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THE n-DUAL SPACE OF THE SPACE OF p-SUMMABLE SEQUENCES

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Cordially dedicated to the late Professor Moedomo

Abstract. In the theory of normed spaces, we have the concept of bounded linear functionals and dual spaces. Now, given an n-normed space, we are interested in bounded multilinear n-functionals and n-dual spaces. The concept of bounded multilinear n-functionals on an n-normed space was initially intoduced by White (1969), and studied further by Batkunde et al., and Gozali et al. (2010). In this paper, we revisit the definition of bounded multilinear n-functionals, introduce the concept of n-dual spaces, and then determine the n-dual spaces of ℓ^p spaces, when these spaces are not only equipped with the usual norm but also with some n-norms.

Keywords: ℓ^p space; n-normed space; n-dual space

MSC 2010: 46B20, 46C05, 46C15, 46B99, 46C99

1. Introduction

Let n be a nonnegative integer and X a real vector space of dimension $d \geqslant n$. A real-valued function $\|\cdot, \ldots, \cdot\|$ on X^n satisfying the following four properties,

(1) $\|x_1, \ldots, x_n\| = 0$ if and only if $x_1, \ldots, x_n$ are linearly dependent,
(2) $\|x_1, \ldots, x_n\|$ is invariant under permutation,
(3) $\|\alpha x_1, \ldots, x_n\| = |\alpha| \|x_1, \ldots, x_n\|$ for all $\alpha \in \mathbb{R}$,
(4) $\|x_1 + x_1', \ldots, x_n\| \leqslant \|x_1, x_2, \ldots, x_n\| + \|x_1', x_2, \ldots, x_n\|$,

is called an *n-norm* on X, and the pair $(X, \|\cdot, \ldots, \cdot\|)$ is called an *n-normed space* [2], [3], [4]. Note that on an n-normed space $(X, \|\cdot, \ldots, \cdot\|)$ we have $\|x_1, x_2, \ldots, x_n\| = \|x_1 + y, x_2, \ldots, x_n\|$ for any linear combination y of $x_2, \ldots, x_2 \in X$.

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To give an example, let $1 \leqslant p < \infty$ and $1/p + 1/q = 1$. Then we can equip the space ℓ^p of p-summable sequences with an n-norm $\|\cdot, \dots, \cdot\|_p^G$ which is given by

$$\|x_1, \dots, x_n\|_p^G := \sup_{y_j \in \ell^q,\, \|y_j\|_q \leqslant 1} \left| \det \left[\sum_{k=1}^{\infty} x_{ik} y_{jk} \right]_{i,j} \right|, \quad x_1, \dots, x_n \in \ell^p.$$

Here ℓ^q is the dual space of ℓ^p, and $\|\cdot\|_q$ denotes the usual norm on ℓ^q (see, for instance, [8]). The above n-norm is due to Gähler [2], [3], [4]. Another n-norm can be defined on ℓ^p, namely

$$\|x_1, \dots, x_n\|_p^H := \left(\frac{1}{n!} \sum_{k_1=1}^{\infty} \dots \sum_{k_n=1}^{\infty} |\det[x_{ik_j}]_{i,j}|^p \right)^{1/p}, \quad x_1, \dots, x_n \in \ell^p.$$

This n-norm was introduced by Gunawan [6]. As shown in [12], these two n-norms on ℓ^p are equivalent, that is,

$$(1.1) \qquad (n!)^{1/p-1} \|x_1, \dots, x_n\|_p^H \leqslant \|x_1, \dots, x_n\|_p^G \leqslant (n!)^{1/p} \|x_1, \dots, x_n\|_p^H$$

for all $x_1, \dots, x_n \in \ell^p$.

Any real-valued function f on X^n, where X is a real vector space of dimension $d \geqslant n$, is called an *n-functional* on X. Furthermore, an n-functional f satisfying the following two properties:

(1) $f(x_1 + y_1, \dots, x_n + y_n) = \sum_{h_i \in \{x_i, y_i\},\, 1 \leqslant i \leqslant n} f(h_1, \dots, h_n)$,

(2) $f(\alpha_1 x_1, \dots, \alpha_n x_n) = \alpha_1 \dots \alpha_n f(x_1, \dots, x_n)$,

is called a *multilinear n-functional* on X.

Next, suppose that f is an n-functional on a normed space $(X, \|\cdot\|)$ [an n-normed space $(X, \|\cdot, \dots, \cdot\|)$]. If there exists a constant $K > 0$ such that

$$|f(x_1, \dots, x_n)| \leqslant K\|x_1\| \dots \|x_n\| \quad [|f(x_1, \dots, x_n)| \leqslant K\|x_1, \dots, x_n\|]$$

for all $x_1, \dots, x_n \in X$, then f is said to be *bounded* on $(X, \|\cdot\|)$ [*bounded* on $(X, \|\cdot, \dots, \cdot\|)$, respectively], see [5] and [11].

It is easy to check that every bounded multilinear n-functional f on an n-normed space $(X, \|\cdot, \dots, \cdot\|)$ satisfies

$$f(x_1, \dots, x_n) = 0$$

whenever $x_1, \dots, x_n$ are linearly dependent. Further, it is antisymmetric, that is,

$$f(x_1, \dots, x_n) = \operatorname{sgn}(\sigma) f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

for any $x_1, \dots, x_n \in X$ and any permutation σ of $(1, \dots, n)$. Here $\mathrm{sgn}(\sigma) = 1$ if σ is an even permutation and $\mathrm{sgn}(\sigma) = -1$ if σ is an odd permutation. These properties do not hold for bounded multilinear n-functionals on a normed space $(X, \|\cdot\|)$.

Inspired by the concept of the dual space of a normed space, the space of bounded multilinear n-functionals on $(X, \|\cdot\|)$ [on $(X, \|\cdot, \dots, \cdot\|)$] is called the n-*dual space* of $(X, \|\cdot\|)$ [the n-*dual space* of $(X, \|\cdot, \dots, \cdot\|)$, respectively]. This space can be equipped with the norm

$$\|f\|_{n,1} := \sup_{\|x_1\|, \dots, \|x_n\| \neq 0} \frac{|f(x_1, \dots, x_n)|}{\|x_1\| \dots \|x_n\|}$$
$$\Big[\|f\|_{n,n} := \sup_{\|x_1, \dots, x_n\| \neq 0} \frac{|f(x_1, \dots, x_n)|}{\|x_1, \dots, x_n\|}, \text{ respectively}\Big].$$

In the subsequent sections, we shall focus on $X = \ell^p$, where $1 \leqslant p < \infty$. For convenience, we shall first discuss the 2-dual spaces of ℓ^p, and then generalize the result for all $n \geqslant 2$. This work is part of the first author thesis [10].

2. The 2-dual spaces of ℓ^p

We shall here identify the 2-dual space of ℓ^p as a normed space, and then use the result to determine the 2-dual space of ℓ^p as a 2-normed space, equipped with Gähler's 2-norm as well as Gunawan's 2-norm. From now on, we shall always assume that $1 \leqslant p < \infty$ and q is the dual exponent of p, that is, $1/p + 1/q = 1$, unless otherwise stated.

To achieve our goals, we need to introduce the following normed space. We say that a double index sequence $\theta := (\theta_{kj})$ (of real numbers) belongs to the space $Y^q_{\mathbb{N}\times\mathbb{N}}$ if

$$\|\theta\|_{Y^q_{\mathbb{N}\times\mathbb{N}}} := \sup_{\|x\|_p = 1} \left(\sum_{j=1}^{\infty} \left| \sum_{k=1}^{\infty} x_k \theta_{kj} \right|^q \right)^{1/q} < \infty.$$

Here $\|\cdot\|_{Y^q_{\mathbb{N}\times\mathbb{N}}}$ defines a norm on $Y^q_{\mathbb{N}\times\mathbb{N}}$. For $q = \infty$, a double index sequence $\theta := (\theta_{kj})$ is in $Y^\infty_{\mathbb{N}\times\mathbb{N}}$ if

$$\|\theta\|_{Y^\infty_{\mathbb{N}\times\mathbb{N}}} := \sup_{\|x\|_1 = 1} \sup_{j \in \mathbb{N}} \left| \sum_{k=1}^{\infty} x_k \theta_{kj} \right| < \infty.$$

Our first result is

Theorem 2.1. *If* $1 < p < \infty$*, then the* 2*-dual space of* $(\ell^p, \|\cdot\|_p)$ *is identified by* $(Y^q_{\mathbb{N}\times\mathbb{N}}, \|\cdot\|_{Y^q_{\mathbb{N}\times\mathbb{N}}})$*. Moreover, the mapping* $f \mapsto \theta := (f(e_k, e_j))$ *is an isometric bijection from the* 2*-dual space of* $(\ell^p, \|\cdot\|_p)$ *to* $(Y^q_{\mathbb{N}\times\mathbb{N}}, \|\cdot\|_{Y^q_{\mathbb{N}\times\mathbb{N}}})$*.*

P r o o f. For $\theta := (\theta_{kj}) \in Y^q_{\mathbb{N}\times\mathbb{N}}$, we define a 2-functional f on ℓ^p by

$$f(x,y) := \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} x_k y_j \theta_{kj},$$

where $x := (x_i) = \sum\limits_{i=1}^{\infty} x_i e_i$ and $y := (y_i) = \sum\limits_{i=1}^{\infty} y_i e_i$. Note that $f(e_k, e_j) = \theta_{kj}$ for $k, j \in \mathbb{N}$. Further, f is a bilinear 2-functional on $(l^p, \|\cdot\|_p)$, and for $x, y \in \ell^p$ with $\|x\|_p = \|y\|_p = 1$, we have

$$\begin{aligned}|f(x,y)| &= \left|\sum_{j=1}^{\infty}\left(y_j \sum_{k=1}^{\infty} x_k\theta_{kj}\right)\right| \leqslant \left(\sum_{j=1}^{\infty}|y_j|^p\right)^{1/p}\left(\sum_{j=1}^{\infty}\left|\sum_{k=1}^{\infty} x_k\theta_{kj}\right|^q\right)^{1/q}\\ &= \left(\sum_{j=1}^{\infty}\left|\sum_{k=1}^{\infty} x_k\theta_{kj}\right|^q\right)^{1/q} \leqslant \sup_{\|z\|_p=1}\left(\sum_{j=1}^{\infty}\left|\sum_{k=1}^{\infty} z_k\theta_{kj}\right|^q\right)^{1/q} = \|\theta\|_{Y^q_{\mathbb{N}\times\mathbb{N}}}.\end{aligned}$$

Hence, for $x, y \neq 0$ we have

$$\frac{|f(x,y)|}{\|x\|_p\|y\|_p} \leqslant \|\theta\|_{Y^q_{\mathbb{N}\times\mathbb{N}}}.$$

This means that f is a bounded bilinear 2-functional on $(\ell^p, \|\cdot\|_p)$ with

$$\|f\|_{2,1} \leqslant \|\theta\|_{Y^q_{\mathbb{N}\times\mathbb{N}}}. \tag{2.1}$$

Conversely, let f be a bounded bilinear 2-functional on $(\ell^p, \|\cdot\|_p)$. We claim that $\theta := (f(e_k, e_j)) \in Y^q_{\mathbb{N}\times\mathbb{N}}$. For each $x \in \ell^p$ with $\|x\|_p = 1$, let f_x be the functional on $(\ell^p, \|\cdot\|_p)$ given by

$$f_x(y) := f(x,y), \quad y \in \ell^p.$$

It is clear that f_x is a linear functional on $(\ell^p, \|\cdot\|_p)$. Moreover, if $y \neq 0$, then

$$\frac{|f_x(y)|}{\|y\|_p} = \frac{|f(x,y)|}{\|x\|_p\|y\|_p} \leqslant \|f\|_{2,1}.$$

Hence f_x is bounded with $\|f_x\| \leqslant \|f\|_{2,1}$. Since the dual space of $(\ell^p, \|\cdot\|_p)$ is $(\ell^q, \|\cdot\|_q)$, the bounded linear functional f_x is identified by $(f_x(e_j)) = (f(x, e_j))$ with

$$\left(\sum_{j=1}^{\infty}|f(x,e_j)|^q\right)^{1/q} = \|f_x\| \leqslant \|f\|_{2,1}.$$

Therefore, we obtain

$$\|\theta\|_{Y^q_{\mathbb{N}\times\mathbb{N}}} = \sup_{\|x\|_p=1} \left(\sum_{j=1}^{\infty} \left| \sum_{k=1}^{\infty} x_k f(e_k, e_j) \right|^q \right)^{1/q} \leqslant \|f\|_{2,1}, \tag{2.2}$$

and this proves our claim.

It follows from (2.1) and (2.2) that the mapping $f \mapsto \theta := (f(e_k, e_j))$ is an isometric bijection from the 2-dual space of $(\ell^p, \|\cdot\|_p)$ to $(Y^q_{\mathbb{N}\times\mathbb{N}}, \|\cdot\|_{Y^q_{\mathbb{N}\times\mathbb{N}}})$. $\square$

For $p = 1$, we can also prove easily that the 2-dual space of $(\ell^1, \|\cdot\|_1)$ is identified by $(Y^\infty_{\mathbb{N}\times\mathbb{N}}, \|\cdot\|_{Y^\infty_{\mathbb{N}\times\mathbb{N}}})$. Hence we have the following corollary.

Corollary 2.2. *For $1 \leqslant p < \infty$ and $1/p + 1/q = 1$, the 2-dual space of $(\ell^p, \|\cdot\|_p)$ is identified by $(Y^q_{\mathbb{N}\times\mathbb{N}}, \|\cdot\|_{Y^q_{\mathbb{N}\times\mathbb{N}}})$.*

Now we shall discuss the 2-dual space of $(\ell^p, \|\cdot,\cdot\|^G_p)$. For this purpose, we need to invoke the concept of g-orthogonality on ℓ^p, where g is the semi-inner product on ℓ^p given by the formula

$$g(x, y) := \|x\|_p^{2-p} \sum_{j=1}^{\infty} |x_j|^{p-1} \operatorname{sgn}(x_j) y_j, \quad x := (x_j), y := (y_j).$$

If $g(x, y) = 0$, then we say that x and y are *g-orthogonal*, and we write $x \perp_g y$. (See [9] for some properties of g-orthogonality.)

As in [7], we may define the "volume" of the parallelepiped spanned by linearly independent $x_1, \dots, x_n \in \ell^p$ by the formula

$$V(x_1, \dots, x_n) := \|x_1^\circ\|_p \dots \|x_n^\circ\|_p,$$

where $\{x_1^\circ, \dots, x_n^\circ\}$ is the left g-orthogonal sequence obtained from $\{x_1, \dots, x_n\}$ through a Gram-Schmidt process. If $x_1, \dots, x_n$ are linearly dependent, then we simply define $V(x_1, \dots, x_n) = 0$.

In [12] it is shown that

$$V(x_{i_1}, \dots, x_{i_n}) \leqslant \|x_1, \dots, x_n\|^G_p \tag{2.3}$$

for all $x_1, \dots, x_n \in \ell^p$ and any permutation $(i_1, \dots, i_n)$ of $(1, \dots, n)$. Using this fact (for the case $n = 2$), we get the following theorem.

Theorem 2.3. *A bilinear 2-functional f is bounded on $(\ell^p, \|\cdot,\cdot\|^G_p)$ if and only if f is antisymmetric and bounded on $(\ell^p, \|\cdot\|_p)$. Furthermore, we have*

$$\frac{1}{2}\|f\|_{2,1} \leqslant \|f\|^G_{2,2} \leqslant \|f\|_{2,1},$$

where $\|\cdot\|^G_{2,2}$ is the norm on the 2-dual space of $(\ell^p, \|\cdot,\cdot\|^G_p)$.

P r o o f. Suppose that f is bounded on $(\ell^p, \|\cdot,\cdot\|_p^G)$. It is clear that f is antisymmetric, that is, $f(x,y) = -f(y,x)$ for all $x, y \in \ell^p$. Next, for $x, y \in \ell^p$ we have $\|x,y\|_p^G \leqslant 2^{1/p}\|x,y\|_p^H$ (by (1.1) for $n=2$) and $\|x,y\|_p^H \leqslant 2^{1-1/p}\|x\|_p\|y\|_p$ (see [6]), so that $\|x,y\|_p^G \leqslant 2\,\|x\|_p\|y\|_p$. Thus, for any linearly independent $x, y \in \ell^p$ we obtain

$$\frac{1}{2}\frac{|f(x,y)|}{\|x\|_p\|y\|_p} \leqslant \frac{|f(x,y)|}{\|x,y\|_p^G} \leqslant \|f\|_{2,2}^G.$$

Hence f is bounded on $(\ell^p, \|\cdot\|_p)$ with

$$\frac{1}{2}\|f\|_{2,1} \leqslant \|f\|_{2,2}^G. \tag{2.4}$$

Conversely, suppose that f is antisymmetric and bounded on $(\ell^p, \|\cdot\|_p)$. Given linearly independent $x, y \in \ell^p$, we observe that $f(x,y) = f(x^\circ, y^\circ)$ where $\{x^\circ, y^\circ\}$ is the left g-orthogonal set obtained from $\{x,y\}$. Moreover, we have

$$\frac{|f(x,y)|}{\|x,y\|_p^G} \leqslant \frac{|f(x,y)|}{V(x,y)} = \frac{|f(x^\circ,y^\circ)|}{\|x^\circ\|_p\|y^\circ\|_p} \leqslant \|f\|_{2,1}.$$

Since f is also antisymmetric, we have

$$|f(x,y)| \leqslant \|f\|_{2,1}\|x,y\|_p^G$$

for all $x, y \in \ell^p$, that is, f is bounded on $(\ell^p, \|\cdot,\cdot\|_p^G)$ with

$$\|f\|_{2,2}^G \leqslant \|f\|_{2,1}. \tag{2.5}$$

Finally, from (2.4) and (2.5) we conclude that

$$\frac{1}{2}\|f\|_{2,1} \leqslant \|f\|_{2,2}^G \leqslant \|f\|_{2,1},$$

as desired. □

To identify the 2-dual space of $(\ell^p, \|\cdot,\cdot\|_p^G)$, we consider some subspace of $Y^q_{\mathbb{N}\times\mathbb{N}}$. A double index sequence $\theta := (\theta_{kj})$ belongs to $Z^q_{\mathbb{N}\times\mathbb{N}}$ if $\theta \in Y^q_{\mathbb{N}\times\mathbb{N}}$ and $\theta_{kj} = -\theta_{jk}$ for all $k, j \in \mathbb{N}$. Note that $Z^q_{\mathbb{N}\times\mathbb{N}}$ can be viewed as a normed space equipped with the norm inherited from $Y^q_{\mathbb{N}\times\mathbb{N}}$.

Previously, we have shown that the 2-dual space of $(\ell^p, \|\cdot\|_p)$ is identified by $(Y^q_{\mathbb{N}\times\mathbb{N}}, \|\cdot\|_{Y^q_{\mathbb{N}\times\mathbb{N}}})$. Hence the space of all antisymmetric bounded bilinear 2-functionals on $(\ell^p, \|\cdot\|_p)$ can be identified by $(Z^q_{\mathbb{N}\times\mathbb{N}}, \|\cdot\|_{Y^q_{\mathbb{N}\times\mathbb{N}}})$. From this and the previous theorem we get the following corollaries.

Corollary 2.4. *The function* $\|\cdot\|^G_{Z^q_{\mathbb{N}\times\mathbb{N}}}$ *on* $Z^q_{\mathbb{N}\times\mathbb{N}}$ *defined by*

$$\|\theta\|^G_{Z^q_{\mathbb{N}\times\mathbb{N}}} := \sup_{\|x,y\|^G_p\neq 0} \frac{\left|\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} x_k y_j \theta_{kj}\right|}{\|x,y\|^G_p}$$

defines a norm on $Z^q_{\mathbb{N}\times\mathbb{N}}$. *Furthermore,* $\|\cdot\|^G_{Z^q_{\mathbb{N}\times\mathbb{N}}}$ *and* $\|\cdot\|_{Y^q_{\mathbb{N}\times\mathbb{N}}}$ *are equivalent norms on* $Z^q_{\mathbb{N}\times\mathbb{N}}$, *with*

$$\frac{1}{2}\|\theta\|_{Y^q_{\mathbb{N}\times\mathbb{N}}} \leqslant \|\theta\|^G_{Z^q_{\mathbb{N}\times\mathbb{N}}} \leqslant \|\theta\|_{Y^q_{\mathbb{N}\times\mathbb{N}}}$$

for all $\theta \in Z^q_{\mathbb{N}\times\mathbb{N}}$.

Corollary 2.5. *The* 2*-dual space of* $(\ell^p, \|\cdot,\cdot\|^G_p)$ *is identified by* $(Z^q_{\mathbb{N}\times\mathbb{N}}, \|\cdot\|^G_{Z^q_{\mathbb{N}\times\mathbb{N}}})$.

Using (1.1) for the case $n = 2$, we obtain the following corollaries.

Corollary 2.6. *The function* $\|\cdot\|^H_{Z^q_{\mathbb{N}\times\mathbb{N}}}$ *on* $Z^q_{\mathbb{N}\times\mathbb{N}}$ *defined by*

$$\|\theta\|^H_{Z^q_{\mathbb{N}\times\mathbb{N}}} := \sup_{\|x,y\|^H_p\neq 0} \frac{\left|\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} x_k y_j \theta_{kj}\right|}{\|x,y\|^H_p}$$

defines a norm on $Z^q_{\mathbb{N}\times\mathbb{N}}$. *Furthermore,* $\|\cdot\|^H_{Z^q_{\mathbb{N}\times\mathbb{N}}}$ *and* $\|\cdot\|^G_{Z^q_{\mathbb{N}\times\mathbb{N}}}$ *are equivalent norms on* $Z^q_{\mathbb{N}\times\mathbb{N}}$, *with*

$$2^{1/p-1}\|\theta\|^G_{Z^q_{\mathbb{N}\times\mathbb{N}}} \leqslant \|\theta\|^H_{Z^q_{\mathbb{N}\times\mathbb{N}}} \leqslant 2^{1/p}\|\theta\|^G_{Z^q_{\mathbb{N}\times\mathbb{N}}}$$

for all $\theta \in Z^q_{\mathbb{N}\times\mathbb{N}}$.

Corollary 2.7. *The* 2*-dual space of* $(\ell^p, \|\cdot,\cdot\|^H_p)$ *is identified by* $(Z^q_{\mathbb{N}\times\mathbb{N}}, \|\cdot\|^H_{Z^q_{\mathbb{N}\times\mathbb{N}}})$.

Remark. Here $\|\cdot\|^H_{Z^q_{\mathbb{N}\times\mathbb{N}}}$, $\|\cdot\|^G_{Z^q_{\mathbb{N}\times\mathbb{N}}}$, and $\|\cdot\|_{Y^q_{\mathbb{N}\times\mathbb{N}}}$ are three equivalent norms on $Z^q_{\mathbb{N}\times\mathbb{N}}$.

3. The n-dual spaces of ℓ^p

The results for the case $n = 2$ can be extended easily to the case $n \geqslant 2$. For $1 \leqslant p < \infty$ and $1/p + 1/q = 1$, we define $Y^q_{\mathbb{N}^n}$ to be the space of all (real) n-index sequence $\theta := (\theta_{k_1 \dots k_n})$ where

$$\|\theta\|_{Y^q_{\mathbb{N}^n}} := \sup_{\|a_1\|_p = \dots = \|a_{n-1}\|_p = 1} \left[\sum_{k_n=1}^{\infty} \left| \sum_{k_1,\dots,k_{n-1}=1}^{\infty} a_{1k_1} \dots a_{n-1,k_{n-1}} \theta_{k_1 \dots k_n} \right|^q \right]^{1/q} < \infty.$$

For $q = \infty$, an n-index sequence $\theta := (\theta_{k_1 \dots k_n})$ belongs to the space $Y^\infty_{\mathbb{N}^n}$ if

$$\|\theta\|_{Y^\infty_{\mathbb{N}^n}} := \sup_{\|a_1\|_1 = \dots = \|a_{n-1}\|_1 = 1} \sup_{k_n \in \mathbb{N}} \left| \sum_{k_1,\dots,k_{n-1}=1}^{\infty} a_{1k_1} \dots a_{n-1,k_{n-1}} \theta_{k_1 \dots k_n} \right| < \infty.$$

Here $\mathbb{N}^n := \mathbb{N} \times \dots \times \mathbb{N}$ (n factors). Note also that the inner sum above is a multiple sum.

We also define the generalization of $Z^q_{\mathbb{N}\times\mathbb{N}}$ spaces as follows. An n-index sequence $\theta := (\theta_{k_1 \dots k_n})$ belongs to the space $Z^q_{\mathbb{N}^n}$ if $\theta \in Y^q_{\mathbb{N}^n}$ and $\theta_{k_1 \dots k_n} = \mathrm{sgn}(\sigma)\theta_{\sigma(k_1)\dots\sigma(k_n)}$, for all $k_1, \dots, k_n \in \mathbb{N}$ and any permutation σ of $(k_1, \dots, k_n)$.

Analogously to the case $n = 2$, we have the following result for $n \geqslant 2$. (We leave the proof to the reader.)

Theorem 3.1. *The n-dual space of $(\ell^p, \|\cdot\|_p)$ is identified by $(Y^q_{\mathbb{N}^n}, \|\cdot\|_{Y^q_{\mathbb{N}^n}})$. Moreover, the mapping $f \mapsto \theta := (f(e_{k_1}, \dots, e_{k_n}))$ is an isometric bijection from the n-dual space of $(\ell^p, \|\cdot\|_p)$ to $(Y^q_{\mathbb{N}^n}, \|\cdot\|_{Y^q_{\mathbb{N}^n}})$.*

Using (2.3) and the following two inequalities from [6], [12]:

$$\|x_1, \dots, x_n\|^G_p \leqslant (n!)^{1/p} \|x_1, \dots, x_n\|^H_p$$

and

$$\|x_1, \dots, x_n\|^H_p \leqslant (n!)^{1-1/p} \|x_1\|_p \dots \|x_n\|_p,$$

we can prove the following theorem by using arguments similar the case $n = 2$.

Theorem 3.2. *A multilinear n-functional f is bounded on $(\ell^p, \|\cdot, \dots, \cdot\|^G_p)$ if and only if it is antisymmetric and bounded on $(\ell^p, \|\cdot\|_p)$. Furthermore, we have*

$$\frac{1}{n!}\|f\|_{n,1} \leqslant \|f\|^G_{n,n} \leqslant \|f\|_{n,1}$$

where $\|\cdot\|^G_{n,n}$ is the norm on the n-dual space of $(\ell^p, \|\cdot, \dots, \cdot\|^G_p)$.

From Theorems 3.1 and 3.2 we get the following result.

Corollary 3.3. *The n-dual space of $(\ell^p, \|\cdot, \dots, \cdot\|_p^G)$ is identified by $(Z^q_{\mathbb{N}^n}, \|\cdot\|^G_{Z^q_{\mathbb{N}^n}})$, where $\|\cdot\|^G_{Z^q_{\mathbb{N}^n}}$ is given by*

$$\|\theta\|^G_{Z^q_{\mathbb{N}^n}} := \sup_{\|x_1,\dots,x_n\|_p^G \neq 0} \frac{\left|\sum_{k_1,\dots,k_n=1}^{\infty} x_{1k_1} \dots x_{nk_n} \theta_{k_1\dots k_n}\right|}{\|x_1, \dots, x_n\|_p^G}.$$

Using (1.1), we also get the following theorem.

Corollary 3.4. *The n-dual space of $(\ell^p, \|\cdot, \dots, \cdot\|_p^H)$ is identified by $(Z^q_{\mathbb{N}^n}, \|\cdot\|^H_{Z^q_{\mathbb{N}^n}})$, where $\|\cdot\|^H_{Z^q_{\mathbb{N}^n}}$ is given by*

$$\|\theta\|^H_{Z^q_{\mathbb{N}^n}} := \sup_{\|x_1,\dots,x_n\|_p^H \neq 0} \frac{\left|\sum_{k_1,\dots,k_n=1}^{\infty} x_{1k_1} \dots x_{nk_n} \theta_{k_1\dots k_n}\right|}{\|x_1, \dots, x_n\|_p^H}.$$

4. Concluding remarks

In the theory of normed spaces, we know that the dual space of $(\ell^p, \|\cdot\|_p)$ is (identified by) $(\ell^q, \|\cdot\|_q)$, where $1 \leqslant p < \infty$ and $1/p + 1/q = 1$. Here we show that the n-dual space of $(\ell^p, \|\cdot\|_p)$ is identified by $(Y^q_{\mathbb{N}^n}, \|\cdot\|_{Y^q_{\mathbb{N}^n}})$. We see similarities between the two results. Similar relations also occur for the n-dual space of ℓ^p when ℓ^p is viewed as an n-normed space with Gähler's n-norm or Gunawan's n-norm. All these results are identical in the case where $n = 1$. For $n \geqslant 2$, however, we still have a question whether the norm $\|\cdot\|_{Y^q_{\mathbb{N}^n}}$ on $Y^q_{\mathbb{N}^n}$, as well as the norms $\|\cdot\|^H_{Z^q_{\mathbb{N}^n}}$ and $\|\cdot\|^G_{Z^q_{\mathbb{N}^n}}$ on $Z^q_{\mathbb{N}^n}$, can be reduced to

$$\|\theta\|^*_{Y^q_{\mathbb{N}^n}} := \left(\sum_{k_1,\dots,k_n=1}^{\infty} |\theta_{k_1\dots k_n}|^q \right)^{1/q}$$

and

$$\|\theta\|^*_{Z^q_{\mathbb{N}^n}} := \left(\sum_{k_1,\dots,k_n=1}^{\infty} |\theta_{k_1\dots k_n}|^q \right)^{1/q}.$$

One may easily check that if $\theta := (\theta_{k_1\dots k_n})$ satisfies

$$\left(\sum_{k_1,\dots,k_n=1}^{\infty} |\theta_{k_1\dots k_n}|^q \right)^{1/q} < \infty,$$

then $\|\theta\|_{Y^q_{\mathbb{N}^n}}$, $\|\theta\|^H_{Z^q_{\mathbb{N}^n}}$, and $\|\theta\|^G_{Z^q_{\mathbb{N}^n}}$ are all dominated by $\left(\sum_{k_1,\dots,k_n=1}^{\infty} |\theta_{k_1\dots k_n}|^q \right)^{1/q}$. We just do not know whether the converse is also true. See [1] for related problems.

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