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Haiyan Su; Pengzhan Huang; Xinlong Feng
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TWO-LEVEL STABILIZED NONCONFORMING FINITE ELEMENT METHOD FOR THE STOKES EQUATIONS

Haiyan Su, Pengzhan Huang, Xinlong Feng, Urumqi

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Abstract. In this article, we present a new two-level stabilized nonconforming finite elements method for the two dimensional Stokes problem. This method is based on a local Gauss integration technique and the mixed nonconforming finite element of the $NCP_1 - P_1$ pair (nonconforming linear element for the velocity, conforming linear element for the pressure). The two-level stabilized finite element method involves solving a small stabilized Stokes problem on a coarse mesh with mesh size H and a large stabilized Stokes problem on a fine mesh size $h = H/3$. Numerical results are presented to show the convergence performance of this combined algorithm.

Keywords: Stokes problem; two-level method; nonconforming finite element; error estimate; numerical result

MSC 2010: 65M60, 76D07, 65M12

1. Introduction

Finite element methods are widely used in computational fluid dynamics. In particular, some stable mixed finite element methods are a basic component in search for efficient numerical methods for solving the Stokes and Navier-Stokes equations governing incompressible flows. Considering the mixed element methods, the equal-order velocity-pressure pairs are quite practical in finite element approximations of the Stokes problem. However, they violate the inf-sup condition and the compatibility between the velocity and pressure spaces.

Therefore, using a primitive variable formulation, the importance of ensuring the compatibility of the component approximations of velocity and pressure by satisfying

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the so-called inf-sup condition is widely understood. It is also well known that the simplest conforming lowest-order element pair $P_1 - P_1$ (linear velocity, linear pressure) is not stable. This impinges on efficiency, since the simple logic and regular data structure associated with low-order finite element methods make them particularly attractive on modern vector and parallel architectures. So it is crucially important to establish stabilized mixed formulation for the Stokes equations to adapt to the need of the development of the fast iterative solution algorithms. Recently many stabilized finite element methods have been proposed for Stokes equations. These methods are based on Gauss integrations [11], [1], [10], [3], [6]. They have some important features over traditional stabilized mixed finite methods: simple, efficient, and independent of stabilization parameters. However, compared with conforming finite element methods, nonconforming finite methods for incompressible flows are more popular due to their simplicity and small support sets of basis functions. Furthermore, they seem much easier to fulfill the discrete inf-sup condition. Moreover, nonconforming finite element can easily relax the high-order continuity requirement for conforming finite element. As a result, in practice, the nonconforming finite element methods seem superior to the conforming finite element methods.

The basic idea of two-level discretization method is to capture the "large eddies", "low modes", or "global solution envelope" by computing an initial approximation on a very coarse mesh. The fine structures are captured by solving the linear system on a fine mesh.

Some details on the two-level approach can be found in the papers of He [4], Xu [12], [13], Layton [9], Layton and Lenferink [7], Ervin et al. [2], Layton and Tobiska [8].

In this paper, we propose a two-level stabilized nonconforming finite element method for the Stokes equations which uses the nonconforming and conforming piecewise linear polynomial approximations for the velocity and pressure, respectively. The method we study combines the stabilized finite element with the two-level discretization for solving the two-dimensional Stokes problem. The method is of the convergence rate of the same order as the usual stabilized finite element method. Meanwhile, with the certain relationship between coarse mesh and fine mesh, there is no need to establish a coarse-to-fine intergrid operator. Hence, our method is more efficient and simpler.

This paper is organized as follows. In the next section, we introduce some notation, the Stokes equations, and their finite element discretizations. One-level and two-level stabilized finite element approximations are presented in Section 3. Then, stability and optimal order estimates for the two-level stabilized element method are obtained in the fourth section. Finally, in Section 5, a suite of numerical experiments are given to demonstrate the theoretical results obtained.

2. Functional setting of the Stokes problem

In this paper, we consider the Stokes problem

$$-\nu\Delta u + \nabla p = f \quad \text{in } \Omega, \tag{2.1}$$

$$\nabla\cdot u = 0 \quad \text{in } \Omega, \tag{2.2}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{2.3}$$

where Ω represents a bounded, convex and open subset of $\mathbb{R}^2$ with Lipschitz continuous boundary $\Gamma = \partial\Omega$, f is a prescribed body force, ν is the dynamic viscosity, and $u = (u_1, u_2)$ and p denote the velocity and pressure fields, respectively.

For the mathematical setting of problem (2.1)–(2.3), we introduce the following Hilbert spaces

$$X = (H_0^1(\Omega))^2, Y = (L^2(\Omega))^2,$$
$$M = L_0^2(\Omega) = \left\{q \in L^2(\Omega)\colon \int_\Omega q\,\mathrm{d}x = 0\right\}.$$

We use the standard definitions for the Sobolev space $W^{m,r}(\Omega)$ and their associated norm $\|\cdot\|_{m,r}$ and seminorm $|\cdot|_{m,r}$, $m, r \geqslant 0$. We will write $H^m(\Omega)$ for $W^{m,2}(\Omega)$ and $\|\cdot\|_m$ for $\|\cdot\|_{m,2}$. The spaces $(L^2(\Omega))^m$, $m = 1, 2$ or 4, are endowed with the usual L^2-scalar product $(\cdot,\cdot)$ and L^2-norm $\|\cdot\|_0$, as appropriate. Also, the space X is equipped with the scalar product $(\nabla u, \nabla v)$ and the norm $|u|_1$, $u, v \in X$. Due to the norm equivalence between $\|\cdot\|_1$ and $|\cdot|_1$, on X, we sometimes use the same notation for them. Finally, for notational convenience, we set $\nu = 1$.

Under the above notation, the variational formulation of the problem (2.1)–(2.3) reads as: find $(u, p) \in (X, M)$ such that for all $(v, q) \in (X, M)$

$$B((u,p);(v,q)) = (f, v), \tag{2.4}$$

where

$$B((u,p);(v,q)) = a(u,v) - d(v,p) + d(u,q),$$
$$a(u,v) = (\nabla u, \nabla v),$$
$$d(v,q) = (\nabla\cdot u, q).$$

It is well known that the well-posedness of the model problem (2.1)–(2.3) follows from the Lax-Milgram Lemma [10]. If the domain Ω is convex, the $H^2 - H^1$-regularity of the solution of (2.1)–(2.3) holds; i.e., the unique solution $(u, p) \in ((H^2(\Omega))^2, H^1(\Omega))$ of (2.1)–(2.3) satisfies the following a priori estimate:

$$\|u\|_2 + \|p\|_1 \leqslant C\|f\|_0, \tag{2.5}$$

where $C > 0$ is a constant depending only on Ω. Subsequently, the constant $C > 0$ will depend only on the data (ν, Ω, f).

It is also well known that the bilinear forms $a(\cdot,\cdot)$ and $d(\cdot,\cdot)$ are continuous on (X, X) and (X, M), respectively. Moreover, the bilinear form $d(\cdot,\cdot)$ satisfies the inf-sup condition [10]:

$$\sup_{0\neq v\in X} \frac{|d(v,q)|}{\|v\|_1} \geqslant \beta \|q\|_0, \tag{2.6}$$

where β is a positive constant depending only on Ω.

3. One-level and two-level stabilized finite element approximations

Let $\tau_h = \bigcup_{j=1}^{J} K_j$ be the regular triangulation of the domain Ω with the mesh parameter $h = \max\{\mathrm{diam}(K_j)\}$. We denote the boundary edge of K_j by $\gamma_j = \partial\Omega \cap \partial K_j$, the interface between elements K_j and K_k by $\gamma_{k,j} = \gamma_{j,k} = \partial K_j = \partial K_k$ and the center of γ_j, $\gamma_{j,k}$ by ξ_j, $\xi_{j,k}$, respectively.

Then we define the nonconforming Crouzeix-Raviart finite element space of velocity and the conforming finite element space of pressure, respectively, by

$$NC_h = \{v \in Y\colon\ v|_{K_k} \in (P_1(k))^2, v(\xi_{k,j}) = v(\xi_{j,k}), v(\xi_j) = 0\ \forall K_j, K_k \in \tau_h\}, \tag{3.1}$$
$$M_h = \{v \in H^1(\Omega)\colon\ v|_{K_k} \in P_1(k)\ \forall K_k \in \tau_h\}. \tag{3.2}$$

And we define another nonconforming finite element space:

$$V_h = \{v_h \in NC_h\colon\ d_h(v_h, q_h) = 0\ \forall q_h \in M_h\}. \tag{3.3}$$

Note that the nonconforming finite element space NC_h is not a subspace of X. For any v in NC_h, the following compatibility conditions hold for all j and k:

$$\int_{\gamma_{jk}} [v]\,\mathrm{d}s = 0,$$

and

$$\int_{\gamma_j} v\,\mathrm{d}s = 0,$$

where $[v] = v|_{\gamma_{jk}} - v|_{\gamma_{kj}}$ denotes the jump of the function v across the interface γ_{jk}.

The two finite element spaces NC_h and M_h satisfy the following approximation property: for any $(v, q) \in (H^2(\Omega) \cap X, H^1(\Omega) \cap M)$ there exists $(v_I, q_I) \in (NC_h, M_h)$ such that [10]

$$\|v - v_I\|_0 + h(\|v - v_I\|_{1,h} + \|q - q_I\|_0) \leqslant Ch^2(\|v\|_2 + \|q\|_1), \tag{3.4}$$

where $\|\cdot\|_{1,h}$ denotes the energy norm of the nonconforming finite element space NC_h,

$$\|v\|_{1,h} = \Big(\sum_j |v|^2_{1,K_j}\Big)^{1/2}, \quad v \in NC_h. \tag{3.5}$$

We set $(\cdot,\cdot)_j = (\cdot,\cdot)_{K_j}$, $\langle\cdot,\cdot\rangle = (\cdot,\cdot)_{\partial K_j}$, and $|\cdot|_{m,j} = |\cdot|_{m,K_j}$. Then the standard discretization of the problem (2.4) is to find $(u_h, p_h) \in (NC_h, M_h)$ such that

$$B_h((u_h,p_h);(v,q)) = (f,v) \quad \forall q \in M_h, \tag{3.6}$$

where

$$B_h((u,p);(v,q)) = a_h(u,v) - d_h(v,p) + d_h(u,q),$$

with

$$a_h(u,v) = \sum_j (\nabla u, \nabla v)_j,$$
$$d_h(v,q) = \sum_j (\nabla\cdot v, q)_j.$$

Nevertheless, this formulation is not stable owing to the violation of the inf-sup condition for velocity and pressure approximations. However, based on the early analysis and knowledge, we can obtain stability and optimal estimates for this formulation by adding a simple, local, and effective stability term $G_h(\cdot,\cdot)$ [10]:

$$G_h(p,q) = \sum_{K_j\in\tau_h}\Big\{\int_{K_{j,2}} pq\,\mathrm{d}x - \int_{K_{j,1}} pq\,\mathrm{d}x\Big\}, \quad p,q \in L^2(\Omega),$$

where $\int_{K_{j,i}} pq\,\mathrm{d}x$ indicates an appropriate Gauss integral over K_j that is exact for polynomials of degree $i (i = 1,2)$, and pq is a polynomial of degree not greater than 2. Thus the stabilizing term $G_h(\cdot,\cdot)$ defined by the difference of Gauss quadratures, must be exact for all test functions $q \in M_h$ and the trial function $p \in P_0$ (piecewise constants) when $i = 1$.

Consequently, we define the L^2-projection operator $\pi_h\colon L^2(\Omega) \to W_h$ by

$$(p, q_h) = (\pi_h p, q_h) \quad \forall p \in L^2(\Omega),\ q_h \in W_h, \tag{3.7}$$

where $W_h \subset L^2(\Omega)$ denotes the piecewise constant space associated with τ_h. The projection operator π_h has the following properties [11], [1], [10]:

$$\|\pi_h p\|_0 \leqslant C\|p\|_0, \quad \forall p \in L^2(\Omega), \tag{3.8}$$

$$\|p - \pi_h p\|_0 \leqslant Ch\|p\|_1, \quad \forall p \in H^1(\Omega). \tag{3.9}$$

Specifically, we define the stability term as follows:

$$G_h(p,q) = (p - \pi_h p, q - \pi_h q). \tag{3.10}$$

Finally, the one-level stabilized discrete weak formulation of the Stokes equation (2.1)–(2.3) is to find $(u_h, p_h) \in (NC_h, M_h)$ such that

$$B_h((u_h,p_h);(v,q)) + G_h(p_h,q) = (f,v) \quad \forall (v,q) \in (NC_h, M_h). \tag{3.11}$$

Next, the stability and error estimates of the discrete problem (3.11) are given as follows:

Theorem 3.1 ([10]). *The bilinear form $B_h((\cdot,\cdot),(\cdot,\cdot))$ satisfies the continuity property*

$$|B_h((u_h,p_h);(v,q)) + G_h(p_h,q)| \leqslant C(\|u_h\|_{1,h} + \|p_h\|_0)(\|v\|_{1,h} + \|q\|_0), \quad (u_h,p_h),(v,q) \in (NC_h, M_h), \tag{3.12}$$

and the coercivity property

$$\sup_{0 \neq (v,q) \in (NC_h, M_h)} \frac{|B_h((u_h,p_h);(v,q)) + G_h(p_h,q)|}{\|v\|_{1,h} + \|q\|_0} \geqslant \beta(\|u_h\|_{1,h} + \|p_h\|_0), \quad (u_h,p_h) \in (NC_h, M_h), \tag{3.13}$$

where β and C are positive constants depending only on Ω, ν.

Theorem 3.2 ([10]). *Let (u,p) and (u_h,p_h) be the respective solution of* (2.1)–(2.3) *and* (3.11). *Then*

$$\|u - u_h\|_0 + h(\|u - u_h\|_{1,h} + \|p - p_h\|_0) \leqslant Ch^2(\|u\|_2 + \|p\|_1), \tag{3.14}$$

where C is a positive constant depending only on Ω, ν.

Then let τ_H be the coarser mesh obtained by coarsening of τ_h and $H = 3h$ (see Fig. 1).

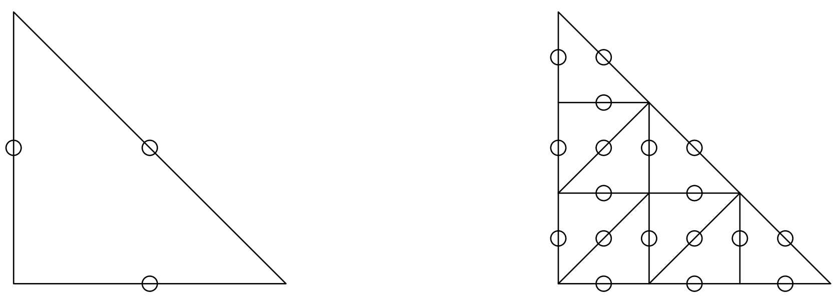

Figure 1. Left, τ_H mesh; right, τ_h mesh.

From Fig. 1, we know that with the special relationship between the coarse mesh and the fine mesh, the nonconforming finite element space pair $(NC_H, M_H) \subset (NC_h, M_h)$ based on the triangulations $\tau_H(\Omega)$ and $\tau_h(\Omega)$. With the above finite element space pairs, we will consider the following two-level stabilized finite element method.

Step I: Solve the Stokes problem on the coarse mesh, i.e. find $(u_H, p_H) \in (NC_H, M_H)$ such that for all $(v, q) \in (NC_H, M_H)$

$$B_H((u_H, p_H); (v, q)) + G_H(p_H, q) = (f, v). \tag{3.15}$$

Step II: Solve the Stokes problem on the fine mesh, i.e. find $(u^h, p^h) \in (NC_h, M_h)$ such that for all $(v, q) \in (NC_h, M_h)$

$$B_h((u^h, p^h); (v, q)) + G_H(p_H, q) = (f, v). \tag{3.16}$$

Next, we will study the stability and error estimates for (u^h, p^h) in some norms.

4. Stability and error estimates for the two-level FEM

In this section, we will present the stability analysis and error estimate for the stabilized two-level finite element method.

Theorem 4.1. *Let (u, p) and (u^h, p^h) be the respective solution of (2.1)–(2.3) and (3.16). Then*

$$\|u^h\|_{1,h} \leqslant \frac{C}{\beta}\|f\|_{-1}, \tag{4.1}$$

$$\|p^h\|_0 \leqslant \frac{C}{\beta}\|f\|_{-1}, \tag{4.2}$$

where $\|f\|_{-1} = \sup\limits_{v \in NC_h} (f, v)/\|v\|_{1,h}$, and β and C are positive constants depending only on Ω, ν.

P r o o f. Combining (3.11) and Theorem 3.1, we arrive at

$$\|u_H\|_{1,h} \leqslant \frac{\|f\|_{-1}}{\beta}, \quad \|p_H\|_0 \leqslant \frac{\|f\|_{-1}}{\beta}.$$

Owing to

$$(NC_H, M_H) \subset (NC_h, M_h),$$

we obtain

$$\|u^h\|_{1,h} \leqslant C\|u_H\|_{1,h} \leqslant \frac{C}{\beta}\|f\|_{-1}, \tag{4.3}$$

and

$$\|p^h\|_0 \leqslant C\|p_H\|_0 \leqslant \frac{C}{\beta}\|f\|_{-1}. \tag{4.4}$$

So, the proof is complete. □

Next we present error estimates for the two-level stabilized finite element method using the (NC_h, M_h) pair. We set

$$\widetilde{B}_h((u,p);(v,q)) = B_h((u,p);(v,q)) - G_H(p_H, q). \tag{4.5}$$

And we introduce the projection operator:

$$(R_h, Q_h)\colon\ (X, M) \to (NC_h, M_h), \tag{4.6}$$

by

$$B_h((R_h(v,q), Q_h(v,q));(v,q)) = \widetilde{B}_h((v,q);(v,q)), \quad \forall (v,q) \in (NC_h, M_h). \tag{4.7}$$

From the definition of the projection operator and the finite element interpolation error estimation theory, we have the following results:

Lemma 4.1. *For any $(v,q) \in ((H^2(\Omega))^2 \cap X, H^1(\Omega) \cap M)$, we have*

$$\|v - R_h(v,q)\|_0 + h(\|v - R_h(v,q)\|_{1,h} + \|q - Q_h(v,q)\|_0) \leqslant Ch^2(\|v\|_2 + \|q\|_1).$$

Lemma 4.2. *For any $s, w \in X \cup NC_h$,*

$$\Big|\sum_j \Big\langle \frac{\partial w}{\partial n_j}, s\Big\rangle\Big| \leqslant Ch\|w\|_2\|s\|_{1,h},\ \forall w \in X \cap (H^2(\Omega))^2,$$

$$\Big|\sum_j \langle q, s\cdot n_j\rangle\Big| \leqslant Ch\|q\|_1\|s\|_{1,h},\ \forall q \in H^1(\Omega).$$

Theorem 4.2. *Let* (u,p) *and* (u^h,p^h) *be the respective solutions of* (2.1)–(2.3) *and* (3.16). *Then*

$$\|u-u^h\|_{1,h} \leqslant Ch(\|u\|_2+\|p\|_1). \tag{4.8}$$

Proof. Multiplying (2.1) and (2.2) by $v \in NC_h$, $q \in M_h$, respectively, and integrating by parts over Ω, we see that

$$a_h(u,v)-d_h(v,p)+d_h(u,q)-\sum_j \left\langle \frac{\partial u}{\partial n}, v\right\rangle_j + \sum_j \langle\, v\cdot n, p\rangle_j = (f,v). \tag{4.9}$$

Combining (3.16) and (4.8), we find that

$$\begin{aligned} a_h(u-u^h,v)-d_h(v,p-p^h)+d_h(u-u^h,q)\\ -\sum_j \left\langle \frac{\partial u}{\partial n}, v\right\rangle_j + \sum_j \langle v\cdot n, p\rangle_j - G_H(p_H,q)=0. \end{aligned} \tag{4.10}$$

Using (4.4) and (4.6), we obtain

$$\begin{aligned} B_h((R_h(u,p),Q_h(u,p));(v,q)) &= \widetilde{B}_h((u,p);(v,q))\\ &= B_h((u,p);(v,q))-G_H(p_H,q), \end{aligned} \tag{4.11}$$

and thus,

$$B_h((R_h(u,p)-u,Q_h(u,p)-p);(v,q)) = -G_H(p_H,q), \tag{4.12}$$

i.e.

$$a_h(R_h(u,p)-u,v)-d_h(v,Q_h(u,p)-p)+d_h(R_h(u,p)-u,q) = -G_H(p_H,q). \tag{4.13}$$

Obviously, from (4.9) and (4.12), we have

$$\begin{aligned} a_h(R_h(u,p)-u^h,v)-d_h(v,Q_h(u,p)-p^h)+d_h(R_h(u,p)-u^h,q)\\ -\sum_j \left\langle \frac{\partial u}{\partial n}, v\right\rangle_j + \sum_j \langle v\cdot n, p\rangle_j = 0. \end{aligned} \tag{4.14}$$

Now taking $v = R_h(u,p)-u^h \in V_h$ in (4.13) and $q=0$, we obtain

$$\begin{aligned} &a_h(R_h(u,p)-u^h, R_h(u,p)-u^h)\\ &\quad = \sum_j \left\langle \frac{\partial u}{\partial n}, R_h(u,p)-u^h\right\rangle_j - \sum_j \left\langle (R_h(u,p)-u^h)\cdot n, p\right\rangle_j. \end{aligned} \tag{4.15}$$

Hence, owing to Lemma 4.2, we know that

$$\Big|\sum_j \Big\langle \frac{\partial u}{\partial n}, R_h(u,p) - u^h \Big\rangle_j \Big| \leqslant Ch\|R_h(u,p) - u^h\|_{1,h}\|u\|_2, \tag{4.16}$$

$$\Big|\sum_j \langle (R(u,p) - u^h)\cdot n, p\rangle_j \Big| \leqslant Ch\|R_h(u,p) - u^h\|_{1,h}\|p\|_1. \tag{4.17}$$

So we get

$$\|R_h(u,p) - u^h\|^2_{1,h} \leqslant Ch\|R_h(u,p) - u^h\|_{1,h}(\|u\|_2 + \|p\|_1), \tag{4.18}$$

i.e.

$$\|R_h(u,p) - u^h\|_{1,h} \leqslant Ch(\|u\|_2 + \|p\|_1). \tag{4.19}$$

Finally, using (4.18), the following error estimates holds

$$\|u - u^h\|_{1,h} = \|R_h(u,p) - u^h + u - R_h(u,p)\|_{1,h} \tag{4.20}$$

$$\leqslant \|R_h(u,p) - u^h\|_{1,h} + \|R_h(u,p) - u\|_{1,h} \tag{4.21}$$

$$\leqslant Ch(\|u\|_2 + \|p\|_1). \tag{4.22}$$

□

Theorem 4.3. *Let* (u,p) *and* (u^h,p^h) *be the respective solutions of* (2.1)–(2.3) *and* (3.16). *Then*

$$\|p - p^h\|_0 \leqslant Ch(\|u\|_2 + \|p\|_1). \tag{4.23}$$

Proof. Combing (3.12), (3.13), Theorem 3.1, and using (4.14)–(4.19) yields

$$\beta\|p - p^h\|_0 \leqslant \sup_{0\neq(v,q)\in(NC_h,M_h)} \frac{B_h((u-u^h);(v,q))}{\|v\|_{1,h} + \|q\|_0} \tag{4.24}$$

$$\leqslant \frac{|a_h(u-u^h,v) - d_h(v,p-p^h) + d_h(u-u^h,q)|}{\|v\|_{1,h} + \|q\|_0} \tag{4.25}$$

$$\leqslant Ch\Big(\frac{\|u-u^h\|_{1,h}\|v\|_{1,h}}{\|v\|_{1,h}} + \frac{\|p-p^h\|_0\|v\|_{1,h}}{\|v\|_{1,h}} + \frac{\|u-u^h\|_{1,h}\|q\|_0}{\|q\|_0}\Big). \tag{4.26}$$

Finally, we get

$$\|p - p^h\|_0 \leqslant Ch(\|u\|_2 + \|p\|_1). \tag{4.27}$$

□

5. Numerical experiment

In this section, we will evaluate the performance of the two-level stabilized nonconforming finite element method described. Then we report two experiments for Stokes problems using this method with $H = 3h$: the problems with known polynomial solution and trigonometric function. To show the desirable feature of our method, we compare it with the one-level nonconforming finite element method in the first example and compare with the two-level conforming finite element method in the second example. In both examples we consider a unite-square domain in $\mathbb{R}^2$. The pressure and velocity are approximated using the same uniform triangulation of Ω into triangles. Moreover, the algorithms are implemented using public domain finite element software—FreeFem++ [5] with some our additional codes. In all experiments, the nonlinear systems are linearised by allowing the nonlinearities to lag one step behind. Besides, concerning iterative solver, we have chosen the standard conjugate gradient method. Furthermore, if one wants to get a simpler and cheaper method, a preconditioner should be used for Step II. Finally the experimental rates of convergence with respect to the mesh size h are calculated by the formula $\log(E_i/E_{i+1})/\log(h_i/h_{i+1})$ where E_i and E_{i+1} are the relative errors corresponding to the meshes of size h_i and h_{i+1}, respectively.

Example 1. We take an example of the Stokes equations (2.1)–(2.3) where the right-hand side $f(x,y)$ is determined by the prescribed exact velocity $u = (u_1, u_2)$ and pressure p:

$$u_1(x,y) = 10x^2(x-1)^2y(y-1)(2y-1), \tag{5.1}$$

$$u_2(x,y) = -10x(2x-1)y^2(y-1)^2, \tag{5.2}$$

$$p(x,y) = 10(2x-1)(2y-1). \tag{5.3}$$

In Fig. 2, we show the H^1 error for the velocity and L^2 error for the pressure obtained by using different methods. Furthermore, from Fig. 2 it can be seen that the L^2-convergence for the pressure is clearly faster than the indicated convergence of order 1. The performance of super convergence stems from the stability term $G_h(\cdot,\cdot)$ which uses the Gauss integration. Finally, we can see that the results confirm a convergence rate of order $O(h)$ for velocity in the H^1-norm and pressure in the L^2-norm.

From Tab. 1, we can see that the method we propose costs less computational time than the one-level method. Obviously, the computed time of our method is not much less than the one-level method, because of the linear relationship between the coarse mesh and the fine mesh. We can save much time if the relationship between the coarse mesh and the fine mesh is square, but the coarse-to-fine intergrid operator

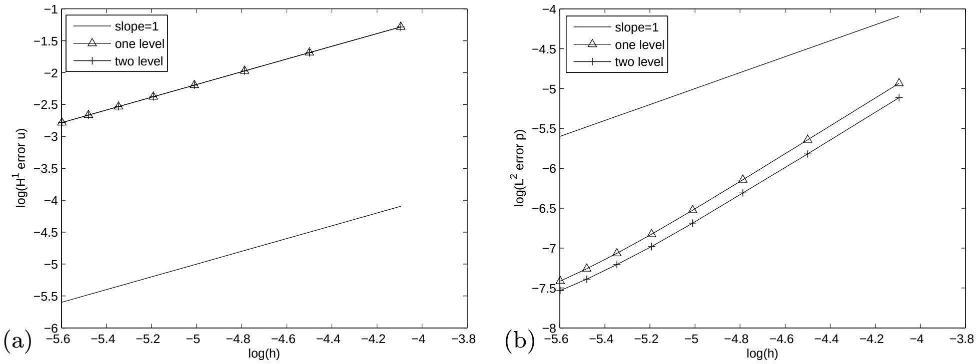

Figure 2. Convergence analysis using one and two-level methods. (a): H^1 error for the velocity; (b): L^2 error for the pressure.

must be established. However, with a certain relationship between the coarse mesh and the fine mesh, the method we proposed does not need the coarse-to-fine intergrid operator. So our method is simpler. In brief, our method can achieve the same convergence rate of the one-level nonconforming finite element method and with less time.

$1/h$	two-level	one-level
60	5.422	6.094
90	12.437	14.516
120	24.109	27.672
150	36.265	41.609
180	64.906	71.001
210	80.703	90.328
240	104.515	115.453
270	150.141	178.547

Table 1. The comparison of the two-level nonconforming method with the one-level nonconforming method

Example 2. We take an example of the Stokes equation (2.1)–(2.3), where the right-hand side $f(x,y)$ is determined by the prescribed exact velocity $u=(u_1,u_2)$ and pressure p:

$$u_1(x,y) = 2\pi\sin^2(\pi x)\sin(\pi y)\cos(\pi x), \tag{5.4}$$

$$u_2(x,y) = -2\pi\sin(\pi x)\sin^2(\pi y)\cos(\pi x), \tag{5.5}$$

$$p(x,y) = \cos(\pi x)\cos(\pi y). \tag{5.6}$$

The error performance of our method and the two-level conforming method can be clearly seen from Tab. 2, which shows that the two-level nonconforming method in this paper has a smaller error than the two-level conforming method. Because the number of the degrees of freedom of the nonconforming method is nearly three times larger than that of the conforming one on the given mesh.

		nonconforming two-level		conforming two-level	
$1/h$	$1/H$	$\Vert u-u^h\Vert_{1,h}/\Vert u\Vert_1$	$\Vert p-p^h\Vert_0/\Vert p\Vert_0$	$\Vert u-u^h\Vert_{1,h}/\Vert u\Vert_1$	$\Vert p-p^h\Vert_0/\Vert p\Vert_0$
60	20	0.036992	0.034488	0.042765	0.079421
90	30	0.024672	0.018687	0.028510	0.042003
120	40	0.018507	0.012108	0.021382	0.026874
150	50	0.014808	0.0086508	0.017105	0.019052
180	60	0.012340	0.0065743	0.014253	0.014403
210	70	0.010578	0.0052132	0.012217	0.011378
240	80	0.0092560	0.0042647	0.010690	0.0092806
270	90	0.0082278	0.0035725	0.0095017	0.0077568

Table 2. Comparison of the nonconforming two-level method with the conforming two-level method.

R e m a r k. Consider the uniform triangulation of Ω as illustrated in Fig. 1, right. Here, '$\square$' and '$\bigcirc$' denote the nodes of conforming and nonconforming method, respectively. From this figure, we can deduce that the number of nodes for the nonconforming method is

$$(n-1)+n(2n-1)=3n^2-2n,$$

and that for the conforming one is

$$(n-1)^2=n^2-2n+1,$$

where $n=1/h$. Thus, the number of the degrees of freedom of the nonconforming method is nearly three times larger than that of the conforming one on the given mesh. Hence, it is natural that the two-level nonconforming P_1-P_1 method has better accuracy and costs more CPU time.

6. Concluding remarks

In this paper, we have extended and studied the two-level stabilized nonconforming finite method based on two local Gauss integrals for the Stokes problem. Error estimates and stability have been obtained, and numerical results agreeing with these estimates have been presented. Also, further developments can extend these techniques and ideas to general nonlinear problems.

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Authors' addresses: *H. Su*, Xinjiang University, Urumqi 830046, P. R. China, e-mail: `yanzi880308xju@sina.com`; *P. Huang*, Xinjiang University, Urumqi 830046, P. R. China, e-mail: `hpz007@yahoo.cn`; *X. Feng*, Xinjiang University, Urumqi 830046, P. R. China, e-mail: `fxlmath@gmail.com`.