilon/9, f(x_0) + 4\varepsilon/9)$ for $x \in \breve{C} \setminus \breve{C}^*$. This means that the function $\tau \restriction \breve{C}$ has no continuity point, so τ is not a Baire 1 function. Since $\tau \in B(g, \varepsilon/9)$ is arbitrary, we conclude that $B(g, \varepsilon/9) \cap \mathrm{DB}_1([0, 1]) = \emptyset$. $\square$

4. Rings of H-connected functions

Many papers (e.g. [3], [4], [12], [13], [16], [17], [18]) deal with algebraic properties of some classes of functions (connected with generalizations of continuity and Darboux-like functions) or with the possibility of building algebraic structures with respect to one operation consisting of functions belonging to a fixed class. In this section we examine a possibility of constructing more complex algebraic structures containing a fixed function. Simultaneously, our aim is to analyse structures containing all continuous functions.

The additive group G is called an *l-additive group* if the following condition is fulfilled:

$$\text{if } f,g\in G,\ \text{then } \max(f,g)\in G \text{ and } \min(f,g)\in G.$$

The ring $\Re$ of real functions defined on $[0,1]$ is called an *l-ring* if the following condition is fulfilled:

$$\text{if } f,g\in \Re,\ \text{then } \max(f,g)\in \Re \text{ and } \min(f,g)\in \Re.$$

The ring $\Re$ of real functions defined on $[0,1]$ is called a *complete ring* if it is an l-ring containing the class of all continuous functions.

The question about existence of rings including the class of all continuous functions and a fixed function and consisting of functions belonging only to some fixed family, is an interesting and frequently considered problem. The next theorem is the result obtained for the class of H-connected functions with respect to an H-trajectory $\bar{q}$, when an od-set H and an H-trajectory $\bar{q}$ are fixed.

Theorem 4.1. *Let H be an od-set, let $\bar{q}=\{q_n\}_{n\in\mathbb{N}}$ be an H-trajectory and $f\colon [0,1]\to\mathbb{R}$ an H-connected function with respect to $\bar{q}$. Then there exists a complete ring $\Re$ consisting of H-connected functions with respect to $\bar{q}$ such that $f\in\Re$.*

P r o o f. Assume first that $[0,1]\setminus H\neq\emptyset$. For every point $x\in[0,1]\setminus H$ and for each positive integer n, fix a number $\delta_x(n)\in(0,1/n)$ such that for every component I of the set H the following implication holds if $I\cap(x-\delta_x(n),x+\delta_x(n))\neq\emptyset$, then

$$(4.1)\quad f(\{q_k\colon\ k=1,2,\ldots\}\cap I\cap(x-\delta_x(n),x+\delta_x(n)))\cap\Big(f(x)-\frac{1}{n},f(x)+\frac{1}{n}\Big)\neq\emptyset.$$

For each pair $(x,n)\in([0,1]\setminus H)\times\mathbb{N}$ let

$$C_n^x=\{I\in\mathrm{Cp}(H)\colon\ I\cap(x-\delta_x(n),x+\delta_x(n))\neq\emptyset\}.$$

Fix $x \in [0,1] \setminus H$ and $n \in \mathbb{N}$. For any component $I \in C_n^x$ fix a point

$$(*) \qquad y_{x,n}^I \in \{q_k\colon\ k = 1,2,\ldots\} \cap I \cap (x - \delta_x(n), x + \delta_x(n)) \\ \text{such that } f(y_{x,n}^I) \in \Bigl(f(x) - \frac{1}{n}, f(x) + \frac{1}{n}\Bigr).$$

Denote $D(x,n) = \{y_{x,n}^I\colon\ I \in C_n^x\}$. For each pair $(x,n) \in ([0,1] \setminus H) \times \mathbb{N}$ we can choose a set $D(x,n)$ in the way presented above.

Let $\Re$ be the family of all functions $g\colon\ [0,1] \to \mathbb{R}$ fulfilling the following conditions:

1.1 $g \in \mathrm{FRC}(H, \bar{q})$;

1.2 for any $x \in [0,1] \setminus H$ and for any $\varepsilon > 0$ there exists $n(g,x,\varepsilon) \in \mathbb{N}$ such that for any integer $n \geqslant n(g,x,\varepsilon)$ we have $g(D(x,n)) \subset (g(x) - \varepsilon, g(x) + \varepsilon)$. It is easy to show that the family $\Re$ is the required ring.

In the case $H = [0,1]$ it is enough to put $\Re = \mathrm{FRC}(H, \bar{q})$. □

Notice that in the proof of Theorem 4.1 two methods of constructing a complete ring containing a fixed function and consisting of functions H-connected with respect to an H-trajectory $\bar{q}$ are presented. The first method concerns the situation when $[0,1] \setminus H \neq \emptyset$ and the second the case when $H = [0,1]$. For the od-set $H \subset [0,1]$ and the function $f\colon\ [0,1] \to \mathbb{R}$, H-connected with respect to the H-trajectory $\bar{q}$, the symbol $\Re_f(H, \bar{q})$ will stand for the family of all rings constructed by use of one of the methods presented in the proof of Theorem 4.1, chosen adequately to the form of the set H. Moreover, in the first part of the proof of Theorem 4.1 (concerning the situation when $[0,1] \setminus H \neq \emptyset$), for a function f, H-connected with respect to the H-trajectory $\bar{q}$ and for each pair $(x,n) \in ([0,1] \setminus H) \times \mathbb{N}$ some sets $D(x,n)$ are created (by the method denoted by $(*)$). The choice of points constituting these sets is not fixed. In the further part of the paper, the family of all possible sets $D(x,n)$ created by the method $(*)$ for a function f and for a pair $(x,n) \in ([0,1] \setminus H) \times \mathbb{N}$ will be denoted by $\mathscr{D}^f(x,n)$.

The following lemma will be useful in further considerations.

Lemma 4.2. *Let $H \subset [0,1]$ be an od-set such that $[0,1] \setminus H \neq \emptyset$. Let $\bar{q}$ be an H-trajectory, let $f\colon\ [0,1] \to \mathbb{R}$ be H-connected with respect to $\bar{q}$ and let $(x,n) \in ([0,1] \setminus H) \times \mathbb{N}$. If $D(x,n) \in \mathscr{D}^f(x,n)$, then $D(x,n) \in \mathscr{D}^{-f}(x,n)$.*

The proof of the above lemma is easy but long, so we omit it.

Signalizing the above ring construction methods is not accidental. The next theorem will show that these methods are optimal; we mean that if there exists a ring containing functions f and g, then there exists a ring common to functions f and g and such that it can be constructed by the method presented in the proof of Theorem 4.1. This theorem will also show that, under some additional assumptions, the existence of an additive group containing two fixed Darboux functions implies the existence of a Darboux ring (constructed by the methods presented in the proof of Theorem 4.1) containing these functions. Notice that this is interesting because in the case of the class DB_1 the existence of an additive group consisting of Darboux functions usually does not imply the existence of a Darboux ring containing these two functions. Indeed, let $\{x\}_{n\in\mathbb{N}}$ and $\{y\}_{n\in\mathbb{N}}$ be sequences converging to 0 and such that $1 = x_1 > y_1 > x_2 > y_2 > x_3 > y_3 > \ldots$. Let f and g be functions fulfilling the following conditions: $f(0) = \frac{1}{2}$, $f(y_{2n}) = 1$, $f([x_{2n}, x_{2n-1}]) = 0$, f is linear on the intervals $[y_{2n-1}, x_{2n-1}]$, $[x_{2n}, y_{2n-1}]$ $(n \in \mathbb{N})$, and $g(0) = \frac{1}{2}$, $g(1) = 0$, $g(y_{2n-1}) = 1$, $g([x_{2n+1}, x_{2n}]) = 0$, g is linear on the intervals $[y_{2n}, x_{2n}]$, $[x_{2n+1}, y_{2n}]$ $(n \in \mathbb{N})$. Of course $f, g \in \mathrm{DB}_1$. It is easy to notice that $f \cdot g \notin \mathrm{D}\mathrm{B}x([0,1])$, so there is no ring of Darboux Baire 1 functions containing functions f and g. Consider the family G of functions $h\colon [0,1] \to \mathbb{R}$ continuous on $(0,1]$ and satisfying the following conditions:

(i) $h(x_n) = 0$ for $n \in \mathbb{N}$;

(ii) there exist $a, b \in \mathbb{R}$ such that $h(0) = (a+b)/2$ and $h(y_{2n}) = a$, $h(y_{2n-1}) = b$ for $n \in \mathbb{N}$.

It is easy to show that $f, g \in G$ and $G \subset \mathrm{DB}_1$ is an additive group.

Theorem 4.3. *Let $f, g\colon [0,1] \to \mathbb{R}$ be Darboux functions which do not intersect the axis and such that $D(f) \cup D(g) \subset Z(f) \cap Z(g)$. Then the following conditions are equivalent:*

(i) *there exists an additive group G of Darboux functions such that $f, g \in G$;*

(ii) *there exists a ring $\Re_0$ of Darboux functions such that $f, g \in \Re_0$;*

(iii) *there exist an od-set H, an H-trajectory $\bar{q}$ and a ring $\Re \in \Re_f(H, \bar{q}) \cap \Re_g(H, \bar{q})$.*

P r o o f. The proof will proceed in the following way: (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).

To deduce (iii) from (i), assume that $D(f) = D(g) = \emptyset$. Let $H = [0,1]$ and let $\bar{q}$ be an H-trajectory. Put $\Re = \mathrm{FRC}(H, \bar{q})$. Then $\Re \in \Re_f(H, \bar{q}) \cap \Re_g(H, \bar{q})$.

We now turn to the case $D(f) \neq \emptyset$ or $D(g) \neq \emptyset$. We need only consider 3 cases:

1. $f \geqslant 0$ and $g \geqslant 0$;
2. $f \leqslant 0$ and $g \leqslant 0$;
3. ($f \geqslant 0$ and $g \leqslant 0$) or ($f \leqslant 0$ and $g \geqslant 0$).

Case 1. Notice that $f, g \in \mathrm{B}_1^{**}$.

This means that the sets $D(f)$ and $D(g)$ are nowhere dense ([20], Lemma 2). It is easy to notice that $\mathrm{Cl}(D(f)) \cup \mathrm{Cl}(D(g))$ is a nowhere dense set and

$$\mathrm{Cl}(D(f)) \cup \mathrm{Cl}(D(g)) \subset Z(f) \cap Z(g). \tag{4.2}$$

Let $H = [0,1] \setminus (\mathrm{Cl}(D(f)) \cup \mathrm{Cl}(D(g)))$. Of course H is an od-set. Put $h = f + g$.

We will show that

$$h \in \mathrm{DB}_1([0,1]). \tag{4.3}$$

Of course $h \in \mathrm{DB}x([0,1])$. From the fact that $f, g \in \mathrm{B}_1^{**}$ and from Proposition 1 in [20] we obtain $f, g \in B_1([0,1])$. Hence $h \in B_1([0,1])$ and consequently $h \in \mathrm{DB}_1([0,1])$.

Introduce some notation. Let L be the set of left endpoints of all components of H being intervals open from the left and let P be the set of right endpoints of all components of H being intervals open from the right.

Fix $a \in L$. Denote by J_l^a the component of H such that a is the left end of it. Then $a \in \mathrm{Cl}(D(f)) \cup \mathrm{Cl}(D(g))$. Because of the inclusion (4.2) we have $f(a) = g(a) = 0$, so $h(a) = 0$.

From (4.3) and by the Young condition ([5], Theorem 1.1) we deduce that there exists a decreasing sequence $\{z_n^a\}_{n \in N} \subset J_l^a$ such that $\lim\limits_{n\to\infty} z_n^a = a$ and $\lim\limits_{n\to\infty} h(z_n^a) = h(a) = 0$. Let $\varepsilon_1 > 0$. Then there exists $n_0 \in \mathbb{N}$ such that for $n \geqslant n_0$ we have $|f(z_n^a) + g(z_n^a)| < \varepsilon_1$. Hence (using the assumption that $f \geqslant 0$ and $g \geqslant 0$) we deduce that $f(z_n^a) < \varepsilon_1$ and $g(z_n^a) < \varepsilon_1$ for $n \geqslant n_0$. Since ε_1 is arbitrary, we have $\lim\limits_{n\to\infty} f(z_n^a) = 0 = f(a)$ and $\lim\limits_{n\to\infty} g(z_n^a) = 0 = g(a)$.

Let $b \in P$. Denote by J_r^b a component of the set H such that b is the right end of it. Of course $h(b) = 0$. Analogously as above we choose an increasing sequence $\{y_n^b\}_{n\in\mathbb{N}} \subset J_r^b$ such that $\lim\limits_{n\to\infty} y_n^b = b$, $\lim\limits_{n\to\infty} h(y_n^b) = h(b) = 0$, $\lim\limits_{n\to\infty} f(y_n^b) = 0 = f(b)$ and $\lim\limits_{n\to\infty} g(y_n^b) = 0 = g(b)$.

Let

$$A = (H \cap \mathbb{Q}) \cup \{z_n^a \colon\ a \in L,\ n \in \mathbb{N}\} \cup \{y_n^b \colon\ b \in P,\ n \in \mathbb{N}\}.$$

Of course it would be sufficient to put $A = H \cap \mathbb{Q}$, but considering the set A in the above form makes the notation of the further part of the proof easier and shorter.

The set A is countable and dense in $[0,1]$.

We will show that the set

$$W = \{(x, h(x)) \colon\ x \in A\} \text{ is dense in the set } G(h). \tag{4.4}$$

Fix $(x^*, h(x^*)) \in G(h)$ and let $\varepsilon_2 > 0$. Consider an open cube $K = (x^* - \varepsilon_2, x^* + \varepsilon_2) \times (h(x^*) - \varepsilon_2, h(x^*) + \varepsilon_2)$. We will show that $W \cap K \neq \emptyset$.

Assume first that x^* is a continuity point of h. Then there exists $\delta_2 \in (0, \varepsilon_2)$ such that $h((x^* - \delta_2, x^* + \delta_2) \cap [0,1]) \subset (h(x^*) - \varepsilon_2, h(x^*) + \varepsilon_2)$. Since A is dense in $[0,1]$, there exists a point $x_1 \in A \cap (x^* - \delta_2, x^* + \delta_2) \subset A \cap (x^* - \varepsilon_2, x^* + \varepsilon_2)$. So we have $(x_1, h(x_1)) \in W \cap K$.

Now let $x^* \in D(h)$. Then $x^* \in D(f)$ or $x^* \in D(g)$. Of course $h(x^*) = 0$. The following cases are possible:

1.1 $x^* \in L$;

1.2 $x^* \in P$;

1.3 $x^* \in [0,1] \setminus (H \cup L \cup P)$.

In the case 1.1 we have $z_n^{x^*} \searrow x^*$ and $\lim\limits_{n\to\infty} h(z_n^{x^*}) = 0$. Thus there exists $n_1 \in \mathbb{N}$ such that $z_n^{x^*} \in (x^*, x^* + \varepsilon_2)$ and $h(z_n^{x^*}) \in (-\varepsilon_2, \varepsilon_2)$, for $n \geqslant n_1$.

We have $(z_n^{x^*}, h(z_n^{x^*})) \in W \cap K$ for $n \geqslant n_1$.

In the case 1.2, analogously to 1.1, we find $n_2 \in \mathbb{N}$ such that $(y_n^{x^*}, h(y_n^{x^*})) \in W \cap K$ for $n \geqslant n_2$.

We now turn to the case 1.3. Assume first that $x^* \neq 1$. Since $x \notin L$ and the set H is dense in $[0,1]$, there exists a point $a^* \in L \cap (x^*, x^* + \varepsilon_2)$, which means that $J_l^{a^*} \cap (x^*, x^* + \varepsilon_2) \neq \emptyset$. Analogously to the case 1.1 we find $n_3 \in \mathbb{N}$ such that $(z_n^{a^*}, h(z_n^{a^*})) \in W \cap K$ for $n \geqslant n_3$. In the case $x^* = 1$ one can notice that there exists a point $b^* \in P \cap (x^* - \varepsilon_2, x^*)$. Further we carry out considerations analogous to those in the case $x^* \neq 1$.

The proof of (4.4) is completed.

By Lemma 2.2 there exists such an ordering $\bar{q} = \{q_n\}_{n\in\mathbb{N}}$ of the set A that $h \in \mathrm{FRC}([0,1], \bar{q})$. It is easy to notice that $\bar{q}$ is an H-trajectory. Of course

(4.5) the function h is H-connected with respect to $\bar{q}$.

We will show that

(4.6) the functions f and g are H-connected with respect to $\bar{q}$.

Let $x \in H$. Then $x \in C(f) \cap C(g)$, so f and g are first return continuous at x with respect to $\bar{q}$.

Now let $x \in L$. Then $f(x) = g(x) = h(x) = 0$. We will show that f and g are first return continuous from the right at x with respect to $\bar{q}$. Let $\varepsilon_3 > 0$. Since the function h is first return continuous from the right at x with respect to $\bar{q}$, there exists $\delta_3 > 0$ such that

$$\text{if } 0 < |s - x| < \delta_3, \text{ then } |h(s)| < \varepsilon_3 \text{ for every point } s \in P_r(x, \bar{q}).$$

Let $s \in P_r(x, \bar{q})$ be such a point that $0 < |s - x| < \delta_3$. Then we have $|h(s)| < \varepsilon_3$. Hence and from the fact that $f \geqslant 0$ and $g \geqslant 0$ we conclude that $|f(s)| < \varepsilon_3$ and $|g(s)| < \varepsilon_3$, so f and g are first return continuous from the right at x with respect to $\bar{q}$. Similarly, one can show that f and g are first return continuous from the left at each point of P. Thus $f, g \in \mathrm{FRC}(H, \bar{q})$.

To prove (4.6) it remains to show that

$$\text{functions } f \text{ and } g \text{ have the } D(H,\bar{q}) \text{ property at each point of the set } \mathrm{Cl}(D(f)) \cup \mathrm{Cl}(D(g)). \tag{4.7}$$

Let $x \in \mathrm{Cl}(D(f)) \cup \mathrm{Cl}(D(g))$. Then $h(x) = 0$. Let $\varepsilon_4 > 0$. Since, by (4.5), the function h has the $D(H, \bar{q})$ property at x, there exists $\delta_4 \in (0, \varepsilon_4)$ such that for every $I \in \mathrm{Cp}(H)$ the following implication is true if $I \cap (x - \delta_4, x + \delta_4) \neq \emptyset$, then

$$h(\{q_k \colon\ k = 1, 2, \ldots\} \cap I \cap (x - \delta_4, x + \delta_4)) \cap (-\varepsilon_4, \varepsilon_4) \neq \emptyset. \tag{4.8}$$

So let I be such a component of H that $I \cap (x - \delta_4, x + \delta_4) \neq \emptyset$. By (4.8) there exists $n_4 \in \mathbb{N}$ such that $q_{n_4} \in I \cap (x - \delta_4, x + \delta_4)$ and $h(q_{n_4}) \in (-\varepsilon_4, \varepsilon_4)$. Hence and from the fact that $f(q_{n_4}) \geqslant 0$ and $g(q_{n_4}) \geqslant 0$ we obtain $f(q_{n_4}) \in (-\varepsilon_4, \varepsilon_4)$ and $g(q_{n_4}) \in (-\varepsilon_4, \varepsilon_4)$. Since $x \in \mathrm{Cl}(D(f)) \cup \mathrm{Cl}(D(g))$ is arbitrary, we deduce that f and g have the $D(H, \bar{q})$ property at each point of the set $\mathrm{Cl}(D(f)) \cup \mathrm{Cl}(D(g))$.

The proof of (4.6) is completed.

Now we will construct the desired ring $\mathfrak{R}$.

Let $x \in [0, 1] \setminus H$. Using the properties of h, similarly to the proof of (4.7), for each $n \in \mathbb{N}$ we choose $\delta_x(n) \in (0, 1/n)$ and for each $I \in \mathrm{Cp}(H)$ such that $I \cap (x - \delta_x(n), x + \delta_x(n)) \neq \emptyset$ we choose a point $w^I_{x,n} \in \{q_k \colon\ k = 1, 2, \ldots\} \cap I \cap (x - \delta_x(n), x + \delta_x(n))$ such that $f(w^I_{x,n}) \in (-1/n, 1/n)$ and $g(w^I_{x,n}) \in (-1/n, 1/n)$. For each pair $(x, n) \in ([0, 1] \setminus H) \times \mathbb{N}$, denote by $D(x, n)$ the set of all elements $w^I_{x,n}$ chosen in that way. Then $D(x, n) \in \mathscr{D}^f(x, n) \cap \mathscr{D}^g(x, n)$. Considering the family $\{D(x, n) \colon\ (x, n) \in ([0, 1] \setminus H) \times \mathbb{N}\}$ we can construct the ring $\mathfrak{R} \in \mathfrak{R}_f(H, \bar{q}) \cap \mathfrak{R}_g(H, \bar{q})$ (using the method presented in the proof of Theorem 4.1).

Case 2. ($f \leqslant 0$ and $g \leqslant 0$). Of course $-f \geqslant 0$, $-g \geqslant 0$, $-f, -g \in G$, $D(-f) = D(f)$, $D(-g) = D(g)$, $Z(-f) = Z(f)$ and $Z(-g) = Z(g)$. So $D(-f) \cup D(-g) \subset Z(-f) \cap Z(-g)$. Let $h_1 = -f - g$. By Case 1 considered above, there exist an od-set H and an H-trajectory $\bar{q}$ such that the functions h_1, $-f$ and $-g$ are H-connected with respect to $\bar{q}$. Of course the functions f and g are also H-connected with respect to $\bar{q}$. For each pair $(x, n) \in ([0, 1] \setminus H) \times \mathbb{N}$, similarly to Case 1, we can create sets $D(x, n) \in \mathscr{D}^{-f}(x, n) \cap \mathscr{D}^{-g}(x, n)$. Lemma 4.2 allows to conclude that $D(x, n) \in \mathscr{D}^f(x, n) \cap \mathscr{D}^g(x, n)$ for $(x, n) \in ([0, 1] \setminus H) \times \mathbb{N}$. Considering the family $\{D(x, n) \colon\ (x, n) \in ([0, 1] \setminus H) \times \mathbb{N}\}$, we can construct the ring $\mathfrak{R} \in \mathfrak{R}_f(H, \bar{q}) \cap \mathfrak{R}_g(H, \bar{q})$ (using the first method presented in the proof of Theorem 4.1).

Case 3. One can apply the same reasoning as in Case 2.

The implications (iii) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i) are obvious. $\square$

Notice that conditions (ii) and (iii) of Theorem 4.3 regard the existence of a Darboux ring containing two functions fulfilling the assumptions of the theorem. However, separating these conditions is justifiable, which is illustrated by the following example. Let $\tau_1, \tau_2\colon [0,1] \to \mathbb{R}$ be defined as follows: $\tau_1(x) = 0$ for $x \in \{0\} \cup \{1/2^k\colon k = 0,1,2,\ldots\}$; $\tau_1(x) = 1$ for $x \in \{3/2^{k+2}\colon k = 0,1,2,\ldots\}$; τ_1 linear on the intervals $[1/2^{k+1}, 3/2^{k+2}]$ and $[3/2^{k+2}, 1/2^k]$, $k = 0,1,2,\ldots$; $\tau_2(x) = 0$ for $x \in \{0\} \cup \{1/2^k\colon k = 0,1,2,\ldots\}$; $\tau_2(x) = -1$ for $x \in \{3/2^{k+2}\colon k = 0,1,2,\ldots\}$; τ_2 linear on the intervals $[1/2^{k+1}, 3/2^{k+2}]$ and $[3/2^{k+2}, 1/2^k]$, $k = 0,1,2,\ldots$. Of course τ_1 and τ_2 fulfil the assumptions of Theorem 4.3. Let $\Re_0$ be the family of all functions $\xi\colon [0,1] \to \mathbb{R}$ continuous on $(0,1]$ and such that $\{0\} \cup \{1/2^k\colon k = 0,1,\ldots\} \subset Z(\xi)$. It is easy to show that $\Re_0$ is a Darboux ring and $\tau_1, \tau_2 \in \Re_0$. By Theorem 4.3 there exist an od-set H and an H-trajectory $\bar{q}$ such that $\Re_{\tau_1}(H,\bar{q}) \cap \Re_{\tau_2}(H,\bar{q}) \neq \emptyset$. Let $\Re \in \Re_{\tau_1}(H,\bar{q}) \cap \Re_{\tau_2}(H,\bar{q})$. Consider $g\colon [0,1] \to \mathbb{R}$ defined by the formula $g(x) = x$. Of course $g \in \Re \setminus \Re_0$, so no ring belonging to $\Re_{\tau_1}(H,\bar{q}) \cap \Re_{\tau_2}(H,\bar{q})$ is equal to $\Re_0$.

5. Rings of iteratively H-connected functions

Considerations dealing with real functions of real variable and results presented in [15] motivate us to make an attempt of generalizing the notion of H-connected functions to the case of functions defined on topological spaces. A typical generalization is not possible because building one-sided first return paths is closely connected with the ordering of the real line. Therefore, it seems to be natural to examine compositions $h = g \circ f$, where g is H-connected, $f\colon X \to [0,1]$ and X is a topological space. One question still unanswered deals with assumptions which need to be imposed on the function f. The assumption that f is continuous seems to be the best adapted to our theory.

Let X be a topological space, H an od-set and $\bar{q}$ an H-trajectory. A function $h\colon X \to \mathbb{R}$ is called iteratively H-connected with respect to $\bar{q}$ if there exist a continuous function $f\colon X \to [0,1]$ and a function $g\colon [0,1] \to \mathbb{R}$ H-connected with respect to $\bar{q}$ such that $h = g \circ f$. The family of all iteratively H-connected functions with respect to $\bar{q}$ defined on X will be denoted by $\mathrm{It}(X,H,\bar{q})$.

The symbol $\mathrm{It}(X)$ will stand for the family of all functions $h\colon X \to \mathbb{R}$ for which there exist an od-set H_h and an H_h-trajectory $\bar{q}$ such that $h \in \mathrm{It}(X,H_h,\bar{q})$.

Of course, if $h \in \mathrm{It}(X)$, then h is a Darboux function.

From Theorem 1 in [15] one can immediately conclude that there exists a connected, uncountable, Hausdorff space X such that every function $f \in \mathrm{It}(X)$ is constant. Hence it is easy to deduce:

Proposition 5.1. *There exists a connected, uncountable, Hausdorff space X such that there is no D-ring $\Re \subset \mathrm{It}(X)$.*

Theorem 5.2. *Let X be nonsingleton, connected and locally connected Tychonoff space. Then for each od-set H and each H-trajectory $\bar{q}$ there exists a D-ring $\Re \subset \mathrm{It}(X, H, \bar{q})$.*

P r o o f. Let $x_1, x_2 \in X$. There exists a continuous function $f\colon X \to [0,1]$ such that $f(x_1) = 0$ and $f(x_2) = 1$. We will construct an H-connected function $\tau_0\colon [0,1] \to [0,1]$ discontinuous at 0 and a sequence $\{y_l\}_{l\in\mathbb{N}}$ such that $y_l \searrow 0$ and $\tau_0(y_l) = 1$ for $l \in \mathbb{N}$.

Consider two cases.

Case 1. There exists $\delta > 0$ such that $(0,\delta] \subset H$. Let $P_r(0,\bar{q}) = \{t_k\}_{k\in\mathbb{N}}$ be the first return path to 0 based on $\bar{q}$. Put $k_0 = \min\{k\colon\ t_k \in (0,\delta)\}$. Consider a sequence of intervals $\{(c_l, d_l)\}_{l\in\mathbb{N}}$ such that $t_{l+k_0} < c_l < d_l < t_{l+k_0-1}$ for $l \in \mathbb{N}$ and put $y_l = (c_l + d_l)/2$ for $l \in \mathbb{N}$. Of course the sequence $\{y_l\}_{l\in\mathbb{N}}$ converges decreasingly to 0.

Define a function τ_0 in the following way: $\tau_0(x) = 1$ for $x = y_l, l \in \mathbb{N}$; $\tau_0(x) = 0$ for $x \in \{c_l, d_l\}, l \in \mathbb{N}$; linear on the intervals $[c_l, y_l]$ and $[y_l, d_l], l \in \mathbb{N}$; $\tau_0(x) = 0$ in the other cases.

Case 2. There is no $\delta > 0$ such that $(0,\delta] \subset H$. Let $\{I_n\}_{n\in\mathbb{N}}$ be a sequence of all components of H and let a_n and b_n be the left and the right endpoint of I_n, respectively. Consider a sequence of intervals $\{(c_n, d_n)\}_{n\in\mathbb{N}}$ such that $a_n < c_n < d_n < b_n$ for $n \in \mathbb{N}$.

Define a function τ_0 in the following way: $\tau_0(x) = 1$ for $x = (c_n + d_n)/2$, $n \in \mathbb{N}$; $\tau_0(x) = 0$ for $x \in \{c_n, d_n\}$, $n \in \mathbb{N}$; linear on the intervals $[c_n, (c_n + d_n)/2]$ and $[(c_n + d_n)/2, d_n]$, $n \in \mathbb{N}$; $\tau_0(x) = 0$ in the other cases.

Let $\{y_l\}_{l\in\mathbb{N}} \subset \{(c_n + d_n)/2\colon\ n \in \mathbb{N}\}$ be a sequence converging decreasingly to 0.

In both the above cases it is easy to notice that τ_0 is H-connected with respect to $\bar{q}$, discontinuous at 0 and $\tau_0(y_l) = 1$ for $l \in \mathbb{N}$. By Theorem 4.1 there exists a complete ring $\Re_0$ of functions H-connected with respect to $\bar{q}$ such that $\tau_0 \in \Re_0$. Of course $\Re_0$ is a D-ring.

Now consider the family $\Re = \{\tau \circ f\colon\ \tau \in \Re_0\}$. It is easy to show that $\Re$ is a ring and $\Re \subset \mathrm{It}(X, H, \bar{q})$.

To complete the proof we need to show that $D(\Re) \neq \emptyset$.

Notice that $X \neq f^{-1}(\{0\}) \neq \emptyset$. X is connected, so we can choose $x_0 \in Fr(f^{-1}(\{0\}))$.

Let $\{U_t\colon\ t \in T\}$ be a local base of X at x_0 consisting of connected sets.

It is easy to show that for each $t \in T$ there exists $l_0 \in \mathbb{N}$ such that for $l \geqslant l_0$ we have

$$U_t \cap f^{-1}(y_l) \neq \emptyset. \tag{5.1}$$

Consider a set Σ consisting of all ordered pairs $(t, l) \in T \times \mathbb{N}$ fulfilling the condition (5.1). Let us define the following relation:

$$(t_1, l_1) \ll (t_2, l_2) \Leftrightarrow (U_{t_2} \subset U_{t_1} \wedge l_1 \leqslant l_2).$$

One can easily verify that Σ with the above relation is a directed set.

For $\sigma = (t, l) \in \Sigma$ let x_σ be a fixed point of the intersection $U_t \cap f^{-1}(y_l)$. In this way we obtain a net $\{x_\sigma\}_{\sigma \in \Sigma}$.

We will show that

$$x_0 \in \lim_{\sigma \in \Sigma} x_\sigma \text{ and } \lim_{\sigma \in \Sigma} (\tau_0 \circ f)(x_\sigma) = 1. \tag{5.2}$$

Let W be an open neighborhood of x_0. There exists $t_0 \in T$ such that $U_{t_0} \subset W$. Let $l_0 \in \mathbb{N}$ be such that $U_{t_0} \cap f^{-1}(y_{l_0}) \neq \emptyset$. Put $\sigma_0 = (t_0, l_0)$. One can easily show that $x_\sigma \in W$ for $\sigma \gg \sigma_0$, and so, $x_0 \in \lim_{\sigma \in \Sigma} x_\sigma$.

Let $\sigma = (t_1, l_1) \in \Sigma$. Then $(\tau_0 \circ f)(x_\sigma) = \tau_0(y_{l_1}) = 1$, so $\lim_{\sigma \in \Sigma} (\tau_0 \circ f)(x_\sigma) = 1$. On the other hand, $(\tau_0 \circ f)(x_0) = \tau_0(0) = 0$. Hence and from (5.2) we conclude that $x_0 \in D(\Re_0)$, which completes the proof. $\square$

Theorem 5.3. *Let X be nonsingleton, connected and locally connected, perfectly normal topological space. Then for each point $x_0 \in X$ there exist a $[0,1]$-trajectory $\bar{q}$ and a prime ring $\Re \subset \mathrm{It}(X, [0,1], \bar{q})$ such that $D(\Re) = \{x_0\}$.*

Proof. Let x_0 be a fixed point of X and $U \neq X$ an open neighborhood of x_0. Put $F = X \setminus U$. By [25], there exists a continuous function $f\colon X \to [0,1]$ such that $f^{-1}(\{0\}) = \{x_0\}$, $f^{-1}(\{1\}) = F$ and $f(X) = [0,1]$. Let $\varphi\colon [0,1] \to \mathbb{R}$ be a function defined as follows: $\varphi(x) = \sin(1/x)$ for $x \in (0,1]$ and $\varphi(0) = 0$. Of course $\varphi \in \mathrm{DB}_1([0,1])$. By Lemma 2.1, there exists a $[0,1]$-trajectory $\bar{q}$ such that $\varphi \in \mathrm{FRC}([0,1], \bar{q})$, so φ is $[0,1]$-connected with respect to $\bar{q}$.

Let $\Re_0$ be the family of all functions which are continuous on the interval $(0,1]$ and first return continuous from the right with respect to $\bar{q}$ at 0. Of course $\varphi \in \Re_0$. It is easy to notice that the family $\Re_0$ is a ring of functions such that $D(\Re_0) = \{0\}$.

Consider the following family of functions defined on X:

$$\Re_1 = \{\tau \circ f\colon \tau \in \Re_0\}.$$

Let $\Re$ be a subset of $\Re_1$ consisting of all functions $h \in \Re_1$ such that $x_0 \in Z(h)$. One can easily show that $\Re$ is a ring and $\Re \subset \mathrm{It}(X, [0,1], \bar{q})$.

We will show that

$$D(\Re) = \{x_0\}. \tag{5.3}$$

Let $x \in X$ and $x \neq x_0$. Then x is a continuity point of $\tau \circ f$ for each $\tau \in \Re_0$, so $D(\Re) \subset \{x_0\}$. Conversely, since $D(\varphi) = \{0\}$, one can show (similarly to the proof of Theorem 5.2) that $x_0 \in D(\varphi \circ f)$, and so $\{x_0\} \subset D(\Re)$, which gives (5.3).

Easy observation that $D(h) \subset Z(h)$ for each function $h \in \Re$ completes the proof. □

Open problem. What assumptions need to be imposed on the set $A \subset X$ so that the theorem analogous to Theorem 5.3 with $\{x_0\}$ replaced by A be true?

References

[1] *A. Bąkowska, A. Loranty, R. J. Pawlak*: On the topological entropy of continuous and almost continuous functions. Topology Appl. *158* (2011), 2022–2033.

[2] *A. Biś, H. Nakayama, P. Walczak*: Modelling minimal foliated spaces with positive entropy. Hokkaido Math. J. *36* (2007), 283–310.

[3] *J. Borsík*: Algebraic structures generated by real quasicontinuous functions. Tatra Mt. Math. Publ. *8* (1996), 175–184.

[4] *J. Borsík*: Sums, differences, products and quotients of closed graph functions. Tatra Mt. Math. Publ. *24* (2002), 117–123.

[5] *A. M. Bruckner*: Differentiation of Real Functions. Lecture Notes in Mathematics 659. Springer, Berlin, 1978.

[6] *M. Čiklová*: Dynamical systems generated by functions with connected G_δ graphs. Real Anal. Exch. *30* (2004/2005), 617–638.

[7] *U. B. Darji, M. J. Evans, C. Freiling, R. J. O'Malley*: Fine properties of Baire one functions. Fundam. Math. *155* (1998), 177–188.

[8] *U. B. Darji, M. J. Evans, R. J. O'Malley*: A first return characterization for Baire one functions. Real Anal. Exch. *19* (1994), 510–515.

[9] *U. B. Darji, M. J. Evans, R. J. O'Malley*: First return path systems: Differentiability, continuity, and orderings. Acta Math. Hung. *66* (1995), 83–103.

[10] *M. J. Evans, R. J. O'Malley*: First-return limiting notions in real analysis. Real Anal. Exch. *29* (2003/2004), 503–530.

[11] *R. G. Gibson, T. Natkaniec*: Darboux like functions. Real Anal. Exch. *22* (1996), 492–533.

[12] *Z. Grande*: On the Darboux property of the sum of cliquish functions. Real Anal. Exch. *17* (1992), 571–576.

[13] *Z. Grande*: On the sums and products of Darboux Baire*1 functions. Real Anal. Exch. *18* (1993), 237–240.

[14] *K. R. Kellum*: Iterates of almost continuous functions and Sarkovskii's theorem. Real Anal. Exch. *14* (1989), 420–422.

[15] *E. Korczak, R. J. Pawlak*: On some properties of essential Darboux rings of real functions defined on topological spaces. Real Anal. Exch. *30* (2004/2005), 495–506.

[16] *A. Maliszewski*: Maximums of almost continuous functions. Real Anal. Exch. *30* (2004/2005), 813–818.
[17] *A. Maliszewski*: Maximums of Darboux Baire one functions. Math. Slovaca *56* (2006), 427–431.
[18] *A. Mikucka*: Graph quasi-continuity. Demonstr. Math. *36* (2003), 483–494.
[19] *R. J. O'Malley*: First return path derivatives. Proc. Am. Math. Soc. *116* (1992), 73–77.
[20] *R. J. Pawlak*: On some class of functions intermediate between the class B_1^* and the family of continuous functions. Tatra Mt. Math. Publ. *19* (2000), 135–144.
[21] *H. Pawlak, R. J. Pawlak*: First-return limiting notions and rings of Sharkovsky functions. Real Anal. Exch. *34* (2009), 549–563.
[22] *R. J. Pawlak*: On the entropy of Darboux functions. Colloq. Math. *116* (2009), 227–241.
[23] *P. Szuca*: Sharkovskiĭ's theorem holds for some discontinuous functions. Fundam. Math. *179* (2003), 27–41.
[24] *P. Szuca*: Connected G_δ functions of arbitrarily high Borel class. Tatra Mt. Math. Publ. *35* (2007), 41–45.
[25] *N. Vedenissoff*: Sur les fonctions continues dans des espaces topologiques. Fundam. Math. *27* (1936), 234–238. (In French.)

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