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ON THE KOLÁŘ CONNECTION

Włodzimierz M. Mikulski

To the memory of my father Jan Mikulski on his 100th birthday

Abstract. Let \( Y \to M \) be a fibred manifold with \( m \)-dimensional base and \( n \)-dimensional fibres and \( E \to M \) be a vector bundle with the same base \( M \) and with \( n \)-dimensional fibres (the same \( n \)). If \( m \geq 2 \) and \( n \geq 3 \), we classify all canonical constructions of a classical linear connection \( A(\Gamma, \Lambda, \Phi, \Delta) \) on \( Y \) from a system \((\Gamma, \Lambda, \Phi, \Delta)\) consisting of a general connection \( \Gamma \) on \( Y \to M \), a torsion free classical linear connection \( \Lambda \) on \( M \), a vertical parallelism \( \Phi : Y \times M E \to VY \) on \( Y \) and a linear connection \( \Delta \) on \( E \to M \). An example of such \( A(\Gamma, \Lambda, \Phi, \Delta) \) is the connection \((\Gamma, \Lambda, \Phi, \Delta)\) by I. Kolář.

0. Introduction

A general connection on a fibred manifold \( Y \to M \) is a section \( \Gamma : Y \to J^1Y \) of the first jet prolongation \( J^1Y \) of \( Y \to M \). Equivalently, \( \Gamma : Y \times_M TM \to TY \) is a lifting map or a projection tensor field \( \Gamma : TY \to TY \) or it is a decomposition \( TY = VY \oplus H^1 \), etc. If \( Y \) is a vector bundle and \( \Gamma : Y \to J^1Y \) is a vector bundle map (over \( \text{id}_M \)), then \( \Gamma \) is called a linear connection on \( Y \to M \). A linear connection on \( Y = TM \to M \) (the tangent bundle of \( M \)) is called a classical linear connection on \( M \). There are several equivalent definitions of classical linear connection on \( M \) (a differentiation \( X(M) \times X(M) \to X(M) \), a right invariant connection \( PM \to J^1PM \) on the linear frame bundle \( PM \) of \( M \), a system of Christoffel symbols, etc.). A classical linear connection \( \nabla \) is torsion free if its torsion tensor \( T(X_1, X_2) = \nabla_{X_1}X_2 - \nabla_{X_2}X_1 - [X_1, X_2] \) is equal to 0.

If \( N \) is a manifold and \( V \) is a vector bundle, \( \dim(N) = \dim(V) \), a parallelism on \( N \), is a fibred diffeomorphism \( P : N \times V \to TN \) over \( \text{id}_N \) such that for any \( z \in N \) the map \( P_z : V \to T_z N, P_z(v) = P(z, v) \), is linear.

If \( Y \to M \) is a fibred manifold and \( E \to M \) is a vector bundle such that \( \dim Y_x = \dim E_x, x \in M \), a vertical parallelism on \( Y \to M \) is a vector bundle isomorphism \( \Phi : Y \times_M E \to VY \), i.e. it is a system of parallelism \( \Phi_x : Y_x \times E_x \to TY_x \) on \( Y_x \) for any \( x \in M \).

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We take a vector field \( \mathbf{v} \) on a manifold \( M \), and consider the induced vertical parallelism \( \Phi : Y \rightarrow VY \) on \( Y \). Suppose \( \Phi \) is a torsion free classical linear connection on \( Y \). We decompose \( Z \in T_yY \) into the horizontal part \( \tau Z = \Phi(y, Z) \), and the vertical part \( \nu Z = \Phi(y, Z_1) \), where \( Z_1 \in E_x \). We take a vector field \( X \) on \( M \) and consider its lift \( \Gamma X \) on \( Y \). We construct a section \( s \) of \( E \) such that \( j^1_x s = \Delta(Z_1) \). For every \( Z \in T_yY \) we define
\[
\psi(Z) = j^1_y(\Gamma X + \varphi(s)).
\]

Here \( \varphi(s) : Y \rightarrow VY \) is defined by \( \varphi(s)(y) = \Phi(y, s(p(y))) \).

The above construction is a generalization of the construction \( H \) of a classical linear connection \( H(D, \Lambda) \) on \( E \) from a linear connection \( D \) in a vector bundle \( E \rightarrow M \) by means of a classical linear connection \( \Lambda \) on \( M \) presented by J. Gancarzewicz [2]. It is also a generalization of a construction \( N \) of a classical linear connection \( N(\Gamma, \Lambda) \) on \( P \) from a principal (right invariant) connection on a principal bundle \( P \rightarrow M \) by means of a classical linear connection \( \Lambda \) considered in [5, p. 415].

In the present paper we study the problem how to construct a classical linear connection \( A(\Gamma, \Lambda, \Delta, \Phi) \) on \( Y \) from a system \( (\Gamma, \Lambda, \Phi, \Delta) \) consisting of a general connection \( \Gamma \) on \( Y \rightarrow M \), a torsion free classical linear connection \( \Lambda \) on \( M \), a vertical parallelism \( \Phi : Y \times_M E \rightarrow VY \) and a linear connection \( \Delta \) on \( E \rightarrow M \).

In Section 2 modifying the torsion tensor field \( Tor(\Gamma, \Lambda, \Phi, \Delta) \) of the Kolář connection \( (\Gamma, \Lambda, \Phi, \Delta) \), the torsion field \( \tau \Phi \) on \( Y \rightarrow \bigwedge^2 V^*Y \otimes VY \) of \( \Phi \) and the covariant differential \( D_{(\Gamma, \Delta)} \Phi : Y \times_M E \rightarrow VY \otimes T^*M \), we construct tensor fields \( \tau_i(\Gamma, \Lambda, \Phi, \Delta) \) of type \( T^* \otimes T^* \otimes T \) on \( Y \) canonically depending on \( (\Gamma, \Lambda, \Phi, \Delta) \), \( i = 1, \ldots, 12 \).

The main result of the present paper can be written in the form of the following theorem.

**Theorem A.** If \( m \geq 2 \) and \( n \geq 3 \), any canonical construction \( A \) in question is of the form
\[
A(\Gamma, \Lambda, \Phi, \Delta) = (\Gamma, \Lambda, \Phi, \Delta) + \sum_{i=1}^{12} \lambda_i \tau_i(\Gamma, \Lambda, \Phi, \Delta)
\]
for some (uniquely determined by \( A \)) real numbers \( \lambda_1, \ldots, \lambda_{12} \).

Classifications of constructions on connections has been studied in many papers, e.g. [2], [1], e.t.c.

All manifolds considered in the paper are assumed to be Hausdorff, second countable, without boundary, finite dimensional and smooth (of class \( C^\infty \)). Maps between manifolds are assumed to be smooth (infinitely differentiable).
1. Natural operators

Let \( \mathcal{FM}_{m,n} \) be the category of fibred manifolds with \( m \)-dimensional bases and \( n \)-dimensional fibres and their fibred (local) diffeomorphisms. Let \( \mathcal{VB}_{m,n} \) be the category of vector bundles with \( m \)-dimensional bases and \( n \)-dimensional fibres and their (local) vector bundle isomorphisms.

**Definition 1.** A (gauge) \( \mathcal{FM}_{m,n} \times \mathcal{VB}_{m,n} \)-natural operator \( A \) sending systems \((\Gamma, \Lambda, \Phi, \Delta)\) consisting of general connections on fibred manifolds \( Y \to M \), torsion free classical linear connections \( \Lambda \) on \( M \), vertical parallelisms \( \Phi: Y \times_M E \to VY \) on \( Y \) and linear connections \( \Delta \) on vector bundles \( E \to M \) into classical linear connections \( A_{Y,E}(\Gamma, \Lambda, \Phi, \Delta) \) on \( Y \) is an \( \mathcal{FM}_{m,n} \times \mathcal{VB}_{m,n} \)-invariant system of regular operators

\[
A_{Y,E}: \text{Con}(Y) \times \text{Con}_{\text{clas}}(M) \times \text{Par}(Y \times_M E) \times \text{Con}_{\text{lin}}(E) \to \text{Con}_{\text{clas}}(Y)
\]

for any pair \((Y, E)\) consisting of a \( \mathcal{FM}_{m,n} \)-object \( Y = (p_Y: Y \to M) \) and a \( \mathcal{VB}_{m,n} \)-object \( E = (p_E: E \to M) \) (the same base \( M \)), where \( \text{Con}(Y) \) is the set of general connections \( \Gamma \) on \( p_Y: Y \to M \), \( \text{Con}_{\text{clas}}(M) \) is the set of torsion free classical linear connections \( \Lambda \) on \( M \), \( \text{Par}(Y \times_M E) \) is the set of vertical parallelisms \( \Phi: Y \times_M E \to VY \) on \( Y \), \( \text{Con}_{\text{lin}}(E) \) is the set of linear connections \( \Delta \) on \( p_E: E \to M \) and \( \text{Con}_{\text{clas}}(Y) \) is the set of classical linear connections on \( Y \).

**Remark 1.** The invariance of \( A \) means that if \((\Gamma, \Lambda, \Phi, \Delta) \in \text{Con}(Y) \times \text{Con}_{\text{clas}}(M) \times \text{Par}(Y \times_M E) \times \text{Con}_{\text{lin}}(E)\) is \((f, g)\)-related to \((\Gamma_1, \Lambda_1, \Phi_1, \Delta_1) \in \text{Con}(Y_1) \times \text{Con}_{\text{clas}}(M_1) \times \text{Par}(Y_1 \times_{M_1} E_1) \times \text{Con}_{\text{lin}}(E_1)\), where \( f: Y \to Y_1 \) is a \( \mathcal{FM}_{m,n} \)-map covering \( f: M \to M_1 \) and \( g: E \to E_1 \) is a \( \mathcal{VB}_{m,n} \)-map covering also \( f: M \to M_1 \), then \( A_{Y,E}(\Gamma, \Lambda, \Phi, \Delta) \) and \( A_{Y_1,E_1}(\Gamma_1, \Lambda_1, \Phi_1, \Delta_1) \) are \((f, g)\)-related. A tuple \((\Gamma, \Lambda, \Phi, \Delta)\) is \((f, g)\)-related to \((\Gamma_1, \Lambda_1, \Phi_1, \Delta_1)\) if \( \Gamma \) is \( f \)-related to \( \Gamma_1 \), \( \Lambda \) is \( f \)-related to \( \Lambda_1 \), \( \Phi \) is \((f, g)\)-related to \( \Phi_1 \) and \( \Delta \) is \( g \)-related to \( \Delta_1 \). In particular, \( \Phi \) is \((f, g)\)-related to \( \Phi_1 \) if \( Vf \circ \Phi = \Phi_1 \circ (f \times g) \).

**Remark 2.** The regularity of \( A \) means that \( A_{Y,E} \) transforms smoothly parametrized families into smoothly parametrized families.

For simplicity, we will omit the indexes \( Y \) and \( E \) on \( A_{Y,E} \).

**Remark 3.** One can show standardly, that if \( \text{germ}_y(\Gamma_1) = \text{germ}_y(\Gamma) \), \( \text{germ}_x(\Lambda_1) = \text{germ}_x(\Lambda) \), \( \text{germ}_y(\Phi_1) = \text{germ}_y(\Phi) \), \( \text{germ}_x(\Delta_1) = \text{germ}_x(\Delta) \), \( y \in Y_x \), \( x \in M \), then \( A(\Gamma_1, \Lambda_1, \Phi_1, \Delta_1)(y) = A(\Gamma, \Lambda, \Phi, \Delta)(y) \). That is why, \( A \) is in fact defined for locally defined \((\Gamma, \Lambda, \Phi, \Delta)\), too.

One can verify that the Kolář connection \((\Gamma, \Lambda, \Phi, \Delta)\) (mentioned in Introduction) defines a natural operator \( A \) in the sense of Definition 1, where \( A(\Gamma, \Lambda, \Phi, \Delta) := (\Gamma, \Lambda, \Phi, \Delta) \).

So, to classify all natural operators in the sense of Definition it suffices to classify all natural operators in the sense of the following definition.
**Definition 2.** A (gauge) $\mathcal{FM}_{m,n} \times \mathcal{VB}_{m,n}$-natural operator $A$ sending systems $(\Gamma, \Lambda, \Phi, \Delta)$ consisting of general connections $\Gamma$ on fibred manifolds $Y \to M$, torsion free classical linear connections $\Lambda$ on $M$, vertical parallelisms $\Phi : Y \times E \to VY$ on $Y$ and linear connections $\Delta$ on vector bundles $E \to M$ into tensor fields $A(\Gamma, \Lambda, \Phi, \Delta)$ of type $T^* \otimes T^* \otimes T$ on $Y$ is an $\mathcal{FM}_{m,n} \times \mathcal{VB}_{m,n}$-invariant system of regular operators

$$A: \text{Con}(Y) \times \text{Con}^0_{\text{clas}}(M) \times \text{Par}(Y \times_M E) \times \text{Con}_{\text{lin}}(E) \to \text{Ten}^{(1,2)}(Y)$$

for any $\mathcal{FM}_{m,n}$-object $Y \to M$ and any $\mathcal{VB}_{m,n}$-object $E \to M$ (the same $M$), where $\text{Ten}^{(1,2)}(Y)$ is the space of tensor fields of type $\otimes^2 T^* \otimes T$ on $Y$.

A simple example of a natural operator $A$ in the sense of Definition 2 is given by the torsion of the Kolář connection $(\Gamma, \Lambda, \Phi, \Delta)$ (mentioned above).

Any natural operator $A$ in the sense of Definition 1 is of the form

$$A(\Gamma, \Lambda, \Phi, \Delta) = (\Gamma, \Lambda, \Phi, \Delta) + A^1(\Gamma, \Lambda, \Phi, \Delta),$$

where $A^1$ is a (uniquely determined) natural operator in the sense of Definition 2. That is why, from now on we study natural operators in the sense of Definition 2 only. Several examples of natural operators in the sense of Definition 2 are presented in the next section.

From now on, we can understand any natural operator $A$ in the extended version as in Remark 3.

2. THE MAIN EXAMPLES OF NATURAL OPERATORS

Let $p_Y : Y \to M$ be a fibred manifold and $p_E : E \to M$ be a vector bundle. Let $(\Gamma, \Lambda, \Phi, \Delta)$ be a 4-tuple consisting of a general connection $\Gamma$ on $p_Y : Y \to M$, a classical linear connection $\Lambda$ on $M$, a vertical parallelism $\Phi : Y \times_M E \to VY$ and of a linear general connection $\Delta$ on $p_E : E \to M$.

According to the usual $\Gamma$-decomposition $TY = VY \oplus_Y H^\Gamma Y$ we have the decomposition

$$T^*Y \otimes TY = (V^*Y \otimes VY) \oplus_Y (V^*Y \otimes H^\Gamma Y)$$

$$\oplus_Y ((H^\Gamma)^* \otimes VY) \oplus_Y ((H^\Gamma)^* \otimes H^\Gamma).$$

Let $\text{id}_{HY}$ be the tensor field of type $T^* \otimes T$ on $Y$ being the $(H^\Gamma Y)^* \otimes H^\Gamma Y$-component of the identity tensor field $\text{id}_{TY}$ on $Y$ (the other 3 component of $\text{id}_{HY}$ are zero). Let $\text{id}_{VY}$ be the tensor field of type $T^* \otimes T$ on $Y$ being the $V^*Y \otimes VY$-component of $\text{id}_{TY}$ (the other 3 component of $\text{id}_{VY}$ are zero).

Quite similarly, we have the decomposition

$$T^*Y \otimes T^*Y \otimes TY = (V^*Y \otimes V^*Y \otimes VY) \oplus_Y (V^*Y \otimes V^*Y \otimes H^\Gamma Y)$$

$$\oplus_Y (V^*Y \otimes (H^\Gamma Y)^* \otimes VY) \oplus_Y (V^*Y \otimes (H^\Gamma Y)^* \otimes H^\Gamma Y)$$

$$\oplus_Y ((H^\Gamma Y)^* \otimes V^*Y \otimes VY) \oplus_Y ((H^\Gamma Y)^* \otimes V^*Y \otimes H^\Gamma Y)$$

$$\oplus_Y ((H^\Gamma Y)^* \otimes (H^\Gamma Y)^* \otimes VY) \oplus_Y ((H^\Gamma Y)^* \otimes (H^\Gamma Y)^* \otimes H^\Gamma Y).$$
Let $T or^{H^* \otimes V^* \otimes V}(\Gamma, \Lambda, \Phi, \Delta)$ be the $(H^*Y)^* \otimes V^*Y \otimes VY$-component of the torsion tensor field $T or(\Gamma, \Lambda, \Phi, \Delta)$ of the Kolář connection $(\Gamma, \Lambda, \Phi, \Delta)$ (mentioned in Introduction). This components can be treated as the tensor field of type $T^* \otimes T^* \otimes T$ on $Y$ (the other 7 components of it are zero). Taking contraction $C_1^1$ we produce tensor field $C_1^1T or^{H^* \otimes V^* \otimes V}(\Gamma, \Lambda, \Phi, \Delta)$ of type $T^*$ on $Y$. Let $T or^{H^* \otimes H^* \otimes V}(\Gamma, \Lambda, \Phi, \Delta)$ be the $(H^*Y)^* \otimes VY$-component of $T or(\Gamma, \Lambda, \Phi, \Delta)$. Thus we have the following tensor fields of type $T^* \otimes T^* \otimes T$ on $Y$ canonically depending on $(\Gamma, \Lambda, \Phi, \Delta)$ (i.e. we have the corresponding natural operators in the sense of Definition 2).

Example 1. $\tau_1(\Gamma, \Lambda, \Phi, \Delta) := T or^{H^* \otimes H^* \otimes V}(\Gamma, \Lambda, \Phi, \Delta)$.

Example 2. $\tau_2(\Gamma, \Lambda, \Phi, \Delta) := T or^{H^* \otimes V^* \otimes V}(\Gamma, \Lambda, \Phi, \Delta)$.

Example 3. $\tau_3(\Gamma, \Lambda, \Phi, \Delta) := id_{HY} \otimes C_1^1T or^{H^* \otimes V^* \otimes V}(\Gamma, \Lambda, \Phi, \Delta)$.

Example 4. $\tau_4(\Gamma, \Lambda, \Phi, \Delta) := C_1^1T or^{H^* \otimes V^* \otimes V}(\Gamma, \Lambda, \Phi, \Delta) \otimes id_{HY}$.

Example 5. $\tau_5(\Gamma, \Lambda, \Phi, \Delta) := C_1^1T or^{H^* \otimes V^* \otimes V}(\Gamma, \Lambda, \Phi, \Delta) \otimes id_{HY}$.

Example 6. $\tau_6(\Gamma, \Lambda, \Phi, \Delta) := id_{VY} \otimes C_1^1T or^{H^* \otimes V^* \otimes V}(\Gamma, \Lambda, \Phi, \Delta)$.

In the above examples (and from now on) we identify tensor fields $\tau$ of type $T^* \otimes T \otimes T^*$ with tensor fields $\bar{\tau}$ of type $T^* \otimes T^* \otimes T$ and with tensor fields $\tilde{\tau}$ of type $T \otimes T^* \otimes T^*$ by $\tilde{\tau}(\omega, X_1, X_2) = \bar{\tau}(X_1, X_2, \omega) = \tau(X_1, \omega, X_2)$. Moreover, tensor fields of types $T \otimes T^* \otimes T^*$ or $T^* \otimes T \otimes T^*$, we will always understand as the equivalent ones of type $T^* \otimes T^* \otimes T$. That is why, the contraction $C_2^1$ is clear.

In general, if $P : N \times V \rightarrow TN$ is a parallelism on a manifold $N$ and $v \in N$, the vector field $\bar{v} : N \rightarrow TN$, $\bar{v}(z) = P(z, v)$ is called the constant vector field corresponding to $v$. One can show easily that there is a unique classical linear connection $\nabla = \nabla^P$ on $N$ such that $\nabla_{\bar{v}} \bar{v} = 0$ for any constant vector fields on $N$. The torsion tensor of $\nabla$ will be denoted by $\tau(P)$ and called the torsion tensor field of $P$ (thus $\tau(P)(X_1, X_2) = \nabla_{X_1} X_2 - \nabla_{X_2} X_1 - [X_1, X_2]$). If $\Phi : Y \times_M E \rightarrow VY$ is a vertical parallelism, we have the torsion tensor field $\tau\Phi$ of $\Phi$ given by

$$\tau\Phi = \bigcup_{x \in M} \tau(\Phi_x) : Y \rightarrow V^*Y \otimes VY.$$  

(The concept of a vertical parallelism and its torsion was introduced by I. Kolář in [K].) We can treat $\tau\Phi$ as the tensor field of type $T^* \otimes T^* \otimes T$ on $Y$ (the other components of it in the decomposition we define to be 0). Thus we have the following tensor fields of type $T^* \otimes T^* \otimes T$ on $Y$ canonically depending on $(\Gamma, \Lambda, \Phi, \Delta)$.

Example 7. $\tau_7(\Gamma, \Lambda, \Phi, \Delta) = \tau\Phi$.

Example 8. $\tau_8(\Gamma, \Lambda, \Phi, \Delta) := id_{HY} \otimes C_2^1\tau\Phi$.

Example 9. $\tau_9(\Gamma, \Lambda, \Phi, \Delta) := C_2^1\tau\Phi \otimes id_{HY}$.

Example 10. $\tau_{10}(\Gamma, \Lambda, \Phi, \Delta) := id_{VY} \otimes C_2^1\tau\Phi$. 
Example 11. \( \tau_{11}(\Gamma, \Lambda, \Phi, \Delta) := C_2^1 \tau \Phi \otimes \text{id}_V \).

By Section 3 (Corollary 1), if \( \Lambda \) is torsion free, then (eventually modulo signum) \( \tau \Phi = T_{or} V^* \otimes V^* \otimes V^* \otimes V^* (\Gamma, \Lambda, \Phi, \Delta) \), the \( V^* Y \otimes V^* Y \otimes VY \)-component of \( T_{or}(\Gamma, \Lambda, \Phi, \Delta) \) in the \( \Gamma \)-decomposition.

In general, a Lie derivative of an arbitrary map \( g: N \to N_1 \) with respect to vector fields \( \xi: N \to TN \) and \( \eta: N_1 \to TN_1 \) is the map

\[
L_{(\xi, \eta)} g = Tg \circ \xi - \eta \circ g: N \to TN_1.
\]

If we have another fibred manifold \( Z \to M \) with general connection \( \Omega \) and a base preserving morphism \( f: Y \to Z \), then the covariant derivative \( D_{\Gamma, \Omega} f: Y \to VZ \otimes T^* M \) is defined by

\[
(D_{\Gamma, \Omega} f)(\xi) := L_{(\Gamma, \Omega, \xi)} f.
\]

Consider \( \Phi: Y \times_M E \to VY \). According to [5, p.55], \( \Gamma \) induces a general connection \( \nabla \) on \( VY \to Y \). Further we construct the product connection \( \Gamma \times \Delta \) on \( Y \times_M E \). Then \( D_{\Gamma \times \Delta, V \nabla} \Phi: Y \times_M E \to VY \times VY \otimes T^* M \). The values lie in a sub-bundle characterized by \( V \pi = 0 \), where \( \pi: VY \to Y \) is the bundle projection. This sub-bundle coincides with \( VY \times_Y VY \). The covariant differential \( D_{\Gamma \times \Delta, V \nabla} \Phi: Y \times_M E \to VY \otimes T^* M \) is the second component of \( D_{\Gamma \times \Delta, V \nabla} \Phi \). (This construction of the covariant differential was proposed by I. Kolář in [4].)

We can consider the covariant differential as the corresponding map \( D_{\Gamma \times \Delta, V \nabla} \Phi: (Y \times_M E) \times_M TM \to VY \). Then we define the modified covariant differential \( \tilde{D}_{\Gamma \times \Delta, V \nabla} \Phi: Y \to V^* Y \otimes T^* Y \otimes VY \) by

\[
(\tilde{D}_{\Gamma \times \Delta, V \nabla} \Phi)(y)(X_1, X_2) := D_{\Gamma \times \Delta, V \nabla} \Phi(\Phi^{-1}(X_1), Tp_Y(X_2)) \in V_y Y,
\]

where \( X_1 \in V_y Y, X_2 \in T_y Y \). We can treat it as the tensor field of type \( T^* \otimes T^* \otimes T \) on \( Y \) (the other parts of it in the decomposition we define to be 0). Thus we have the following tensor field of type \( T^* \otimes T^* \otimes T \) on \( Y \) canonically induced by \( (\Gamma, \Lambda, \Phi, \Delta) \).

Example 12. \( \tau_{12}(\Gamma, \Lambda, \Phi, \Delta) := \tilde{D}_{\Gamma \times \Delta, V \nabla} \Phi \).

By Section 3 (Corollary 2), if \( \Lambda \) is torsion free, then (eventually modulo signum) \( \tau_{12}(\Gamma, \Lambda, \Phi, \Delta) = T_{or} V^* \otimes H^* \otimes V^* (\Gamma, \Lambda, \Phi, \Delta) \), the \( V^* Y \otimes (H^* Y)^* \otimes VY \)-part of \( T_{or}(\Gamma, \Lambda, \Phi, \Delta) \) in the \( \Gamma \)-decomposition.

3. ESTIMATION OF DIMENSION OF THE VECTOR SPACE OF NATURAL OPERATORS

Let \( x^1, \ldots, x^m \) be the usual coordinates on \( \mathbb{R}^m \). Let \( \mathbb{R}^{m,n} \) be the trivial bundle over \( \mathbb{R}^m \) with the standard fiber \( \mathbb{R}^n \) and \( x^1, \ldots, x^m, y^1, \ldots, y^n \) be the usual fiber coordinates on \( \mathbb{R}^{m,n} \). Let \( \mathbb{R}^{m,n} \) be also the trivial vector bundle over \( \mathbb{R}^m \) and \( x^1, \ldots, x^m, v^1, \ldots, v^n \) be the usual vector bundle coordinates on \( \mathbb{R}^{m,n} \).

Let

\[
\Gamma^o = \sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i}, \quad \Lambda^o = (0), \quad \Phi^o = \sum_{p=1}^n v^p \frac{\partial}{\partial y^p}, \quad \Delta^o = \sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i}
\]
be the trivial general connection on $\mathbb{R}^{m,n}$, the torsion free flat classical linear connection on $\mathbb{R}^m$, the canonical parallelism on $\mathbb{R}^{m,m}$ and the trivial linear connection on $\mathbb{R}^{m,n}$, respectively.

In this section we study a natural operator $A$ in the sense of Definition 2.

From Corollary 19.8 in [5], we get immediately the following proposition.

**Proposition 1.** Let $p_Y : Y \to M$ be an $\mathcal{F}\mathcal{M}_{m,n}$-object and $p_E: E \to M$ be a VB$_{m,n}$-object, $y \in Y_x$, $x \in M$. Let $(\Gamma, \Lambda, \Phi, \Delta) \in \text{Con}(Y) \times \text{Con}^o_{\text{clas}}(M) \times \text{Par}(Y \times_M E) \times \text{Con}_{\text{lin}}(E)$. There exists a finite number $r = r(\Gamma, \Lambda, \Phi, \Delta, y)$ such that for any $(\Gamma_1, \Lambda_1, \Phi_1, \Delta_1) \in \text{Con}(Y) \times \text{Con}^o_{\text{clas}}(M) \times \text{Par}(Y \times_M E) \times \text{Con}_{\text{lin}}(E)$ we have the following implication

$$
(j^y_y \Gamma_1 = j^y_y \Gamma, \ j^x_x \Lambda_1 = j^x_x \Lambda, \ j^y_y \Phi_1 = j^y_y \Phi, \ j^y_y \Delta_1 = j^y_y \Delta) \implies A(\Gamma_1, \Lambda_1, \Phi_1, \Delta_1)(y) = A(\Gamma, \Lambda, \Phi, \Delta)(y).
$$

It is clear that $A$ is determined by the values

$$
A(\Gamma, \Lambda, \Phi, \Delta)(y) \in T^*_y Y \otimes T^*_y Y \otimes T_y Y
$$

for fibred manifolds $p_Y : Y \to M$ with $m$-dimensional bases and $n$-dimensional fibres, vector bundles $p_E : E \to M$ with $n$-dimensional fibres, general connections $\Gamma$ on $p_Y : Y \to M$, torsion free classical linear connections $\Lambda$ on $M$, vertical parallelisms $\Phi : Y \times_M E \to VY$, linear connections $\Delta$ on $p_E : E \to M$ and $y \in Y_x$, $x \in M$.

Using the invariance of $A$ with respect to (respective) fibred manifold charts and vector bundle charts and Proposition 1, we can assume $E = Y = \mathbb{R}^{m,n}$, $y = (0,0)$,

$$
\Gamma = \Gamma^\alpha + \sum F^p_{j;\alpha \beta} x^\alpha y^\beta \, dx^j \otimes \frac{\partial}{\partial y^p},
$$

where the sum is over all $m$-tuples $\alpha$ and all $n$-tuples $\beta$ of non-negative integers and $j = 1, \ldots, m$ and $p = 1, \ldots, n$ with $1 \leq |\alpha| + |\beta| \leq K$ (i.e. we can assume $F^p_{j;\alpha}(0,0) = 0$),

$$
\Lambda = \left( \sum \Lambda_{jk;\gamma}^i x^\gamma \right)_{i,j,k = 1, \ldots, m}, \quad \Lambda_{jk;\gamma}^i = \Lambda_{k;\gamma}^i,
$$

where the sums are over all $m$-tuples $\gamma$ of non-negative integers with $1 \leq |\gamma| \leq K$ (i.e. we can assume $\Lambda_{jk;\gamma}^i(0,0) = 0$),

$$
\Phi = \Phi^\alpha + \sum a^q_{s;\delta \sigma} x^\delta y^\sigma \, v^q \frac{\partial}{\partial y^s},
$$

where the sum is over all $m$-tuples $\delta$ and all $n$-tuples $\sigma$ of non-negative integers and $s, q = 1, \ldots, n$ with $1 \leq |\delta| + |\sigma| \leq K$ (i.e. we can assume $a^q_{s;\delta}(0,0) = 0$) (we remark that such $\Phi$ can not be defined globally (it may not be a diffeomorphism $\mathbb{R}^{m,n} \times \mathbb{R}^m \mathbb{R}^{m,n} \Rightarrow V \mathbb{R}^{m,n} \Rightarrow VU$), but it is defined locally on some neighborhood of $(0,0)$ (it is a diffeomorphism $U \times U \mathbb{R}^{m,n} \Rightarrow VU$)),

$$
\Delta = \Delta^\alpha + \sum \Delta_{jq}^p x^\rho v^q \, dx^j \otimes \frac{\partial}{\partial v^p},
$$
where the sum is over all \( m \)-tuples \( \rho \) of non-negative integers and \( j = 1, \ldots, m \) and \( p, q = 1, \ldots, n \) with \( 0 \leq |\rho| \leq K \), where \( K \) is an arbitrary positive integer.

Given a positive integer \( K \) we define a smooth (as \( A \) is regular) map \( A_K : \mathbb{R}^{n(K)} \to \mathbb{R}^q = T^*_{(0,0)}T^*_{(0,0)} \mathbb{R}^{m,n} \otimes T_{(0,0)} \mathbb{R}^{m,n} \) by

\[
A_K((F^p_{j;\alpha\beta})_i(A^i_{j\kappa;\gamma}), (a^p_{q\delta\sigma}), (\Delta^i_{j,\rho})) := A(\Gamma, \Lambda, \Phi, \Delta)(0, 0),
\]
where \( \Gamma, \Lambda, \Phi, \Delta \) are as in (2)–(5).

By the homogeneous function theorem, from this homogeneity condition we obtain.

\[
\phi_{p,q} = \frac{1}{230} W. M. MIKULSKI
\]

Lemma 2. Any natural operator \( A \) in the sense of Definition 2 is of order not more than 1.

Using these facts, we prove the following lemma.

Lemma 2. Let \( m \geq 2 \) and \( n \geq 2 \). A natural operator \( A \) in the sense of Definition 2 is fully determined by the collection of values

\[
A^1 := A(\Gamma^0, x^2dx^1 \otimes \frac{\partial}{\partial y^1}, \Lambda^0, \Phi^0, \Delta^0)(0, 0),
\]
\[
A^2 := A(\Gamma^0, y^1dx^1 \otimes \frac{\partial}{\partial y^1}, \Lambda^0, \Phi^0, \Delta^0)(0, 0),
\]
\[
A^3 := A(\Gamma^0, \Lambda^0, \Phi^0 + v^2y^1 \frac{\partial}{\partial y^1}, \Delta^0)(0, 0),
\]

where \( \Gamma^0, \Lambda^0, \Phi^0, \Delta^0 \) are defined in (1).

Proof. (a) We are going to observe that the value \( A(\Gamma^0, x^i, dx^i \otimes \frac{\partial}{\partial y^\rho}, \Lambda^0, \Phi^0, \Delta^0)(0, 0) \) is determined by \( A^1 \).

If \( i_o = j_o \), by the invariance of \( A \) with respect to

\[
((x^1, \ldots, x^m, y^1, \ldots, y^p, + \frac{1}{2}(x^{i_o})^2, \ldots, y^n), (x^1, \ldots, x^m, v^1, \ldots, v^n)),
\]

Clearly, \( A \) is determined by the collection of all \( A_K, K = 1, 2, \ldots \).

Using the invariance of \( A \) with respect to \((\varphi_t \times \phi_t, \varphi_t \times \phi_t) \), \( \varphi_t = t id_{\mathbb{R}^n} \), \( \phi_t = t id_{\mathbb{R}^n} \), \( t > 0 \), we get the homogeneous condition

\[
t_A K((F^p_{j;\alpha\beta})_i(A^i_{j\kappa;\gamma}), (a^p_{q\delta\sigma}), (\Delta^i_{j,\rho})) := A_K((t^{\alpha} + |\beta| F^p_{j;\alpha\beta}), (t^{\gamma} + 1 \Lambda^i_{j\kappa;\gamma}), (t^{\delta} + |\sigma| a^p_{q\delta\sigma}), (t^{\rho} + 1 \Delta^i_{j,\rho})).
\]

By the homogeneous function theorem, from this homogeneity condition we obtain.

Lemma 1. \( A_K \) is independent of \( F^p_{j;\alpha\beta} \) with \(|\alpha| + |\beta| \geq 2 \), \( A_K \) is independent of \( \Lambda^i_{j\kappa;\gamma} \) with \(|\gamma| \geq 1 \), \( A_K \) is independent of \( a^p_{q\delta\sigma} \) with \(|\delta| + |\sigma| \geq 2 \) and \( A_K \) is independent of \( \Delta^i_{j,\rho} \) with \(|\rho| \geq 1 \). Even, \( A_K \) is a linear combination with real coefficients of \( \Lambda^i_{j\kappa;\gamma}(0)(0) \) and \( F^p_{j;\alpha\beta}, a^p_{q\delta\sigma} \) with \(|\alpha| + |\beta| = 1, |\delta| + |\sigma| = 1, i, j, k = 1, \ldots, m, p, q, s = 1, \ldots, n \).

In particular, \( A_K(\Gamma^0, \Lambda^0, \Phi^0, \Delta^0)(0, 0) = 0 \).
from \(A(\Gamma^o, \Lambda^o, \Phi^o, \Delta^o)(0, 0) = 0\) we get \(A(\Gamma^o + x^i dx^i \otimes \frac{\partial}{\partial y^p}, \Lambda^o, \Phi^o, \Delta^o)(0, 0) = 0\).

If \(i_o \neq j_o\), there exists a respective permutation of coordinates sending \(\Gamma^o + x^2 dx^1 \otimes \frac{\partial}{\partial y^o} \) into \(\Gamma^o + x^{j_o} dx^{j_o} \otimes \frac{\partial}{\partial y^o}\) and preserving \(\Lambda^o, \Phi^o, \Delta^o\). Then using the invariance of \(A\) with respect to this permutation, we end the observation.

(b) We are going to observe that \(A^2\) determines the value
\(A(\Gamma^o + y^{k_o} x^l \otimes \frac{\partial}{\partial y^o}, \Lambda^o, \Phi^o, \Delta^o)(0, 0)\).

By the invariance of \(A\) with respect to
\[(f, g) = ((x^1, \ldots, x^m, y^1 + y^2, y^2, \ldots, y^m), (x^1, \ldots, x^m, v^1 + v^2, v^2, \ldots, v^n))\]
we see \(A(\Gamma^o + (y^1 - y^2) dx^1 \otimes \frac{\partial}{\partial y^o}, \Lambda^o, \Phi^o, \Delta^o)(0, 0)\) is the image of \(A^2\) by \((f, g)\), and then it is determined by \(A^2\). Therefore \(A(\Gamma^o + y^1 dx^1 \otimes \frac{\partial}{\partial y^o}, \Lambda^o, \Phi^o, \Delta^o)(0, 0) = A^2 - A(\Gamma^o + y^1 - y^2) dx^1 \otimes \frac{\partial}{\partial y^o}, \Lambda^o, \Phi^o, \Delta^o)(0, 0)\) is determined by \(A^2\).

Now, using the invariance of \(A\) with a respective permutation of coordinates, we end the observation in this case.

(c) We are going to observe that \(A^3\) determines the value
\(A(\Gamma^o, \Lambda^o, \Phi^o + v^{k_o} y^p \otimes \frac{\partial}{\partial y^o}, \Delta^o)(0, 0)\).

If \(p \neq 1\), then
\[(x^1, \ldots, x^m, y^1 + y^2, \ldots, y^n), (x^1, \ldots, x^m, v^1 + v^2, \ldots, v^n)\]
preserves \(\Gamma^o, \Lambda^o, \Delta^o\) and sends \(\Phi^o + v^2 y^1 \frac{\partial}{\partial y^o} \) into \(\Phi^o + v^2 (y^1 - y^2) \frac{\partial}{\partial y^o}\). Then (similarly as in the case (b) of the proof) \(A(\Gamma^o, \Lambda^o, \Phi^o + v^2 y^1 \frac{\partial}{\partial y^o}, \Delta^o)(0, 0)\) is determined by \(A^3\). In particular \(A(\Gamma^o, \Lambda^o, \Phi^o + v^2 y^1 \frac{\partial}{\partial y^o}, \Delta^o)(0, 0)\) and \(A(\Gamma^o, \Lambda^o, \Phi^o + v^2 y^3 \frac{\partial}{\partial y^o}, \Delta^o)(0, 0)\) are determined by \(A^3\).

By the invariance of \(A\) with respect to
\[(x^1, \ldots, x^m, y^1 + y^2, y^3, \ldots, y^n), (x^1, \ldots, x^m, v^1, \ldots, v^n)\]
from \(A(\Gamma^o, \Lambda^o, \Phi^o, \Delta^o)(0, 0) = 0\) we get \(A(\Gamma^o, \Lambda^o, \Phi^o + v^2 y^1 \frac{\partial}{\partial y^o} + v^1 y^2 \frac{\partial}{\partial y^o}, \Delta^o)(0, 0) = 0\) (because \(\Phi^o\) is mapped into \(\Phi^o + v^2 y^1 \frac{\partial}{\partial y^o} + v^1 y^2 \frac{\partial}{\partial y^o} + \ldots\), where the dots have the 1-jet equal to 0, and \(\Gamma^o, \Lambda^o, \Delta^o\) and \(A\) are preserved, and \(A\) is of order not more than 1), i.e. \(A(\Gamma^o, \Lambda^o, \Phi^o + v^1 y^2 \frac{\partial}{\partial y^o}, \Delta^o)(0, 0)\) is determined by \(A^3\) (it is \(-A^3\)). By the invariance of \(A\) with respect to
\[(x^1, \ldots, x^m, y^1 + \frac{1}{2}(y^1)^2, y^2, \ldots, y^n), (x^1, \ldots, x^m, v^1, \ldots, v^n)\]
from \(A(\Gamma^o, \Lambda^o, \Phi^o, \Delta^o) = 0\) we get \(A(\Gamma^o, \Lambda^o, \Phi^o + v^1 y^1 \frac{\partial}{\partial y^o}, \Delta^o)(0, 0) = 0\).

Now, using the invariance of \(A\) with respect to a respective permutation of coordinates, we end the observation.

(d) Let us denote
\begin{equation}
A^4 := A(\Gamma^o, \Lambda^o, \Phi^o, \Delta^o + v^1 dx^1 \otimes \frac{\partial}{\partial v^1})(0, 0) .
\end{equation}
We are going to observe that $A(\Gamma^o, \Lambda^o, \Phi^o, \Delta^o + v^{q_o}d{x^i_o} \otimes \frac{\partial}{\partial v^{p_o}})(0,0)$ is determined by $A^4$.

Using the invariance of $A$ with respect to

$$(f,g) = (\{(x^1, \ldots, x^m, y^1 + y^2, y^2, \ldots, y^n), (x^1, \ldots, x^m, v^1 + v^2, v^2, \ldots, v^n)\})$$

we deduce that $A' = A(\Gamma^o, \Lambda^o, \Phi^o, \Delta^o + (v^1 - v^2)d{x^1} \otimes \frac{\partial}{\partial v^2})(0,0)$ is determined by $A^4$ (it is image of $A^4$ by $(f,g)$). So, $A(\Gamma^o, \Lambda^o, \Phi^o, \Delta^o + v^2d{x^1} \otimes \frac{\partial}{\partial v^2})(0,0)$ is determined by $A^4$ (it is $A^4 - A'$).

Now, using the invariance of $A$ with respect to a respective permutation of coordinates, we end the observation.

(e) We are going to observe that $A^4$ determines the value $A(\Gamma^o, \Lambda^o, \Phi^o + x^i_o v^{q_o} \frac{\partial}{\partial v^{p_o}}, \Delta^o)(0,0)$.

Using the invariance of $A$ with respect to

$$( (x^1, \ldots, x^m, y^1, \ldots, y^n), (x^1, \ldots, x^m, v^1, \ldots, v^{p_o} + x^i_o v^{q_o}, \ldots, v^n) ) ,$$

since $(x^1, \ldots, x^m, v^1, \ldots, v^{p_o} + x^i_o v^{q_o}, \ldots, v^n)^{-1} = (x^1, \ldots, x^m, v^1, \ldots, v^{p_o} - x^i_o v^{q_o} + \tilde{\varphi}(x^i_o)v^{q_o}, \ldots, v^n)$ with $j^1_0 \tilde{\varphi} = 0$ (if $p_o \neq q_o$, $\tilde{\varphi} = 0$), from $A(\Gamma^o, \Lambda^o, \Phi^o, \Delta^o)(0,0) = 0$ we get

$$A(\Gamma^o, \Lambda^o, \Phi^o - x^i_o v^{q_o} \frac{\partial}{\partial y^{p_o}}, \Delta^o + v^{q_o}d{x^i_o} \otimes \frac{\partial}{\partial v^{p_o}})(0,0) = 0,$$

i.e. $A(\Gamma^o, \Lambda^o, \Phi^o + x^i_o v^{q_o} \frac{\partial}{\partial v^{p_o}}, \Delta^o)(0,0) = A(\Gamma^o, \Lambda^o, \Phi^o, \Delta^o + v^{q_o}d{x^i_o} \otimes \frac{\partial}{\partial v^{p_o}})(0,0)$ is determined by $A^4$ because of the part (d) of the proof. In particular (for $i_o = 1$, $p_o = 1$, $q_o = 1$), we proved

$$(11) \quad A^4 = A(\Gamma^o, \Lambda^o, \Phi^o + x^1 v^1 \frac{\partial}{\partial y^{1}}, \Delta^o)(0,0).$$

(g) We are going to prove that $A^4$ is determined by $A^2$.

Using the invariance of $A$ with respect to

$$( (x^1, \ldots, x^m, y^1 + x^1 y^1, y^2, \ldots, y^n), (x^1, x^2, \ldots, x^m, v^1, \ldots, v^n) )$$

from $A(\Gamma^o, \Lambda^o, \Phi^o, \Delta^o) = 0$ we get

$$A(\Gamma^o + y^1d{x^1} \otimes \frac{\partial}{\partial y^1}, \Lambda^o, \Phi^o + x^1 v^1 \frac{\partial}{\partial y^1}, \Delta^o)(0,0) = 0.$$

Hence $A^4 = -A^2$ because of $\text{(11)}$.

The proof of Lemma 2 is complete. $\square$

Now, we prove the following lemma.

**Lemma 3.** Let $m \geq 2$ and $n \geq 3$. Let $A^1$, $A^2$, $A^3$ be the values $\text{(7)} - \text{(9)}$ from Lemma 2. There are real numbers $a_1, \ldots, a_{12}$ such that

$$(12) \quad A^1 = a_1 \left( d_{(0,0)}x^2 \otimes d_{(0,0)}x^1 \otimes \frac{\partial}{\partial y^1}|_{(0,0)} - d_{(0,0)}x^1 \otimes d_{(0,0)}x^2 \otimes \frac{\partial}{\partial y^1}|_{(0,0)} \right),$$
\[ A^2 = a_2 \sum_{p=1}^{n} d_{(0,0)}x^1 \otimes d_{(0,0)}y^p \otimes \frac{\partial}{\partial y^p} \big|_{(0,0)} \]
\[ + a_3 \sum_{p=1}^{n} d_{(0,0)}y^p \otimes d_{(0,0)}x^1 \otimes \frac{\partial}{\partial y^p} \big|_{(0,0)} \]
\[ + a_4 d_{(0,0)}x^1 \otimes d_{(0,0)}y^1 \otimes \frac{\partial}{\partial y^1} \big|_{(0,0)} \]
\[ + a_5 d_{(0,0)}y^1 \otimes d_{(0,0)}x^1 \otimes \frac{\partial}{\partial y^1} \big|_{(0,0)} \]
\[ + a_6 \sum_{i=1}^{m} d_{(0,0)}x^1 \otimes d_{(0,0)}x^i \otimes \frac{\partial}{\partial x^i} \big|_{(0,0)} \]
\[ + a_7 \sum_{i=1}^{m} d_{(0,0)}x^i \otimes d_{(0,0)}x^1 \otimes \frac{\partial}{\partial x^i} \big|_{(0,0)} \]
\[ (13) \]

\[ A^3 = a_8 \sum_{p=1}^{n} d_{(0,0)}y^p \otimes d_{(0,0)}y^2 \otimes \frac{\partial}{\partial y^p} \big|_{(0,0)} \]
\[ + a_9 \sum_{p=1}^{n} d_{(0,0)}y^2 \otimes d_{(0,0)}y^p \otimes \frac{\partial}{\partial y^p} \big|_{(0,0)} \]
\[ + a_{10} \sum_{i=1}^{m} d_{(0,0)}x^i \otimes d_{(0,0)}y^2 \otimes \frac{\partial}{\partial x^i} \big|_{(0,0)} \]
\[ + a_{11} \sum_{i=1}^{m} d_{(0,0)}y^2 \otimes d_{(0,0)}x^i \otimes \frac{\partial}{\partial x^i} \big|_{(0,0)} \]
\[ + a_{12} \left( d_{(0,0)}y^2 \otimes d_{(0,0)}y^1 \otimes \frac{\partial}{\partial y^1} \big|_{(0,0)} \right) \]
\[ - d_{(0,0)}y^1 \otimes d_{(0,0)}y^2 \otimes \frac{\partial}{\partial y^1} \big|_{(0,0)} \] \[ (14) \]

**Proof.** a. By the invariance of $A$ with respect to
\[ (t^1 x^1, \ldots, t^m x^m, \tau^1 y^1, \ldots, \tau^n y^n), \ (t^1 x^1, \ldots, t^m x^m, \tau^1 v^1, \ldots, \tau^n v^n) \]
for $t^1 > 0, \ldots, t^m > 0, \tau^1 > 0, \ldots, \tau^n > 0$ we get immediately
\[ A^1 = b_1 d_{(0,0)}x^2 \otimes d_{(0,0)}x^1 \frac{\partial}{\partial y^1} \big|_{(0,0)} \]
\[ + b_2 d_{(0,0)}x^1 \otimes d_{(0,0)}x^2 \frac{\partial}{\partial y^1} \big|_{(0,0)} \]
for some real numbers $b_1, b_2$. But by the invariance of $A$ with respect to
\[
\left( (x^1, \ldots, x^m, y^1 + x^1 x^2, y^3, \ldots, y^n), (x^1, \ldots, x^m, v^1, \ldots, v^n) \right)
\]
from $A(\Gamma^o, \Lambda^o, \Phi^o, \Delta^o)(0, 0) = 0$ we get
\[
A\left( \Gamma^o + x^2 dx^1 \otimes \frac{\partial}{\partial y^1} + x^1 dx^2 \otimes \frac{\partial}{\partial y^1}, \Lambda^o, \Phi^o, \Delta^o \right)(0, 0) = 0.
\]

Therefore $b_1 = -b_2$. We define $a_1 := b_1 = -b_2$. That is why, formula (12) holds.

b. By the invariance of $A$ with respect to (15) we get immediately
\[
A^2 = \sum_{p=1}^{n} b_p d_{(0,0)} y^p \otimes d_{(0,0)} x^1 \otimes \frac{\partial}{\partial y^p} \big|_{(0,0)}
\]
\[
+ \sum_{p=1}^{n} c_p d_{(0,0)} x^1 \otimes d_{(0,0)} y^p \otimes \frac{\partial}{\partial y^p} \big|_{(0,0)}
\]
\[
+ \sum_{i=1}^{m} d_i d_{(0,0)} x^1 \otimes d_{(0,0)} x^i \otimes \frac{\partial}{\partial x^i} |_{(0,0)}
\]
\[
+ \sum_{i=1}^{m} e_i d_{(0,0)} x^i \otimes d_{(0,0)} x^1 \otimes \frac{\partial}{\partial x^i} |_{(0,0)}.
\]

Next, by the invariance of $A$ with respect to respective permutation of coordinates, we deduce $b_2 = \cdots = b_n, c_2 = \cdots = c_n, d_2 = \cdots = d_m, e_2 = \cdots = e_m$. Then
\[
A^2 = a_2 \sum_{p=1}^{n} d_{(0,0)} x^1 \otimes d_{(0,0)} y^p \otimes \frac{\partial}{\partial y^p} \big|_{(0,0)}
\]
\[
+ a_3 \sum_{p=1}^{n} d_{(0,0)} y^p \otimes d_{(0,0)} x^1 \otimes \frac{\partial}{\partial y^p} \big|_{(0,0)}
\]
\[
+ a_4 d_{(0,0)} x^1 \otimes d_{(0,0)} y^1 \otimes \frac{\partial}{\partial y^1} \big|_{(0,0)}
\]
\[
+ a_5 d_{(0,0)} y^1 \otimes d_{(0,0)} x^1 \otimes \frac{\partial}{\partial y^1} \big|_{(0,0)}
\]
\[
+ a_6 \sum_{i=1}^{m} d_{(0,0)} x^1 \otimes d_{(0,0)} x^i \otimes \frac{\partial}{\partial x^i} \big|_{(0,0)}
\]
\[
+ a_7 \sum_{i=1}^{m} d_{(0,0)} x^i \otimes d_{(0,0)} x^1 \otimes \frac{\partial}{\partial x^i} \big|_{(0,0)}
\]
\[
+ bd_{(0,0)} x^1 \otimes d_{(0,0)} x^1 \otimes \frac{\partial}{\partial x^1} \big|_{(0,0)}.
\]

Then by the invariance of $A$ with respect to
\[
\left( (x^1, x^2 + x^1, x^3, \ldots, x^m, y^1, \ldots, y^n), (x^1, x^2 + x^1, x^3, \ldots, x^m, v^1, \ldots, v^n) \right)
\]
from the last equality we get \( b = 0 \). That is why, formula (13) is true.

c. By the invariance of \( A \) with respect to (15) we get immediately

\[
A^3 = \sum_{p=1}^{n} b_p d_{(0,0)} y^2 \otimes d_{(0,0)} y^p \otimes \frac{\partial}{\partial y^p} |_{(0,0)} \\
+ \sum_{p=1}^{n} c_p d_{(0,0)} y^p \otimes d_{(0,0)} y^2 \otimes \frac{\partial}{\partial y^p} |_{(0,0)} \\
+ \sum_{i=1}^{m} d_i d_{(0,0)} y^2 \otimes d_{(0,0)} x^i \otimes \frac{\partial}{\partial x^i} |_{(0,0)} \\
+ \sum_{i=1}^{m} e_i d_{(0,0)} x^i \otimes d_{(0,0)} y^2 \otimes \frac{\partial}{\partial x^i} |_{(0,0)}. 
\]

Then by the invariance of \( A \) with respect to respective permutation of coordinates, we deduce \( b_3 = \cdots = b_n, c_3 = \cdots = c_n, d_1 = \cdots = d_m \) and \( e_1 = \cdots = e_m \). Then

\[
A^3 = \lambda_1 d_{(0,0)} y^2 \otimes d_{(0,0)} y^1 \otimes \frac{\partial}{\partial y^1} |_{(0,0)} \\
+ \lambda_2 d_{(0,0)} y^1 \otimes d_{(0,0)} y^2 \otimes \frac{\partial}{\partial y^1} |_{(0,0)} \\
+ \lambda_3 d_{(0,0)} y^2 \otimes d_{(0,0)} y^2 \otimes \frac{\partial}{\partial y^2} |_{(0,0)} \\
+ \lambda_4 \sum_{p=3}^{n} d_{(0,0)} y^p \otimes d_{(0,0)} y^2 \otimes \frac{\partial}{\partial y^p} |_{(0,0)} \\
+ \lambda_5 \sum_{p=3}^{n} d_{(0,0)} y^2 \otimes d_{(0,0)} y^p \otimes \frac{\partial}{\partial y^p} |_{(0,0)} \\
+ \lambda_6 \sum_{i=1}^{m} d_{(0,0)} x^i \otimes d_{(0,0)} y^2 \otimes \frac{\partial}{\partial x^i} |_{(0,0)} \\
+ \lambda_7 \sum_{i=1}^{m} d_{(0,0)} y^2 \otimes d_{(0,0)} x^i \otimes \frac{\partial}{\partial x^i} |_{(0,0)}. 
\]

Then by the invariance of \( A \) with respect to

\[
((x^1, \ldots, x^m, y^1 - y^2, \ldots, y^n), (x^1, \ldots, x^m, v^1 - v^2, \ldots, v^n)) 
\]

from (16), we deduce
\[
A^3 + A(\Gamma^o, \Lambda^o, \Phi^o + v^2 y^2 \frac{\partial}{\partial y^1}, \Delta^o)(0, 0) = A^3 + \lambda_1 d_{(0,0)} y^2 \otimes d_{(0,0)} y^2 \otimes \frac{\partial}{\partial y^1}|_{(0,0)} + \lambda_2 d_{(0,0)} y^2 \otimes d_{(0,0)} y^2 \otimes \frac{\partial}{\partial y^1}|_{(0,0)} - \lambda_3 d_{(0,0)} y^2 \otimes d_{(0,0)} y^2 \otimes \frac{\partial}{\partial y^1}|_{(0,0)}.
\]

On the other hand, by the invariance of \(A\) with respect to
\[
((x^1, \ldots, x^m, y^1 + \frac{1}{2}(y^2)^2, y^2, \ldots, y^n), (x^1, \ldots, x^m, v^1, \ldots, v^n))
\]
from \(A(\Gamma^o, \Lambda^o, \Phi^o, \Delta^o)(0,0) = 0\), we obtain
\[
A(\Gamma^o, \Lambda^o, \Phi^o + v^2 y^2 \frac{\partial}{\partial y^1}, \Delta^o)(0,0) = 0.
\]
So, \(\lambda_1 + \lambda_2 - \lambda_3 = 0\).

From the invariance of \(A\) with respect to
\[
((x^1, \ldots, x^m, y^1 - y^3, y^2, \ldots, y^n), (x^1, \ldots, x^m, v^1 - v^3, v^2, \ldots, v^n))
\]
(we assume \(n \geq 3\)) from [16] we get (after cancelling \(A^3\))
\[
A(\Gamma^o, \Lambda^o, \Phi^o + v^2 y^3 \frac{\partial}{\partial y^1}, \Delta^o)(0,0) = (\lambda_1 - \lambda_5) d_{(0,0)} y^3 \otimes d_{(0,0)} y^3 \otimes \frac{\partial}{\partial y^1}|_{(0,0)} + (\lambda_2 - \lambda_4) d_{(0,0)} y^3 \otimes d_{(0,0)} y^2 \otimes \frac{\partial}{\partial y^1}|_{(0,0)}.
\]

Then by the invariance of \(A\) with respect to the switching \((y^2 \text{ and } y^3)\) and \((v^2 \text{ and } v^3)\) we get
\[
A(\Gamma^o, \Lambda^o, \Phi^o + v^3 y^2 \frac{\partial}{\partial y^1}, 0,0) = (\lambda_1 - \lambda_5) d_{(0,0)} y^3 \otimes d_{(0,0)} y^2 \otimes \frac{\partial}{\partial y^1}|_{(0,0)} + (\lambda_2 - \lambda_4) d_{(0,0)} y^3 \otimes d_{(0,0)} y^3 \otimes \frac{\partial}{\partial y^1}|_{(0,0)}.
\]

On the other hand by the invariance of \(A\) with respect to
\[
((x^1, \ldots, x^m, y^1 + y^2 y^3, y^2, \ldots, y^n), (x^1, \ldots, x^m, v^1, \ldots, v^n))
\]
from \(A(\Gamma^o, \Lambda^o, \Phi^o, \Delta^o)(0,0) = 0\) we get
\[
A(\Gamma^o, \Lambda^o, \Phi^o + y^3 v^2 \frac{\partial}{\partial y^1} + y^2 v^3 \frac{\partial}{\partial y^1}, \Delta^o)(0,0) = 0.
\]
So, $\lambda_1 - \lambda_5 = -(\lambda_2 - \lambda_4)$.

That is why, formula (14) holds.

The proof of the lemma is complete. □

From Lemma 3 it follows immediately the following proposition.

**Proposition 3.** If $m \geq 2$ and $n \geq 3$, the dimension of the vector space of all natural operators in the sense of Definition 2 is of the dimension not more than 12.

4. Linear independence of natural operators from Examples 11–12

We prove the following proposition.

**Proposition 4.** Let $m \geq 2$ and $n \geq 2$. The natural operators $\tau_i (i = 1, \ldots, 12)$ in the sense of Definition 2 from Examples 11–12 are linearly independent.

**Proof.** By Lemma 2, it is sufficient to study the values (7)–(9) for $A = \tau_i$, $i = 1, \ldots, 12$. To compute these values, we use Proposition 1 in [4].

a. The case $\Psi = (\Gamma^o + x^2 dx^1 \otimes \frac{\partial}{\partial y^1}, \Lambda^o, \Phi^o, \Delta^o)$.

In this case, we have (in the notation of Proposition 1 in [4]) $F^1_i(x, y) = x^2$ and other $F^p_i(x, y) = 0$, $\Lambda^i_{ij} = 0$, $\frac{\partial \alpha^p}{\partial x^j} = 0$, $\frac{\partial \alpha^p}{\partial y^j} = 0$, $\Delta^i_{sj} = 0$. Then (by Proposition 1 in [4]) $d\eta^1 = \xi^1 dx^2$ and other $d\eta^p = 0$, and $d\xi^i = 0$. Then (modulo signum)

$$T \text{or}(\Psi)(0, 0) = d_{(0,0)}x^1 \otimes d_{(0,0)}x^2 \otimes \frac{\partial}{\partial y^1} \mid_{(0,0)} - d_{(0,0)}x^2 \otimes d_{(0,0)}x^1 \otimes \frac{\partial}{\partial y^1} \mid_{(0,0)}.$$

Hence (modulo signum)

$$\tau_1(\Psi)(0, 0) = d_{(0,0)}x^1 \otimes d_{(0,0)}x^2 \otimes \frac{\partial}{\partial y^1} \mid_{(0,0)} - d_{(0,0)}x^2 \otimes d_{(0,0)}x^1 \otimes \frac{\partial}{\partial y^1} \mid_{(0,0)}$$

and $\tau_i(\Psi)(0, 0) = 0$ for $i = 2, \ldots, 6$.

By the coordinate expression of the torsion tensor of vertical parallelism in Section 3 of [4], $\tau \Phi^o(0, 0) = 0$. Then $\tau_i(\Psi)(0, 0) = 0$ for $i = 7, \ldots, 11$.

By the coordinate expression of the covariant differential in Section 4 in [4], we have $\tau_{12}(0, 0)(\Psi) = 0$.

b. The case $\Psi = (\Gamma^o + y^1 dx^1 \otimes \frac{\partial}{\partial y^1}, \Lambda^o, \Phi^o, \Delta^o)$.

Now, by Proposition 1 in [4], $d\eta^1 = \xi^1 dy^1$ and other $d\eta^p = 0$, and $d\xi^i = 0$. Then (modulo signum)

$$T \text{or}(\Psi)(0, 0) = d_{(0,0)}x^1 \otimes d_{(0,0)}y^1 \otimes \frac{\partial}{\partial y^1} \mid_{(0,0)} - d_{(0,0)}y^1 \otimes d_{(0,0)}x^1 \otimes \frac{\partial}{\partial y^1} \mid_{(0,0)}.$$
Then \( \tau_1(\Psi)(0,0) = 0 \) and (modulo signum)

\[
\tau_2(\Psi)(0,0) = d_{(0,0)}x^1 \otimes d_{(0,0)}y^1 \otimes \left. \frac{\partial}{\partial y^1} \right|_{(0,0)},
\]

\[
\tau_3(\Psi)(0,0) = \sum_{i=1}^{m} d_{(0,0)}x^i \otimes d_{(0,0)}x^1 \otimes \left. \frac{\partial}{\partial x^i} \right|_{(0,0)},
\]

\[
\tau_4(\Psi)(0,0) = \sum_{i=1}^{m} d_{(0,0)}x^1 \otimes d_{(0,0)}x^i \otimes \left. \frac{\partial}{\partial x^1} \right|_{(0,0)},
\]

\[
\tau_5(\Psi)(0,0) = \sum_{p=1}^{n} d_{(0,0)}x^1 \otimes d_{(0,0)}y^p \otimes \left. \frac{\partial}{\partial y^p} \right|_{(0,0)},
\]

\[
\tau_6(\Psi)(0,0) = \sum_{p=1}^{n} d_{(0,0)}y^p \otimes d_{(0,0)}x^1 \otimes \left. \frac{\partial}{\partial y^p} \right|_{(0,0)}.
\]

Since \( \tau_0(\Phi)(0,0) = 0 \) (see the case a of the proof), \( \tau_i(\Psi)(0,0) = 0 \) for \( i = 7, \ldots, 11 \).

By the coordinate expression of the covariant differential,

\[
\tau_{12}(\Psi)(0,0) = -d_{(0,0)}y^1 \otimes d_{(0,0)}x^1 \otimes \left. \frac{\partial}{\partial y^1} \right|_{(0,0)}.
\]

c. The case \( \Psi = (\Gamma^o, \Lambda^o, \Phi^o + v^2 y^1 \frac{\partial}{\partial y^1}, \Delta^o) \).

By Proposition 1 in [4], \( d\eta^1 = \eta^2 dy^1 \) and \( d\eta^p = 0 \) for other \( p \), and \( d\xi^i = 0 \). Then (modulo signum)

\[
\nabla(\Psi)(0,0) = d_{(0,0)}y^2 \otimes d_{(0,0)}y^1 \otimes \left. \frac{\partial}{\partial y^1} \right|_{(0,0)} - d_{(0,0)}y^1 \otimes d_{(0,0)}y^2 \otimes \left. \frac{\partial}{\partial y^1} \right|_{(0,0)}.
\]

Then \( \tau_i(\Psi)(0,0) = 0 \) for \( i = 1, \ldots, 6 \).

By Section 3 of [4], one can compute

\[
\tau(\Phi^o + v^2 y^1 \frac{\partial}{\partial y^1})(0,0) = d_{(0,0)}y^2 \otimes d_{(0,0)}y^1 \otimes \left. \frac{\partial}{\partial y^1} \right|_{(0,0)}
\]

\[
- d_{(0,0)}y^1 \otimes d_{(0,0)}y^2 \otimes \left. \frac{\partial}{\partial y^1} \right|_{(0,0)}.
\]

Then

\[
\tau_7(\Psi)(0,0) = d_{(0,0)}y^2 \otimes d_{(0,0)}y^1 \otimes \left. \frac{\partial}{\partial y^1} \right|_{(0,0)} - d_{(0,0)}y^1 \otimes d_{(0,0)}y^2 \otimes \left. \frac{\partial}{\partial y^1} \right|_{(0,0)},
\]

\[
\tau_8(\Psi)(0,0) = \sum_{i=1}^{m} d_{(0,0)}x^i \otimes d_{(0,0)}y^2 \otimes \left. \frac{\partial}{\partial x^i} \right|_{(0,0)}.
\]
\[ \tau_9(\Psi)(0,0) = \sum_{i=1}^{m} d_{(0,0)} y^2 \otimes d_{(0,0)} x^i \otimes \frac{\partial}{\partial x^i}|_{(0,0)}, \]

\[ \tau_{10}(\Psi)(0,0) = \sum_{p=1}^{n} d_{(0,0)} y^p \otimes d_{(0,0)} y^2 \otimes \frac{\partial}{\partial y^p}|_{(0,0)}, \]

\[ \tau_{11}(\Psi)(0,0) = \sum_{p=1}^{n} d_{(0,0)} y^2 \otimes d_{(0,0)} y^p \otimes \frac{\partial}{\partial y^p}|_{(0,0)}. \]

By the coordinate expression of the covariant differential, \( \tau_{12}(\Psi)(0,0) = 0. \)

Now, it is easily seen that the natural operators \( \tau_1, \ldots, \tau_{12} \) are linearly independent. The proof of Proposition 4 is complete. \( \Box \)

Else, using Lemma 2 from the proof of Proposition 3 we have the following facts.

**Corollary 1.** If \( \Lambda \) is torsion free, then (eventually modulo signum)

\[ \tau \Phi = T_{\text{or}} V^* \otimes V^* \otimes V (\Gamma, \Lambda, \Phi, \Delta), \]

where \( T_{\text{or}} V^* \otimes V^* \otimes V (\Gamma, \Lambda, \Phi, \Delta) \) is the \( V^* Y \otimes V^* Y \otimes V Y \)-part of \( T_{\text{or}} (\Gamma, \Lambda, \Phi, \Delta) \) in the \( \Gamma \)-decomposition of Section 2.

**Corollary 2.** If \( \Lambda \) is torsion free, then (eventually modulo signum)

\[ \tau_{12}(\Gamma, \Lambda, \Phi, \Delta) = T_{\text{or}} V^* \otimes H^* \otimes V (\Gamma, \Lambda, \Phi, \Delta), \]

where \( T_{\text{or}} V^* \otimes H^* \otimes V (\Gamma, \Lambda, \Phi, \Delta) \) is the \( V^* Y \otimes (H^* Y)^* \otimes V Y \)-part of \( T_{\text{or}} (\Gamma, \Lambda, \Phi, \Delta) \) in the \( \Gamma \)-decomposition of Section 2.

### 5. The main result

From Propositions 3 and 4 it follows the main theorem of the paper.

**Theorem 1.** Let \( m \geq 2 \) and \( n \geq 3 \). Any natural operator \( A \) in the sense of Definition 7 is of the form

\[ A_{Y,E}(\Gamma, \Lambda, \Phi, \Delta) = (\Gamma, \Lambda, \Phi, \Delta) + \sum_{i=1}^{12} \lambda_i \tau_i(\Gamma, \Lambda, \Phi, \Delta) \]

for some (uniquely determined by \( A \)) real numbers \( \lambda_i \), where \( \tau_i \) are the operators described in Examples 7 13 and \( (\Gamma, \Lambda, \Phi, \Delta) \) is the connection constructed by I. Kolář in 4.
References


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