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Vector Optimization Results
for $\ell$-Stable Data*

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(Received May 31, 2013)

Abstract

The aim of this paper is to summarize basic facts about $\ell$-stable at a point vector functions and existing results for certain vector constrained programming problem with $\ell$-stable data.

Key words: $\ell$-stable function, generalized second-order directional derivative, Dini derivative, weakly efficient minimizer, isolated minimizer

2010 Mathematics Subject Classification: 49K10, 49J52, 49J50, 90C29, 90C30

1 Introduction

In 2008 the concept of $\ell$-stable at a point scalar functions was introduced in [1] as a generalization of $C^{1,1}$ functions—functions with locally Lipschitz derivative. The main aim was to receive more general optimality conditions than for $C^{1,1}$ functions which were extensively studied previously (see e.g. [8]). In subsequent years the attention was devoted to deriving other properties of $\ell$-stable at a point functions and to extending $\ell$-stability to finite-dimensional spaces in connection with vector optimization ([2, 3, 4, 5, 6, 7]).

In this paper, I try to summarize the most important of this existing results. The basic facts concerning vector $\ell$-stability are recalled in the following section. Section 3 informs about second-order necessary and sufficient conditions for the following programming problem:

$$\begin{array}{ll}
\text{minimize} & f(x) \\
\text{subject to} & C \\
& \text{such that } g(x) \in -K, \\
\end{array} \quad (P)$$

*Supported by the student project PrF-2013-013 of the Palacky University.
where \( f : \mathbb{R}^N \to \mathbb{R}^M \) and \( g : \mathbb{R}^N \to \mathbb{R}^P \), \( M \in \mathbb{N} \), \( N \in \mathbb{N} \), \( P \in \mathbb{N} \), are given functions and \( C \subset \mathbb{R}^M \), \( K \subset \mathbb{R}^P \), are closed, convex, and pointed cones with non-empty interior (for definitions see e.g. [9]).

2 \( \ell \)-stability

First of all I recall several fundamental notations which are used in this paper. The Euclidean norm and the scalar product in \( \mathbb{R}^N \) are denoted by \( \| \cdot \| \) and \( \langle \cdot, \cdot \rangle \), the zero element and the unit sphere of \( \mathbb{R}^N \), i.e. the set \( \{ x \in \mathbb{R}^N; \| x \| = 1 \} \), by \( 0_{\mathbb{R}^N} \) and \( S_{\mathbb{R}^N} \), respectively. For a function \( f : \mathbb{R}^N \to \mathbb{R}^M \) and a point \( x \in \mathbb{R}^N \), the symbol \( f'(x) \) means the Fréchet derivative of \( f \) at \( x \).

The scalar \( \ell \)-stable at a point function was introduced in [1] using a lower directional derivative:

\[
\ell_f(x; h) = \liminf_{t \downarrow 0} \frac{f(x + th) - f(x)}{t}
\]

for \( f : \mathbb{R}^N \to \mathbb{R}^N \), \( x \in \mathbb{R}^N \), \( h \in \mathbb{R}^N \).

**Definition 2.1** We say that a function \( f : \mathbb{R}^N \to \mathbb{R} \) is \( \ell \)-stable at \( x_0 \in \mathbb{R}^N \) if there are a neighbourhood \( U \) of \( x_0 \) and a constant \( K > 0 \) such that

\[
|f^\ell(x; h) - f^\ell(x_0; h)| \leq K\|x - x_0\|, \quad \forall x \in U, \forall h \in S_{\mathbb{R}^N}.
\]

Following example presents a scalar function which is \( \ell \)-stable at a point, but not differentiable on any neighbourhood of this point.

**Example 2.1** Consider the functions \( f : \mathbb{R}^2 \to \mathbb{R} \),

\[
f(x_1, x_2) = \int_0^{[x_1]} \varphi(u) \, du,
\]

where function \( \varphi : \mathbb{R}_0^+ \to \mathbb{R} \) is defined as follows:

\[
\varphi(u) = \begin{cases} 
1 & \text{if } u > 1, \\
2u - \frac{1}{n+1} & \text{if } u \in \left( \frac{1}{n+1}, \frac{1}{n} \right], \ n \in \mathbb{N}, \\
0 & \text{if } u = 0.
\end{cases}
\]

The first-order directional derivatives of function \( f \) at points \( a_n = (\frac{1}{n}, 0) \), \( n \in \mathbb{N} \), \( n > 1 \), in directions \( \bar{d} = (1, 0) \), \( \hat{d} = (-1, 0) \) are

\[
f'(a_n; \bar{d}) = \frac{1}{n}, \quad f'(a_n; \hat{d}) = -\frac{n + 2}{n(n + 1)}.
\]

Hence, \( f \) is not differentiable on any neighbourhood of point \( x_0 \). For every \( v = (v_1, v_2) \in S_{\mathbb{R}^2} \) and for every \( y = (y_1, y_2) \in \mathbb{R}^2 \), \( \|y\| < 1 \), \( y \neq (0, 0) \), it holds:
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Figure 1: Graph of function $\varphi$ on $[0,1]$

- if $y_1 \in \left( \frac{1}{n+1}, \frac{1}{n} \right)$, $n \in \mathbb{N}$

$$\left| \liminf_{t \downarrow 0} \frac{f(y + tv) - f(y)}{t} \right| = \left| \lim_{t \downarrow 0} \frac{1}{t} \left( \int_0^{y_1+tv_1} \varphi(u) \, du - \int_0^{y_1} \varphi(u) \, du \right) \right|$$

$$= \left| \lim_{t \downarrow 0} \frac{1}{t} \left[ u^2 - \frac{u}{n+1} \right]^{y_1+tv_1}_{y_1} \right| = |v_1| \left( 2y_1 - \frac{1}{n+1} \right);$$

- if $y_1 \in \left( -\frac{1}{n}, -\frac{1}{n+1} \right)$, $n \in \mathbb{N}$

$$\left| \liminf_{t \downarrow 0} \frac{f(y + tv) - f(y)}{t} \right| = \left| \lim_{t \downarrow 0} \frac{1}{t} \left( \int_0^{y_1+tv_1} \varphi(u) \, du - \int_0^{y_1} \varphi(u) \, du \right) \right|$$

$$= \left| \lim_{t \downarrow 0} \frac{1}{t} \left[ u^2 - \frac{u}{n+1} \right]^{-y_1-tv_1}_{-y_1} \right| = |v_1| \left( 2|y_1| - \frac{1}{n+1} \right);$$

- if $y_1 = \frac{1}{n}$, $n \in \mathbb{N}$, $n > 1$, $v_1 \geq 0$

$$\left| \liminf_{t \downarrow 0} \frac{f(y + tv) - f(y)}{t} \right| = \left| \lim_{t \downarrow 0} \frac{1}{t} \left( \int_0^{y_1+tv_1} \varphi(u) \, du - \int_0^{y_1} \varphi(u) \, du \right) \right|$$

$$= \left| \lim_{t \downarrow 0} \frac{1}{t} \left[ u^2 - \frac{u}{n} \right]^{y_1+tv_1}_{y_1} \right| = v_1 y_1;$$
• if \( y_1 = \frac{1}{n} \), \( n \in \mathbb{N} \), \( n > 1 \), \( v_1 < 0 \)

\[
\liminf_{t \to 0} \frac{f(y + tv) - f(y)}{t} = \liminf_{t \to 0} \frac{1}{t} \left( \int_0^{y_1 + tv} \varphi(u) \, du - \int_0^{y_1} \varphi(u) \, du \right)
= \lim_{t \to 0} \frac{1}{t} \left[ u^2 - \frac{u}{n+1} \right]^{y_1+tv}_{y_1} = |v_1| \left( 2y_1 - \frac{1}{n+1} \right);
\]

• if \( y_1 = -\frac{1}{n} \), \( n \in \mathbb{N} \), \( n > 1 \), \( v_1 > 0 \)

\[
\liminf_{t \to 0} \frac{f(y + tv) - f(y)}{t} = \liminf_{t \to 0} \frac{1}{t} \left( \int_0^{y_1 + tv} \varphi(u) \, du - \int_0^{y_1} \varphi(u) \, du \right)
= \lim_{t \to 0} \frac{1}{t} \left[ u^2 - \frac{u}{n+1} \right]^{-y_1-tv}_{-y_1} = |v_1| \left( 2|y_1| - \frac{1}{n+1} \right);
\]

• if \( y_1 = -\frac{1}{n} \), \( n \in \mathbb{N} \), \( n > 1 \), \( v_1 \leq 0 \)

\[
\liminf_{t \to 0} \frac{f(y + tv) - f(y)}{t} = \liminf_{t \to 0} \frac{1}{t} \left( \int_0^{y_1 + tv} \varphi(u) \, du - \int_0^{y_1} \varphi(u) \, du \right)
= \lim_{t \to 0} \frac{1}{t} \left[ u^2 - \frac{u}{n+1} \right]^{-y_1-tv}_{-y_1} = v_1 y_1;
\]

• if \( y_1 = 0 \)

\[
\liminf_{t \to 0} \frac{f(y + tv) - f(y)}{t} = 0 \text{ because}
\]

\[
0 \leq \left| \lim_{t \to 0} \frac{f(y + tv) - f(y)}{t} \right| = \left| \lim_{t \to 0} \frac{f(tv)}{t} \right| \leq \left| \lim_{t \to 0} \frac{t^2 v_1^2}{t} \right| = 0
\]

and it also implies \( f'(x_0) = (0, 0) \).

Overall

\[
\left| \liminf_{t \to 0} \frac{f(y + tv) - f(y)}{t} \right| = \begin{cases} 
|v_1| \left( 2|y_1| - \frac{1}{n+1} \right) & \text{if } |y_1| \in \left( \frac{1}{n+1}, \frac{1}{n} \right), \\
v_1 y_1 & \text{if } |y_1| = \frac{1}{n}, \ v_1 y_1 \geq 0, \\
|v_1| \left( 2|y_1| - \frac{1}{n+1} \right) & \text{if } |y_1| = \frac{1}{n}, \ v_1 y_1 < 0, \\
0 & \text{if } y_1 = 0.
\end{cases}
\]

The function \( f \) is \( \ell \)-stable at \( x_0 \) because:

\[
|f^\ell(x_0; v) - f^\ell(y; v)| = \left| \liminf_{t \to 0} \frac{f(y + tv) - f(y)}{t} \right| \leq 2\|y\|,
\]

\( \forall y \in \mathbb{R}^2, \|y\| < 1, \forall v \in S_{\mathbb{R}^2}. \)

There are two approaches how to generalize the concept of \( \ell \)-stability for vector functions. The first one introduced in [2] is stated in Definition 2.3. The
second one was introduced in [6] and since their equivalence was shown in [5], I mention it in Theorem 2.1 as a characterization of \( \ell \)-stability.

In the definition of \( \ell \)-stable at a point vector function, this type of lower directional derivative is needed:

\[
f_\ell^x(x; h) = \liminf_{t \downarrow 0} \frac{\langle \xi, f(x + th) - f(x) \rangle}{t}
\]

for \( f: \mathbb{R}^N \to \mathbb{R}^M \), \( x \in \mathbb{R}^N \), \( h \in \mathbb{R}^N \) and \( \xi \in \mathbb{R}^M \).

**Definition 2.2** For arbitrary cone \( C \subseteq \mathbb{R}^N \), we define a positive polar cone \( C^* \) and a set \( \Gamma_C \):

\[
C^* := \{ \xi \in \mathbb{R}^N; \langle \xi, y \rangle \geq 0, \ y \in C \}, \quad \Gamma_C := C^* \cap S_{\mathbb{R}^N}.
\]

**Definition 2.3** Let \( f: \mathbb{R}^N \to \mathbb{R}^M \) be a function and \( C \subseteq \mathbb{R}^M \) be a closed, convex and pointed cone with non-empty interior. We say that \( f \) is \( \ell \)-stable at \( x_0 \in \mathbb{R}^N \) with respect to \( C \) if there is a neighbourhood \( U \) of \( x_0 \) and a constant \( K > 0 \) such that

\[
|f_\ell^x(y; h) - f_\ell^x(x_0; h)| \leq K\|y - x_0\|, \quad \forall y \in U, \forall h \in S_{\mathbb{R}^N}, \forall \xi \in \Gamma_C.
\]

In [4], it was proved that if any function is \( \ell \)-stable at a point with respect to some closed, convex and pointed cone, it must be \( \ell \)-stable at this point with respect to arbitrary closed, convex and pointed cone. Therefore, we talk in the following text only about \( \ell \)-stability at a point.

**Theorem 2.1** The function \( f: \mathbb{R}^N \to \mathbb{R}^M \) is \( \ell \)-stable at \( x_0 \in \mathbb{R}^N \) if and only if for any \( \xi \in \mathbb{R}^M \) the scalar function

\[
f_\ell(\cdot) = \langle \xi, f(\cdot) \rangle
\]

is \( \ell \)-stable at \( x_0 \).

The next theorems provide characterization of \( \ell \)-stability.

**Theorem 2.2** [6, Theorem 3.3] The function \( f: \mathbb{R}^N \to \mathbb{R}^M \) is \( \ell \)-stable at \( x_0 \in \mathbb{R}^N \) if and only if there exist a neighbourhood \( U \) of \( x_0 \) and a constant \( K > 0 \) such that it holds that

\[
|f_\ell^x(x; h) - f_\ell^x(x_0; h)| \leq K\|\xi\|\|x - x_0\|, \quad \forall x \in U, \forall h \in S_{\mathbb{R}^N}, \forall \xi \in \mathbb{R}^M.
\]

**Theorem 2.3** [6, Theorem 3.4] The function \( f: \mathbb{R}^N \to \mathbb{R}^M \) is \( \ell \)-stable at \( x_0 \in \mathbb{R}^N \) if and only if the Fréchet derivative \( f'(x_0) \) exists, there exists a neighbourhood \( U \) of \( x_0 \) such that \( f \) is Lipschitz on \( U \), and there is a \( K > 0 \) such that it holds that

\[
\|f'(x) - f'(x_0)\| \leq K\|x - x_0\|, \quad a.e. \ x \in U.
\]
At the end of this section, I mention other important properties of $\ell$-stable at a point functions.

**Definition 2.4** We say that a function $f : \mathbb{R}^N \to \mathbb{R}^M$ is *strictly differentiable* at $x \in \mathbb{R}^N$ if there exists a continuous linear operator $A : \mathbb{R}^N \to \mathbb{R}^M$ such that

$$
\lim_{y \to x, t \downarrow 0} \frac{f(y + th) - f(y)}{t} = Ah, \quad \forall h \in S_{\mathbb{R}^N},
$$

and this limit is uniform for $h \in S_{\mathbb{R}^N}$.

**Theorem 2.4** [2, Proposition 2.2] Let a function $f : \mathbb{R}^N \to \mathbb{R}^M$ be $\ell$-stable at $x_0 \in \mathbb{R}^N$. Then $f$ is strictly differentiable at $x_0$.

Theorem 2.4 implies that every function which is $\ell$-stable at some point is continuous near this point and Fréchet differentiable at this point.

**Theorem 2.5** [4, Proposition 1] Let $f = (f_1, f_2, \ldots, f_M) : \mathbb{R}^N \to \mathbb{R}^M$ be a function and $x_0 \in \mathbb{R}^N$. Then $f$ is $\ell$-stable at $x_0$ if and only if the function $f_i$ is $\ell$-stable at $x_0$ for every $i \in \{1, 2, \ldots, M\}$.

**Theorem 2.6** [4, Theorem 1] Let a function $f : \mathbb{R}^M \to \mathbb{R}^N$ be $\ell$-stable at $x_0 \in \mathbb{R}^M$ and let a function $g : \mathbb{R}^N \to \mathbb{R}^P$ be $\ell$-stable at $y_0 = f(x_0) \in \mathbb{R}^N$. Then the composition $g \circ f$ is $\ell$-stable at $x_0$.

### 3 Vector optimization results

In this section, I consider constrained optimization problem (P). Firstly I recall fundamental definitions of vector optimization and second-order directional derivatives which are used in second-order optimality conditions.

**Definition 3.1** Let us consider the problem (P) and define

a) a set of feasible points $\Phi$:

$$
\Phi = \{x \in \mathbb{R}^N; \ g(x) \in -K\};
$$

b) a cone $K(x)$, $x \in -K$:

$$
K(x) = \{\gamma(z + x); \gamma \geq 0, z \in K\}.
$$

Now we introduce two types of minimizers for problem (P).

**Definition 3.2** A feasible point $x_0$ is said

a) a *local weakly efficient point for problem (P)* if there exists a neighbourhood $U$ of $x_0$ such that

$$
(f(U \cap \Phi) - f(x_0)) \cap (- \text{int } C) = \emptyset.
$$
b) an isolated local minimizer of second-order for problem (P), if there exist a neighbourhood $U$ of $x_0$ and a constant $A > 0$ such that

$$
\sup_{\xi \in \mathcal{C}} \langle \xi, f(x) - f(x_0) \rangle \geq A\|x - x_0\|^2, \quad \forall x \in U \cap \Phi.
$$

**Definition 3.3** Let a function $f: \mathbb{R}^N \to \mathbb{R}^M$ be Fréchet differentiable at point $x \in \mathbb{R}^N$. The second-order Dini directional derivative $d_2f(x; h)$ of $f$ at $x \in \mathbb{R}^N$ in the direction $h \in \mathbb{R}^N$ is defined as follows:

$$
d_2f(x; h) = \left\{ y \in \mathbb{R}^M; \exists \{t_n\}_{n=1}^{+\infty} \subset \mathbb{R}^+; \lim_{n \to +\infty} t_n = 0, \right. \\
y = \lim_{n \to +\infty} \frac{f(x + t_nh) - f(x) - t_nf'(x)h}{t_n^2/2} \right\}.
$$

The second-order Hadamard directional derivative $D_2f(x; h)$ of $f$ at $x \in \mathbb{R}^N$ in the direction $h \in \mathbb{R}^N$ is defined as follows:

$$
D_2f(x; h) = \left\{ y \in \mathbb{R}^M; \exists \{t_n\}_{n=1}^{+\infty} \subset \mathbb{R}^+; \exists \{h_n\}_{n=1}^{+\infty} \subset \mathbb{R}^N; \lim_{n \to +\infty} t_n = 0, \right. \\
\lim_{n \to +\infty} h_n = h, \ y = \lim_{n \to +\infty} \frac{f(x + t_nh) - f(x) - t_nf'(x)h}{t_n^2/2} \right\}.
$$

Problem (P) was deeply studied for at least continuously differentiable functions. Khanh and Tuan achieved following results using concept of calm at a point function.

**Definition 3.4** A function $f: \mathbb{R}^N \to \mathbb{R}^M$ is called calm at $x_0 \in \mathbb{R}^N$ if there is a neighbourhood $U$ of $x_0$ and a constant $K > 0$ such that

$$
\|f(x) - f(x_0)\| \leq K\|x - x_0\|, \quad \forall x \in U.
$$

**Theorem 3.1** [10, Theorem 4.1] Let functions $f: \mathbb{R}^N \to \mathbb{R}^M$ and $g: \mathbb{R}^N \to \mathbb{R}^P$ be continuously differentiable at $x_0 \in \mathbb{R}^N$. If $x_0$ is a local weakly efficient point of problem (P), then

(i) there exists $(c^*, k^*) \in C^* \times K^*(g(x_0)) \setminus \{(0_{\mathbb{R}_M}, 0_{\mathbb{R}_P})\}$ such that

$$
c^* \circ f'(x_0) + k^* \circ g'(x_0) = 0_{\mathbb{R}_N}; \quad (3.1)
$$

(ii) for $u \in \mathbb{R}^N$ if $(f, g)'(x_0)u \in -(C \times K(g(x_0))) \cap \text{int} (C \times K(g(x_0))))$, then for every $(y_0, z_0) \in D_2(f, g)(x_0; u)$ there exists $(c^*, k^*) \in C^* \times K^*(g(x_0)) \setminus \{(0_{\mathbb{R}_M}, 0_{\mathbb{R}_P})\}$ such that (3.1) is true and

$$
\langle c^*, y_0 \rangle + \langle k^*, z_0 \rangle \geq 0.
$$

**Theorem 3.2** [10, Theorem 4.2] Let functions $f: \mathbb{R}^N \to \mathbb{R}^M$ and $g: \mathbb{R}^N \to \mathbb{R}^P$ be continuously Fréchet differentiable around $x_0 \in \mathbb{R}^N$ with $f'$ and $g'$ being calm at $x_0$ which is feasible point of problem (P). Then, each of the following conditions is sufficient for $x_0$ to be an isolated local minimizer of second-order for problem (P).
(i) For every \( u \in \mathbb{R}^N \) satisfying \( \|u\| = 1 \) there exists \( (c^*, k^*) \in C^* \times K^*(g(x_0)) \) such that
\[
\langle c^*, f'(x_0)u \rangle + \langle k^*, g'(x_0)u \rangle > 0.
\]

(ii) For every \( u \in \mathbb{R}^N \) satisfying \( \|u\| = 1 \), one has

\[
\begin{align*}
(a) & \quad (f'(x_0)u, g'(x_0)u) \in -(C \times K(g(x_0))) \setminus (C \times K(g(x_0))), \\
(b) & \quad \text{for every } (y_0, z_0) \in d_2(f, g)(x_0; u) \text{ there exists } (c^*, k^*) \in C^* \times K^*(g(x_0)) \text{ such that (3.1) is true and} \\
& \quad \langle c^*, y_0 \rangle + \langle k^*, z_0 \rangle > 0. 
\end{align*}
\]

Theorems 3.1 and 3.2 was strengthened for strictly differentiable and \( \ell \)-stable functions, respectively, by Bednářík and Pastor.

**Theorem 3.3** [3, Theorem 3.1] Let functions \( f: \mathbb{R}^N \to \mathbb{R}^M \) and \( g: \mathbb{R}^N \to \mathbb{R}^P \) be strictly differentiable at \( x_0 \in \mathbb{R}^N \). If \( x_0 \) is a local weakly efficient point of problem \((P)\), then

(i) there exists \( (c^*, k^*) \in C^* \times K^*(g(x_0)) \setminus \{(0_{\mathbb{R}^M}, 0_{\mathbb{R}^P})\} \) such that
\[
c^* \circ f'(x_0) + k^* \circ g'(x_0) = 0_{\mathbb{R}^N}; \tag{3.3}
\]

(ii) for \( u \in \mathbb{R}^N \) if \( (f, g)'(x_0)u \in -(C \times K(g(x_0))) \setminus (C \times K(g(x_0))) \), then for every \( (y_0, z_0) \in D_2(f, g)(x_0; u) \) there exists \( (c^*, k^*) \in C^* \times K^*(g(x_0)) \setminus \{(0_{\mathbb{R}^M}, 0_{\mathbb{R}^P})\} \) such that (3.3) is true and
\[
\langle c^*, y_0 \rangle + \langle k^*, z_0 \rangle \geq 0.
\]

**Theorem 3.4** [3, Theorem 4.1], [6, proof of Thm 5.1] Let functions \( f: \mathbb{R}^N \to \mathbb{R}^M \) and \( g: \mathbb{R}^N \to \mathbb{R}^P \) be \( \ell \)-stable at feasible point \( x_0 \in \mathbb{R}^N \). We suppose that for every \( u \in S_{\mathbb{R}^N} \) one of the following two conditions is satisfied.

(i) There exists \( (c^*, k^*) \in C^* \times K^*(g(x_0)) \) such that
\[
\langle c^*, f'(x_0)u \rangle + \langle k^*, g'(x_0)u \rangle > 0.
\]

(ii)

\[
\begin{align*}
(a) & \quad (f'(x_0)u, g'(x_0)u) \in -(C \times K(g(x_0))) \setminus (C \times K(g(x_0))), \\
b) & \quad \text{for every } (y_0, z_0) \in d_2(f, g)(x_0; u) \text{ there exists } (c^*, k^*) \in C^* \times K^*(g(x_0)) \text{ such that} \\
& \quad c^* \circ f'(x_0) + k^* \circ g'(x_0) = 0_{\mathbb{R}^N}, \tag{3.4}

& \quad \langle c^*, y_0 \rangle + \langle k^*, z_0 \rangle > 0. \tag{3.5}
\end{align*}
\]

Then \( x_0 \) is an isolated local minimizer of second-order for problem \((P)\).
In [6, Theorem 5.1], the condition (3.4) from Theorem 3.4 is substituted by
\[ \langle c^*, f'(x_0)u \rangle + \langle k^*, g'(x_0)u \rangle = 0. \] (3.6)
In [5], the incorrectness of using condition (3.6) was showed on following example.

**Example 3.1** Let us consider the problem (P) with the functions \( f: \mathbb{R}^2 \to \mathbb{R}^2 \), and \( g: \mathbb{R}^2 \to \mathbb{R} \),
\[ f(x_1, x_2) = (x_1 + x_2^2, x_1^2), \quad g(x_1, x_2) = x_1 x_2, \]
\( C = \mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2; x_1 \geq 0, x_2 \geq 0\}, \quad K = [0, +\infty). \)
It can be showed that these functions fulfill at point \( x_0 = (0, 0) \) the assumptions of Theorem 3.4 where (3.4) is replaced by (3.6) but \( x_0 \) is not an isolated local minimizer of second-order for problem (P). The condition (i) is satisfied for \( u = (u_1, u_2) \in S_{\mathbb{R}^2}, u_1 > 0 \), choosing \( c^* = (1, 0) \in C^* \) and arbitrary \( k^* \in K^*(g(x_0)) \):
\[ \langle c^*, f'(x_0)u \rangle + \langle k^*, g'(x_0)u \rangle = \langle (1, 0), (u_1, 0) \rangle + 0 = u_1 > 0. \]
The changed condition (ii) is satisfied for \( u = (u_1, u_2) \in S_{\mathbb{R}^2}, u_1 = 0 \), choosing \( c^* = (1, 0), k^* = 0 \) and for \( u_1 < 0 \), choosing \( c^* = (0, 1), k^* = 0 \):
\[ (f'(x_0)u, g'(x_0)u) = (u_1, 0, 0) \in -(C \times K(g(x_0))) \cap \text{ int } (C \times K(g(x_0))), \]
\[ \langle c^*, f'(x_0)u \rangle + \langle k^*, g'(x_0)u \rangle = \begin{cases} 
\langle (1, 0), (0, 0) \rangle + 0 = 0, & \text{if } u_1 = 0, \\
\langle (0, 1), (u_1, 0) \rangle + 0 = 0, & \text{if } u_1 < 0,
\end{cases} \]
\[ \langle c^*, y_0 \rangle + \langle k^*, z_0 \rangle = \begin{cases} 
\langle (1, 0), (2u_2, 0) \rangle + 0 = 2u_2 > 0, & \text{if } u_1 = 0, \\
\langle (0, 1), (2u_2, 2u_1^2) \rangle + 0 = 2u_1^2 > 0, & \text{if } u_1 < 0.
\end{cases} \]
However, \( x_0 \) is not an isolated local minimizer of second-order, since the sequence of feasible points \( \left\{ \left( -\frac{1}{k^2}, \sqrt{\frac{1}{k^2}} \right) \right\}_{k=1}^{+\infty} \) converges to \( x_0 \) for \( k \to +\infty \), but for every \( A > 0 \), it can be found \( k_0 \in \mathbb{N} \) such that it holds for every \( k \in \mathbb{N}, \ k \geq k_0: \)
\[ \sup_{\xi \in \Gamma} \langle \xi, f(-\frac{1}{k^2}, \sqrt{\frac{1}{k^2}}) \rangle = \langle (0, 1), (0, \frac{1}{k^2}) \rangle = \frac{1}{k^2} < A\|(-\frac{1}{k^2}, \sqrt{\frac{1}{k^2}})\|^2 = \frac{A}{k^2}(1 + k). \]
Thus \( x_0 \) cannot be an isolated local minimizer of second-order for problem (P).

### 4 Conclusion

This paper informed about \( \ell \)-stable at a point vector functions, their characterizations and properties and about their applications in second-order optimality conditions for constrained vector optimization problem. I tried to sum up the most important results to provide insight into these issues. The research of \( \ell \)-stability and its application continues. Currently it is focused on \( \ell \)-stability in infinite-dimensional normed linear spaces.
References


