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COMPUTATION OF TOPOLOGICAL DEGREE IN ORDERED BANACH SPACES WITH LATTICE STRUCTURE AND APPLICATIONS

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Abstract. Using the cone theory and the lattice structure, we establish some methods of computation of the topological degree for the nonlinear operators which are not assumed to be cone mappings. As applications, existence results of nontrivial solutions for singular Sturm-Liouville problems are given. The nonlinearity in the equations can take negative values and may be unbounded from below.

Keywords: cone; lattice; topological degree

MSC 2010: 47H11, 34B15

1. Introduction

Let E be a Banach space with a cone P. Then E becomes an ordered Banach space under the partial ordering $\leqslant$ which is induced by P. A cone P is said to be normal if there exists a positive constant N such that $\theta \leqslant x \leqslant y$ implies $\|x\| \leqslant N\|y\|$. For the concepts and properties concerning the cone we refer to [1], [2].

We call E a lattice under the partial ordering $\leqslant$ if $\sup\{x, y\}$ and $\inf\{x, y\}$ exist for arbitrary $x, y \in E$. For $x \in E$, let

$$x^+ = \sup\{x, \theta\}, \quad x^- = \sup\{-x, \theta\};$$

x^+ and x^- are called the positive part and the negative part of x respectively, and obviously $x = x^+ - x^-$. Take $|x| = x^+ + x^-$, then $|x| \in P$. One can refer to [4] for

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the definition and the properties of the lattice. For convenience, we use the notation:

$$x_+ = x^+, \quad x_- = -x^-$$

and clearly

$$x_+ \in P, \quad x_- \in (-P), \quad x = x_+ + x_-.$$

In an ordered Banach space, much research has been done on computation of topological degree and the fixed point index for cone mappings by using the partial ordered relation and the functional [7], [9], [3], [10], [8]. The use of the partial ordered relation to compute topological degree and the fixed point index goes back to a pioneering paper by M. A. Krasnoselskii [5], which has been so influential as to motivate several authors to develop further the theory of topological degree and the fixed point index. His work made a very significant contribution to the field. For instance, in [8] Sun and Liu gave a computational method of topological degree by applying the theory of cones to studying non-cone mappings, and in [9], [3], [10], the authors established some theorems about computation of the topological degree for nonlinear operators which are not cone mappings, using the partial ordering relation and the lattice structure.

Motivated by [9], [10], we derive some new theorems about computation of the topological degree by means of the partial ordering relation and the lattice structure. As applications of our main results, existence of nontrivial solutions for the singular Sturm-Liouville problem is considered where the nonlinear term f is a sign-changing function and not necessarily bounded from below.

To conclude this section, we present a result which will be used in Section 2.

Lemma 1.1 ([1], [2]). *Let Ω be a bounded open set in a real Banach space E and let $A\colon \overline{\Omega} \to E$ be compact. If there exists a $u_0 \in E$, $u_0 \neq \theta$, such that*

$$x - Ax \neq \mu u_0 \quad \text{for all } x \in \partial\Omega \text{ and } \mu \geqslant 0,$$

then the Leray-Schauder degree is

$$\deg(I - A, \Omega, \theta) = 0.$$

2. Main results

In this section, we always assume that E is a Banach space, P is a normal cone in E and the partial ordering $\leqslant$ in E is induced by P. We also suppose that E is a lattice in the partial ordering $\leqslant$.

Let $B\colon E\to E$ be a positive completely continuous linear operator; $r(B)$ a spectral radius of B; B^* the conjugated operator of B; P^* the conjugated cone of P. According to the famous Krein-Rutman theorem (see [6]), we infer that if $r(B)\neq 0$, then there exist $\varphi\in P\setminus\{\theta\}$ and $h\in P^*\setminus\{\theta\}$ such that

$$B\varphi = r(B)\varphi,\quad B^*h = r(B)h,\quad \|\varphi\| = \|h\| = 1. \tag{2.1}$$

Choose $\delta>0$ and define

$$P(h,\delta)=\{x\in P;\ h(x)\geqslant \delta\|x\|\}.$$

Then $P(h,\delta)$ is also a cone in E.

Definition 2.1 ([9])**.** Let $D\subset E$ and let $F\colon D\to E$ be a nonlinear operator. Then F is said to be quasi-additive on lattice if there exists $y\in E$ such that

$$Fx = Fx_+ + Fx_- + y,\quad \forall x\in D, \tag{2.2}$$

where x_+ and x_- are defined by (1.1).

Remark 2.1. By Remark 3.1 in [3], we know that the condition (2.2) appears naturally in the applications involving nonlinear differential equations and integral equations.

Now we establish the main theorems:

Theorem 2.1. *Let $A\colon E\to E$ be a completely continuous operator satisfying $A=BF$, where F is quasi-additive on lattice with $y=\theta$. Suppose:*

- (H$_1$) *There exist $\varphi\in P\setminus\{\theta\}$ and $h\in P^*\setminus\{\theta\}$ such that* (2.1) *holds and $B(P)\subset P(h,\delta)$.*
- (H$_2$) *There exists $M>0$ such that $\|x\|_1\leqslant M\|x\|$, where $\|x\|_1$ denotes the norm of $|x|$.*
- (H$_3$) *There exist $\eta>0$ and $r^*>0$ such that*

$$BFx\geqslant r^{-1}(B)(1+\eta)Bx,\quad x\in P\cap B_{r^*},$$

where $B_{r^*}=\{x\in E \mid \|x\|<r^*\}$.
- (H$_4$) *There exist $0\leqslant a<\min\{\delta/(M(r(B)+\delta\|B\|)), r^{-1}(B)(1+\eta)\}$ and $r^{**}>0$ such that*

$$Fx+ax\in P\cap B_{r^{**}},\quad x\in P;\ Fx-ax\in P,\quad x\in(-P)\cap B_{r^{**}}.$$

Then there exists $0 < r < \min\{r^*, r^{**}\}$ *such that the topological degree is*

$$\deg(I - A, B_r, \theta) = 0.$$

R e m a r k 2.2. We point out that the condition (H_2) of Theorem 2.1 appears naturally in the applications involving nonlinear differential equations and integral equations.

Let $E = C[0,1] = \{x(t) \mid x\colon [0,1] \to R^1 \text{ is continuous}\}$ and $P = \{x \in C[0,1] \mid x(t) \geqslant 0\}$, then $C[0,1]$ is a lattice under the partial ordering induced by P. For any $x \in C[0,1]$, it is evident that

$$x_+(t) = \begin{cases} x(t), & \text{if } x(t) \geqslant 0, \\ 0, & \text{if } x(t) \leqslant 0, \end{cases}$$
$$x_-(t) = \begin{cases} x(t), & \text{if } x(t) \leqslant 0, \\ 0, & \text{if } x(t) \geqslant 0, \end{cases}$$

and hence $|x|(t) = |x(t)|$, $\|x\|_1 = \|x\|$ and so the condition (H_2) of Theorem 2.1 is a natural condition.

P r o o f. It follows from the normality of the cone P and from (H_2) that there exists $r_0 > 0$ such that $\|x\| \leqslant r_0 (< \min\{r^*, r^{**}\})$ implies that $\|x_+\| < \min\{r^*, r^{**}\}$ and $\|x_-\| < \min\{r^*, r^{**}\}$.

We now claim that there exists $0 < r < r_0$ such that

(2.3) $$x - Ax \neq \tau\varphi, \ \forall x \in \partial B_r \text{ and } \tau \geqslant 0,$$

where φ is the positive eigenfunction of B corresponding to its eigenvalue $r(B)$. If otherwise, then for all $0 < r < r_0$ there exist $x \in \partial B_r$ and $\tau \geqslant 0$ such that

(2.4) $$x = Ax + \tau\varphi.$$

Then, from (2.1), (H_3) and (H_4), we have

$$\begin{aligned} h(x) &\geqslant h(Ax) \geqslant h(BFx_+ + BFx_-) \geqslant h(r^{-1}(B)(1+\eta)Bx_+) + h(aBx_-) \\ &\geqslant h(r^{-1}(B)(1+\eta)Bx) = r^{-1}(B)(1+\eta)(B^*h)(x) = (1+\eta)h(x). \end{aligned}$$

Thus $h(x) \leqslant 0$. This, together with (2.1) and (H_2), implies that

(2.5) $$h(x + aB(|x|)) = h(x) + ar(B)h(|x|) \leqslant ar(B)h(|x|) \leqslant ar(B)M\|x\|.$$

Since $\tau r(B)\varphi = \tau B\varphi$ by virtue of (2.1), we have from conditions (H_1) and (H_4)

$$x + aB(|x|) = Ax + \tau\varphi + aB(|x|) = B(Fx_+ + ax_+) + B(Fx_- - ax_-) + \tau\varphi \in P(h,\delta).$$

So, from the definition of $P(h,\delta)$ we obtain

$$h(x + aB(|x|)) \geqslant \delta\|x + aB(|x|)\| \geqslant \delta\|x\| - \delta aM\|B\|\,\|x\|. \tag{2.6}$$

Thus, by (2.5) and (2.6), we have

$$(\delta - aM(r(B) + \delta\|B\|))\|x\| \leqslant 0.$$

Since $a < \min\{\delta/(M(r(B) + \delta\|B\|)), r^{-1}(B)(1+\eta)\}$, (2.4) cannot hold. Therefore, there exists $0 < r < r_0$ such that (2.3) holds. Note that the operator A is compact. The conclusion now readily follows from Lemma 1.1, and this completes the proof of the theorem. □

Theorem 2.2. *Let $A\colon E \to E$ be a completely continuous operator satisfying $A = BF$, where F is quasi-additive on lattice and B is a positive bounded linear operator satisfying the conditions* (H_1) *and* (H_2) *of Theorem* 2.1. *Suppose in addition that*

(H_5) *there exist $0 < \eta < 1$ and $u_0 \in P$ such that*

$$Fx \geqslant r^{-1}(B)(1+\eta)x - u_0,\ \ u \in P,$$
$$Fx \geqslant r^{-1}(B)(1-\eta)x - u_0,\ \ x \in (-P).$$

Then there exists $R_0 > 0$ such that for $R > R_0$, the topological degree is

$$\deg(I - A, B_R, \theta) = 0.$$

Proof. Setting $D = \{x \in E;\ x - Ax = \tau\varphi,\ \tau \geqslant 0\}$, we claim that D is bounded. Then for $x \in D$ there exists $\tau \geqslant 0$ such that

$$x = Ax + \tau\varphi. \tag{2.7}$$

Then, from (2.2) and (H_5), we have

$$\begin{aligned} x = Ax + \tau\varphi &\geqslant Ax_+ + Ax_- + By \\ &\geqslant r^{-1}(B)(1+\eta)Bx_+ + r^{-1}(B)(1-\eta)Bx_- - 2Bu_0 + By_- \\ &\geqslant r^{-1}(B)(1-\eta)Bx_- - 2Bu_0 + By_-. \end{aligned} \tag{2.8}$$

Since $r^{-1}(B)(1-\eta)Bx_- - 2Bu_0 + By_- \leqslant \theta$, it follows from (2.8) that

$$x_- \geqslant r^{-1}(B)(1-\eta)Bx_- - 2Bu_0 + By_-,$$

and thus

$$(I - r^{-1}(B)(1-\eta)B)x_- \geqslant -2Bu_0 + By_-.$$

This implies that

$$x_- \geqslant (I - r^{-1}(B)(1-\eta)B)^{-1}(-2Bu_0 + By_-) := w, \quad x \in D. \tag{2.9}$$

It follows from (2.7) and (2.9) that

$$\begin{aligned} x_+ \geqslant x = Ax + \tau\varphi &\geqslant BFx_+ + BFx_- + By_- \\ &\geqslant r^{-1}(B)(1+\eta)Bx_+ + r^{-1}(B)(1-\eta)Bx_- - 2Bu_0 + By_- \\ &\geqslant r^{-1}(B)(1+\eta)Bx_+ + r^{-1}(B)(1-\eta)Bw - 2Bu_0 + By_-. \end{aligned}$$

This, together with (2.1) and (H_2), implies that

$$\begin{aligned} h(x_+) &\geqslant r^{-1}(B)(1+\eta)h(Bx_+) + r^{-1}(B)(1-\eta)h(Bw) - 2h(Bu_0) + h(By_-) \\ &= (1+\eta)h(x_+) + (1-\eta)h(w) - 2r(B)h(u_0) + r(B)h(y_-). \end{aligned}$$

Thus

$$h(x_+) \leqslant \eta^{-1}(2r(B)h(u_0) - r(B)h(y_-) - (1-\eta)h(w)) := C_2. \tag{2.10}$$

On account of (2.2), (2.9), and (H_5), we arrive at

$$\begin{aligned} Ax \geqslant Ax_+ + Ax_- + By &\geqslant r^{-1}(B)(1+\eta)Bx_+ - Bu_0 \\ &\quad + r^{-1}(B)(1-\eta)Bx_- - Bu_0 + By_- \\ &\geqslant r^{-1}(B)(1+\eta)Bx_+ + r^{-1}(B)(1-\eta)Bw - 2Bu_0 + By_- \\ &= r^{-1}(B)(1+\eta)Bx_+ + w. \end{aligned}$$

This implies

$$Ax \geqslant w, \quad x \in D.$$

Set $v = r^{-1}(B)(1-\eta)w - 2u_0 + y$. By (2.2), (2.9), and ($H_5$), we have

$$\begin{aligned} Fx &\geqslant Fx_+ + Fx_- + y \geqslant r^{-1}(B)(1+\eta)x_+ - u_0 + r^{-1}(B)(1-\eta)x_- - u_0 + y \\ &\geqslant r^{-1}(B)(1-\eta)w - 2u_0 + y = v, \ x \in D. \end{aligned}$$

So, from the definition of $P(h,\delta)$, we get

$$B(Fx - v) \in P(h,\delta). \tag{2.11}$$

(2.1) gives $\tau\varphi = \tau/(r(B))B(\varphi)$ and this, together with (2.7) and (2.11), yields

$$x - Bv = B(Fx - v) + \tau\varphi \in P(h,\delta).$$

Therefore,

$$h(x - Bv) \geqslant \delta\|x - Bv\| \geqslant \delta\|x\| - \delta\|Bv\|.$$

Hence,

$$\|x\| \leqslant \frac{1}{\delta}(\delta\|Bv\| + h(x) - h(Bv)) \leqslant \frac{1}{\delta}((1+\delta)\|Bv\| + h(x_+)).$$

Then for $x \in D$, by (2.10), we have

$$\|x\| \leqslant \frac{1}{\delta}((1+\delta)\|Bv\| + C_2),$$

which shows that D is bounded.

Let $R_0 = \sup\limits_{x\in D}\|x\|$. For $R > R_0$ we obtain

$$x - Ax \neq \tau\varphi, \quad \forall x \in \partial B_R,\ \tau \geqslant 0. \tag{2.12}$$

Using Lemma 1.1, we infer by (2.12) that the conclusion is true. □

Remark 2.3. In Theorem 2.2, we do not assume that the cone P is necessarily solid. Hence Theorem 2.2 improves the result of Theorem 3.2 in [10] and has a wider range of applications.

By Theorem 2.1 and Theorem 3.3 in [9] we obtain

Theorem 2.3. *Suppose that the conditions in Theorem* 2.1 *hold. If there exist a positive bounded linear operator B_1 with $r(B_1) < 1$ and $v_0 \in P$ such that*

$$|Ax| \leqslant B_1|x| + v_0 \quad \forall x \in E, \tag{2.13}$$

then A has at least one nonzero fixed point.

3. Applications

Consider the singular Sturm-Liouville boundary value problem

$$\begin{cases} -(Lu)(t) = a(t)f(t,u(t)), \quad 0 < t < 1, \\ R_1(u) = \alpha_0 u(0) + \beta_0 u'(0) = 0, \ R_2(u) = \alpha_1 u(1) + \beta_1 u'(1) = 0, \end{cases} \tag{3.1}$$

where $(Lu)(t) = (p(t)u'(t))' + q(t)u(t)$, $a(t)$ is allowed to be singular at both $t = 0$ and $t = 1$. Through this section, we always suppose that

$$p \in C^1[0,1], \quad p(t) > 0, \quad q \in C[0,1], \quad q(t) \leqslant 0;$$
$$\alpha_0 \geqslant 0, \quad \beta_0 \leqslant 0, \quad \alpha_1 \geqslant 0, \quad \beta_1 \geqslant 0, \quad \alpha_0^2 + \beta_0^2 \neq 0, \quad \alpha_1^2 + \beta_1^2 \neq 0;$$

and the homogeneous equation with respect to (3.1)

$$\begin{cases} -(Lu)(t) = 0, \quad 0 < t < 1, \\ R_1(u) = R_2(u) = 0 \end{cases} \tag{3.2}$$

has only the trivial solution.

Let $k(t,s)$ be Green's function with respect to (3.2). According to the Sturm-Liouville theory of ordinary differential equations (see [12]), we have

Lemma 3.1. *Green's function $k(t,s)$ possesses the following form:*

$$k(t,s) = \begin{cases} c^{-1}u_0(t)v_0(s), \quad 0 \leqslant t \leqslant s \leqslant 1, \\ c^{-1}u_0(s)v_0(t), \quad 0 \leqslant s \leqslant t \leqslant 1, \end{cases} \tag{3.3}$$

where c is a positive constant, and $u_0, v_0 \in C^2[0,1]$ satisfy the following conditions:

(i) $k(t,s) = k(s,t) \geqslant 0$ *and* $k(t,t) = u_0(t)v_0(t)/c$ *for* $t, s \in [0,1]$;
(ii) u_0 *is increasing on* $[0,1]$ *with* $u_0(t) > 0$ *for* $t \in (0,1]$;
(iii) v_0 *is decreasing on* $[0,1]$ *with* $v_0(t) > 0$ *for* $t \in [0,1)$;
(iv) $(Lu_0)(t) \equiv 0, \ u_0(0) = -\beta_0, \ u_0'(0) = \alpha_0$;
(v) $(Lv_0)(t) \equiv 0, \ v_0(1) = \beta_1, \ v_0'(1) = -\alpha_1$.

By Lemma 3.1, it is easy to conclude that

$$\frac{ck(t,t)k(s,s)}{u_0(1)v_0(0)} \leqslant k(t,s) \leqslant k(t,t) \ (\text{or } k(s,s)), \quad 0 \leqslant t, s \leqslant 1, \tag{3.4}$$

and

$$k(t,s) \geqslant \frac{ck(t,t)}{u_0(1)v_0(0)} k(\tau,s) \quad \forall t,\tau,s \in [0,1]. \tag{3.5}$$

In this section, we always suppose that

(G_1) $a\colon (0,1)\to[0,+\infty)$ is continuous, $a(t)\not\equiv 0$ and

$$0<\int_0^1 a(t)\,\mathrm{d}t<+\infty;$$

(G_2) $f(t,u)\colon [0,1]\times R^1\to R^1$ is continuous and $f(t,0)=0$ for $t\in[0,1]$.

Let $E=C[0,1]$. Then E is an ordered Banach space with the sup norm $\|u\|=\sup\limits_{0\leqslant t\leqslant 1}|u(t)|$ and

$$P=\{u\in C[0,1]\mid u(t)\geqslant 0,\ t\in[0,1]\}$$

is a cone of E. It is obvious that P is a normal solid cone, and E becomes a lattice under the natural ordering $\leqslant$.

Let us introduce the operators

$$(A\varphi)(t)=\int_0^1 k(t,s)a(s)f(s,\varphi(s))\,\mathrm{d}s,\quad t\in[0,1]; \tag{3.6}$$

$$(B\varphi)(t)=\int_0^1 k(t,s)a(s)\varphi(s)\,\mathrm{d}s,\quad t\in[0,1]; \tag{3.7}$$

$$(F\varphi)(t)=f(t,\varphi(t)),\quad t\in[0,1]. \tag{3.8}$$

We have

Lemma 3.2. *Suppose that* (H_1) *is satisfied. Then for the operator* B *defined by* (3.7),

(i) $B\colon E\to E$ *is a completely continuous linear operator and* $B(P)\subset P_1$, *where* $P_1=\{u\in P;\ u(t)\geqslant ck(t,t)/(u_0(1)v_0(0))\|u\|\}$ *is a cone of* E;

(ii) *the spectral radius* $r(B)\neq 0$ *and* B *has a positive normalized eigenfunction* $\varphi\in P$ *corresponding to its first eigenvalue* $\lambda_1=(r(B))^{-1}$;

(iii) *there exists* $\delta_1>0$ *such that* $\varphi(s)\geqslant\delta_1 k(s,s)\geqslant\delta_1 k(t,s)$ *for* $t,s\in[0,1]$.

Proof. It follows from (3.4), (3.5) and (G_1) that the operator B satisfies (i) and (ii). Since $\varphi\in P$ is positive eigenfunction of B, it follows from (3.4) that $\varphi(s)\geqslant c\lambda_1 k(s,s)/(u_0(1)v_0(0))\int_0^1 k(t,t)a(t)\varphi(t)\,\mathrm{d}t$ and $\varphi(s)\leqslant\lambda_1\int_0^1 k(t,t)a(t)\varphi(t)\,\mathrm{d}t$, therefore $\int_0^1 k(t,t)a(t)\varphi(t)\,\mathrm{d}t>0$. Set

$$\delta_1=\frac{c\lambda_1}{u_0(1)v_0(0)}\int_0^1 k(t,t)a(t)\varphi(t)\,\mathrm{d}t,$$

then we have

$$\varphi(s)\geqslant\delta_1 k(s,s)\geqslant\delta_1 k(t,s),\quad\forall t,s\in[0,1].$$

□

Theorem 3.1. *Let* $\delta = c/u_0(1)v_0(0)\int_0^1 a(t)\,\mathrm{d}t\int_0^1 k(t,t)a(t)\varphi(t)\,\mathrm{d}t$, *and let* (G_1) *and* (G_2) *hold. Suppose in addition that there exist* $\eta > 0$, $r > 0$ *and* $0 \leqslant a < \min\{\delta/(M(r(B)+\delta\|B\|)), r^{-1}(B)(1+\eta)\}$ *such that*

$$f(t,u) - au \geqslant 0, \quad \text{for } (t,u) \in [0,1]\times[-r,0]; \tag{3.9}$$

$$\liminf_{u\to 0^+}\min_{t\in[0,1]}\frac{f(t,u)}{u} > \lambda_1; \tag{3.10}$$

$$\limsup_{u\to +\infty}\max_{t\in[0,1]}\frac{f(t,u)}{u} < \lambda_1. \tag{3.11}$$

Then the singular Sturm-Liouville boundary value problem (3.1) *has at least one nontrivial solution.*

P r o o f. Let $E = C[0,1]$; A, B, F be defined by (3.6), (3.7) and (3.8) respectively. Clearly, $F\colon E\to E$ is continuous and quasi-additive on lattice with $y=\theta$. Since $B\colon E\to E$ is completely continuous, we know that $A\colon E\to E$ is completely continuous.

Let $h^*(x) = \int_0^1 a(t)\varphi(t)x(t)\,\mathrm{d}t$ and $h = h^*/\|h^*\|$. For $x\in P$, by $k(t,s)=k(s,t)$ and Lemma 3.2 (ii) we have

$$\begin{aligned}(B^*h)(x) = h(Bx) &= \frac{1}{\|h^*\|}h^*(Bx) = \frac{1}{\|h^*\|}\int_0^1 a(t)\varphi(t)(Bx)(t)\,\mathrm{d}t \\ &= \frac{1}{\|h^*\|}\int_0^1 a(t)\varphi(t)\,\mathrm{d}t\int_0^1 k(t,s)a(s)x(s)\,\mathrm{d}s \\ &= \frac{1}{\|h^*\|}\int_0^1 a(s)x(s)\,\mathrm{d}s\int_0^1 k(t,s)a(t)\varphi(t)\,\mathrm{d}t \\ &= \frac{1}{\|h^*\|}\int_0^1 a(s)x(s)\,\mathrm{d}s\int_0^1 k(s,t)a(t)\varphi(t)\,\mathrm{d}t \\ &= \frac{1}{\|h^*\|}\int_0^1 a(s)x(s)(B\varphi)(s)\,\mathrm{d}s = \frac{1}{\lambda_1\|h^*\|}\int_0^1 a(s)x(s)\varphi(s)\,\mathrm{d}s \\ &= \frac{1}{\lambda_1}h(x),\end{aligned} \tag{3.12}$$

and thus $B^*h = r(B)h$. For $x\in P$, Lemma 3.2 shows $Bx\in P$. In addition, by virtue of Lemma 3.2 (iii) and (3.12), we get

$$\begin{aligned}h(Bx) &\geqslant \frac{\delta_1}{\lambda_1\|h^*\|}\int_0^1 k(t,s)a(s)x(s)\,\mathrm{d}s \\ &\geqslant \frac{\delta_1}{\lambda_1\int_0^1 a(t)\,\mathrm{d}t}\int_0^1 k(t,s)a(s)x(s)\,\mathrm{d}s = \delta(Bx)(t), \quad t\in[0,1],\end{aligned}$$

which means that $B(P) \subset P(h,\delta)$. This shows that the condition (H_1) in Theorem 2.3 is satisfied.

On account of Remark 2.2, we see that the condition (H_2) in Theorem 2.3 is satisfied.

By virtue of (3.9) and (3.10) there exist $\eta > 0$ and $r^* > 0$ such that

$$f(t,u) - au \geqslant 0 \quad \text{for } (t,u) \in [0,1] \times [-r,0],$$
$$f(t,u) \geqslant \lambda_1(1+\eta)u, \quad t \in [0,1],\ u \in [0,r^*],$$

which clearly implies that

$$Fx \geqslant \lambda_1(1+\eta)x, \quad x \in P \cap B_{r^{**}},$$
$$Fx - ax \in P, \quad x \in P \cap B_{r^{**}}, \quad r^{**} = \min\{r, r^*\}.$$

Hence, (H_3) and (H_4) in Theorem 2.3 are satisfied.

By (3.11) there exist $\varepsilon > 0$ and a sufficiently large number $L_1 > 0$ such that

$$|f(t,u)| \leqslant \lambda_1(1-\varepsilon)|u|, \quad t \in [0,1],\ u > L_1. \tag{3.13}$$

Combining (3.13) with (H_1), we have that there exists $b_1 > 0$ such that

$$|f(t,u)| \leqslant \lambda_1(1-\varepsilon)|u| + b_1, \quad t \in [0,1],\ u \in R,$$

and so

$$|Fx| \leqslant \lambda_1(1-\varepsilon)|x| + b_1 \quad \forall x \in E. \tag{3.14}$$

Since B is a positive linear operator and $r(B) = 1/\lambda_1$, from (3.14) we have

$$|Ax| \leqslant \lambda_1(1-\varepsilon)B|x| + B(b_1) \quad \forall x \in E.$$

So condition (2.13) in Theorem 2.3 is satisfied with $B_1 = \lambda_1(1-\varepsilon)B$.

Thus, all conditions in Theorem 2.3 are satisfied. So Theorem 2.3 guarantees that our conclusion holds. □

Remark 3.1. From (3.9) we know that $f(t,u)$ may take negative values for $(t,u) \in [0,1] \times [-r,0]$, which makes it impossible to apply the methods in [11] to the present paper. So the method is new and the results obtained in this paper improve and extend those in [11].

At the end of this section, we give a rough estimate for a. Since φ is the positive normalized eigenfunction of B corresponding to its first eigenvalue $\lambda_1 = r^{-1}(B)$ and $B(P) \subset P_1$ (see Lemma 3.2), we have

$$\begin{aligned}
\delta &= \frac{c}{u_0(1)v_0(0)\int_0^1 a(t)\,\mathrm{d}t}\int_0^1 k(t,t)a(t)\varphi(t)\,\mathrm{d}t \\
&\geqslant \frac{c}{u_0(1)v_0(0)\int_0^1 a(t)\,\mathrm{d}t}\int_0^1 k(t,t)a(t)\frac{ck(t,t)}{u_0(1)v_0(0)}\|\varphi\|\,\mathrm{d}t \\
&= \frac{c^2}{u_0^2(1)v_0^2(0)\int_0^1 a(t)\,\mathrm{d}t}\int_0^1 k^2(t,t)a(t)\,\mathrm{d}t := \delta_0.
\end{aligned} \tag{3.15}$$

On the other hand, it is easy to see that $\delta_1 > \delta_2 > 0$ implies that $P(h,\delta_1) \subset P(h,\delta_2)$ and $r(B) \leqslant \|B\|$. As a result, by Theorem 3.1, if

$$a \in \Big[0, \min\Big\{\frac{\delta}{M\|B\|(1+\delta)}, r^{-1}(B)\Big\}\Big), \tag{3.16}$$

then the singular Sturm-Liouville boundary value problem (3.1) has at least one nontrivial solution.

4. An example

In this section, we construct an example to demonstrate the application of our result obtained in Section 3.

Let $h(t) = 1/\sqrt{t(1-t)}$ and

$$f(t,u) = \begin{cases} \sqrt{u}, & u \geqslant 0, \\ \sum_{i=1}^{n} a_i u^i, & -1 < u < 0, \\ \sum_{i=1}^{n} (-1)^i a_i - (1+t^2)\ln|u|, & u \leqslant -1, \end{cases} \tag{4.1}$$

where $a_1 \in (-24/131\pi, +\infty)$. Consider the second-order singular Dirichlet two-point boundary value problem

$$\begin{cases} u''(t) + h(t)f(t,u) = 0, & 0 < t < 1, \\ u(0) = u(1) = 0. \end{cases} \tag{4.2}$$

Green's function of the relevant homogeneous equation is

$$k(t,s) = \begin{cases} t(1-s), & 0 \leqslant t \leqslant s \leqslant 1, \\ (1-t)s, & 0 \leqslant s \leqslant t \leqslant 1. \end{cases}$$

Let $(Bu)(t) = \int_0^1 k(t,s)h(s)u(s)\,\mathrm{d}s$, $t \in [0,1]$; $(B_1u)(t) = \int_0^1 k(t,s)u(s)\,\mathrm{d}s$, $t \in [0,1]$; $(B_2u)(t) = \int_0^1 \sqrt{s(1-s)}u(s)\,\mathrm{d}s$, $t \in [0,1]$. It is easy to show that

$$B_1u \leqslant Bu \leqslant B_2u, \quad u \in P = \{u \in C[0,1] \mid u(t) \geqslant 0,\ t \in [0,1]\}.$$

Thus by [12] and $\int_0^1 \sqrt{s(1-s)}\,\mathrm{d}s = \pi/8$, $r(B_1) = 1/\pi^2$, we have $r(B) \geqslant r(B_1) > 0$ and $r(B) \leqslant \|B\| \leqslant \|B_2\| \leqslant \pi/8$. This together with

$$\delta = \frac{\int_0^1 t^{\frac{3}{2}}(1-t)^{\frac{3}{2}}\,\mathrm{d}t}{\int_0^1 \frac{1}{\sqrt{t(1-t)}}\,\mathrm{d}t} = \frac{3}{128}$$

implies

$$\min\Big\{\frac{\delta}{M\|B\|(1+\delta)}, r^{-1}(B)\Big\} \geqslant \frac{24}{131\pi}.$$

It is easy to prove that all the conditions in Theorem 3.1 are satisfied. As a result, BVP (4.1) with the $h(t)$ and $f(t,u)$ given by (4.1) has at least one nontrivial solution.

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