

A. Mohammed Cherif; Djaa Mustapha
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On generalized f -harmonic morphisms

A. MOHAMMED CHERIF, DJAA MUSTAPHA

Abstract. In this paper, we study the characterization of generalized f -harmonic morphisms between Riemannian manifolds. We prove that a map between Riemannian manifolds is an f -harmonic morphism if and only if it is a horizontally weakly conformal map satisfying some further conditions. We present new properties generalizing Fuglede-Ishihara characterization for harmonic morphisms ([Fuglede B., *Harmonic morphisms between Riemannian manifolds*, Ann. Inst. Fourier (Grenoble) **28** (1978), 107–144], [Ishihara T., *A mapping of Riemannian manifolds which preserves harmonic functions*, J. Math. Kyoto Univ. **19** (1979), no. 2, 215–229]).

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1. Introduction

Consider a smooth map $\varphi : (M, g) \rightarrow (N, h)$ between Riemannian manifolds and let $f : M \times N \rightarrow (0, +\infty)$ be a smooth positive function. The map φ is said to be f -harmonic (in a generalized sense) if it is a critical point of the f -energy functional

$$(1.1) \quad E_f(\varphi) = \frac{1}{2} \int_M f(x, \varphi(x)) |d\varphi|^2 v_g,$$

The Euler-Lagrange equation associated to the f -energy functional is

$$(1.2) \quad \tau_f(\varphi) \equiv f_\varphi \tau(\varphi) + d\varphi(\text{grad}^M f_\varphi) - e(\varphi)(\text{grad}^N f) \circ \varphi = 0,$$

where $f_\varphi : M \rightarrow (0, +\infty)$ is a smooth positive function defined by

$$(1.3) \quad f_\varphi(x) = f(x, \varphi(x)), \quad \forall x \in M,$$

$\tau(\varphi) = \text{trace}_g \nabla d\varphi$ is the tension field of φ , and $e(\varphi) = \frac{1}{2}|d\varphi|^2$ is the energy density of φ . $\tau_f(\varphi)$ is called the f -tension field of φ ([4]).

In particular, if $\varphi : M \rightarrow N$ has no critical points, i.e. $|d_x \varphi| \neq 0$, then harmonic maps, p -harmonic maps and F -harmonic maps ([1]) are f -harmonic maps with $f = 1$, $f = |d\varphi|^{p-2}$ and $f = F'(\frac{|d\varphi|^2}{2})$ respectively.

Let $f_1 : M \rightarrow (0, \infty)$ be a smooth function. If $f(x, y) = f_1(x)$ for all $(x, y) \in M \times N$, then $\tau_f(\varphi) = \tau_{f_1}(\varphi) = f_1 \tau(\varphi) + d\varphi(\text{grad}^M f_1)$. Moreover, $\varphi : M \rightarrow N$

is f -harmonic if and only if it is f_1 -harmonic in the sense of A. Lichnerowicz [9] and N. Course [3].

The identity map $Id : (\mathbb{R}^m, \langle \cdot, \cdot \rangle_{\mathbb{R}^m}) \longrightarrow (\mathbb{R}^m, \langle \cdot, \cdot \rangle_{\mathbb{R}^m})$ is f -harmonic if it satisfies the system of differential equation

$$(1.4) \quad \frac{\partial f}{\partial x^i} + \frac{2-m}{2} \frac{\partial f}{\partial y^i} = 0,$$

for all $i = 1, \dots, m$, where $f \in C^\infty(\mathbb{R}^m \times \mathbb{R}^m)$ be a smooth positive function. Let $F \in C^\infty(\mathbb{R}^m)$ be a smooth positive function, then the function of type $f(x^1, \dots, x^m, y^1, \dots, y^m) = F(y^1 - \frac{2-m}{2}x^1, \dots, y^m - \frac{2-m}{2}x^m)$ satisfies the system of differential equation (1.4).

For more details and examples of f -harmonic maps (in a generalized sense), we can refer to [4] and [5].

2. f -harmonic morphisms

Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth mapping between Riemannian manifolds. The critical set of φ is the set $C_\varphi = \{x \in M \mid d_x\varphi = 0\}$. The map φ is said to be horizontally weakly conformal or semi-conformal if for each $x \in M \setminus C_\varphi$, the restriction of $d_x\varphi$ to \mathcal{H}_x is surjective and conformal, where the horizontal space \mathcal{H}_x is the orthogonal complement of $\mathcal{V}_x = Ker d_x\varphi$. The horizontal conformality of φ implies that there exists a function $\lambda : M \setminus C_\varphi \longrightarrow \mathbb{R}_+$ such that for all $x \in M \setminus C_\varphi$ and $X, Y \in \mathcal{H}_x$

$$(2.1) \quad h(d_x\varphi(X), d_x\varphi(Y)) = \lambda(x)^2 g(X, Y).$$

The map φ is horizontally weakly conformal at x with dilation $\lambda(x)$ if and only if in any local coordinates (y^α) on a neighbourhood of $\varphi(x)$,

$$(2.2) \quad g(\text{grad}^M \varphi^\alpha, \text{grad}^M \varphi^\beta) = \lambda^2 (h^{\alpha\beta} \circ \varphi) \quad (\alpha, \beta = 1, \dots, n).$$

Let $f : M \times \mathbb{R} \longrightarrow (0, +\infty)$, $(x, t) \longmapsto f(x, t)$ be a smooth function.

Definition 2.1. A C^2 -function $u : U \longrightarrow \mathbb{R}$ defined on an open subset U of M is called f -harmonic if

$$(2.3) \quad \Delta_f^M u \equiv f_u \Delta^M u + du(\text{grad}^M f_u) - e(u)(f')_u = 0,$$

where $f_u : M \longrightarrow (0, +\infty)$ is a smooth function defined by

$$(2.4) \quad f_u(x) = f(x, u(x)), \quad x \in U,$$

$(f')_u : M \longrightarrow (0, +\infty)$ is a smooth function defined by

$$(2.5) \quad (f')_u(x) = \frac{\partial f}{\partial t}(x, u(x)), \quad x \in U.$$

Definition 2.2. The map $\varphi : (M, g) \longrightarrow (N, h)$ is called a f -harmonic morphism if, for every harmonic function $v : V \longrightarrow \mathbb{R}$ defined on an open subset V of N with $\varphi^{-1}(V)$ non-empty, the composition $v \circ \varphi$ is f -harmonic on $\varphi^{-1}(V)$.

Theorem 2.1. Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map. Let $f : M \times \mathbb{R} \longrightarrow (0, +\infty)$ be a smooth function. Then, the following are equivalent:

- (1) φ is an f -harmonic morphism;
- (2) φ is a horizontally weakly conformal with dilation λ satisfying

$$(2.6) \quad f_{\varphi^\alpha} \tau(\varphi)^\alpha + g(\text{grad}^M f_{\varphi^\alpha}, \text{grad}^M \varphi^\alpha) - \frac{1}{2} \lambda^2 (f')_{\varphi^\alpha} (h^{\alpha\alpha} \circ \varphi) = 0,$$

for all $\alpha = 1, \dots, n$ and in any local coordinates (y^α) on N ;

- (3) there exists a smooth positive function λ on M such that

$$\Delta_f^M (v \circ \varphi) = f_{v \circ \varphi} \lambda^2 (\Delta^N v) \circ \varphi,$$

for every smooth function $v : V \longrightarrow \mathbb{R}$ defined on an open subset V of N .

We will need the following lemma to prove the theorem.

Lemma 2.1 ([8]). Let y_0 be a point in N^n , let (y^γ) be normal coordinates on N centered at y_0 and let $\{c_\gamma, c_{\alpha\beta}\}_{\alpha, \beta, \gamma=1}^n$ be constants with $c_{\alpha\beta} = c_{\beta\alpha}$ and $\sum_\alpha c_{\alpha\alpha} = 0$. Then there exists a neighborhood V of y_0 in N and a harmonic function $v : V \longrightarrow \mathbb{R}$ such that

$$(2.7) \quad \frac{\partial v}{\partial y^\alpha}(y_0) = c_\alpha, \quad \frac{\partial^2 v}{\partial y^\alpha \partial y^\beta}(y_0) = c_{\alpha\beta},$$

for all $\alpha, \beta, \gamma = 1, \dots, n$.

PROOF OF THEOREM 2.1: Suppose $\varphi : (M^m, g) \longrightarrow (N^n, h)$ is a f -harmonic morphism. If $x_0 \in M$, consider systems of local coordinates (x^i) and (y^α) around $x_0, y_0 = \varphi(x_0)$, respectively, where we assume that (y^α) are normal, centered at y_0 . To prove the horizontal conformality of φ , we apply Lemma 2.1, that is, we may for every sequence $(c_{\alpha\beta})_{\alpha, \beta=1}^n$ with $c_{\alpha\beta} = c_{\beta\alpha}$ and $\sum_\alpha c_{\alpha\alpha} = 0$ choose a harmonic function v such that

$$(2.8) \quad \frac{\partial v}{\partial y^\alpha}(y_0) = 0, \quad \frac{\partial^2 v}{\partial y^\alpha \partial y^\beta}(y_0) = c_{\alpha\beta},$$

for all $\alpha, \beta = 1, \dots, n$. By assumption, the function $v \circ \varphi$ is f -harmonic in a neighbourhood of x_0 , so by Definition 2.1

$$(2.9) \quad \begin{aligned} 0 &= \Delta_f^M (v \circ \varphi) \\ &= f_{v \circ \varphi} \Delta^M (v \circ \varphi) + dv(d\varphi(\text{grad}^M f_{v \circ \varphi})) - e(v \circ \varphi) (f')_{v \circ \varphi}. \end{aligned}$$

In particular, since at x_0 we have

$$(2.10) \quad dv(d\varphi(\text{grad}^M f_{v \circ \varphi})) = 0,$$

we get

$$(2.11) \quad e(v \circ \varphi) = 0.$$

By (2.9), (2.10) and (2.11) we have

$$(2.12) \quad \begin{aligned} 0 &= \Delta^M(v \circ \varphi) \\ &= dv(\tau(\varphi)) + \text{trace}_g \nabla dv(d\varphi, d\varphi) \\ &= \text{trace}_g \nabla dv(d\varphi, d\varphi). \end{aligned}$$

Since at x_0 we have

$$(2.13) \quad \nabla dv = \sum_{\alpha, \beta} \frac{\partial^2 v}{\partial y^\alpha \partial y^\beta} dy^\alpha \otimes dy^\beta = \sum_{\alpha, \beta} c_{\alpha\beta} dy^\alpha \otimes dy^\beta,$$

by (2.8), (2.12) and (2.13), we obtain

$$(2.14) \quad \begin{aligned} 0 &= \sum_{\alpha, \beta} g(\text{grad}^M \varphi^\alpha, \text{grad}^M \varphi^\beta) c_{\alpha\beta} \\ &= \sum_{\alpha} g(\text{grad}^M \varphi^\alpha, \text{grad}^M \varphi^\alpha) c_{\alpha\alpha} + \sum_{\alpha \neq \beta} g(\text{grad}^M \varphi^\alpha, \text{grad}^M \varphi^\beta) c_{\alpha\beta}. \end{aligned}$$

We subtract

$$(2.15) \quad 0 = \sum_{\alpha} g(\text{grad}^M \varphi^1, \text{grad}^M \varphi^1) c_{\alpha\alpha}.$$

By (2.14) and (2.15), we obtain

$$(2.16) \quad \begin{aligned} 0 &= \sum_{\alpha} \left[g(\text{grad}^M \varphi^\alpha, \text{grad}^M \varphi^\alpha) - g(\text{grad}^M \varphi^1, \text{grad}^M \varphi^1) \right] c_{\alpha\alpha} \\ &\quad + \sum_{\alpha \neq \beta} g(\text{grad}^M \varphi^\alpha, \text{grad}^M \varphi^\beta) c_{\alpha\beta}. \end{aligned}$$

Let $\alpha_0 \neq 1$ and let

$$c_{\alpha\beta} = \begin{cases} 1, & \text{if } \alpha = \beta = 1; \\ -1, & \text{if } \alpha = \beta = \alpha_0; \\ 0, & \text{if } \alpha = \beta \neq 1, \alpha_0; \\ 0, & \text{if } \alpha \neq \beta. \end{cases}$$

Then by (2.16), we have

$$(2.17) \quad g(\text{grad}^M \varphi^{\alpha_0}, \text{grad}^M \varphi^{\alpha_0}) = g(\text{grad}^M \varphi^1, \text{grad}^M \varphi^1).$$

Then

$$(2.18) \quad g(\text{grad}^M \varphi^\alpha, \text{grad}^M \varphi^\alpha) = g(\text{grad}^M \varphi^1, \text{grad}^M \varphi^1),$$

for all $\alpha = 1, \dots, n$. Let $\alpha_0 \neq \beta_0$ and let

$$c_{\alpha\beta} = \begin{cases} 1, & \text{if } \alpha = \alpha_0 \text{ and } \beta = \beta_0; \\ 0, & \text{if } \alpha \neq \alpha_0 \text{ or } \beta \neq \beta_0; \\ 0, & \text{if } \alpha = \beta. \end{cases}$$

Then by (2.16), we have

$$(2.19) \quad g(\text{grad}^M \varphi^{\alpha_0}, \text{grad}^M \varphi^{\beta_0}) = 0.$$

So we have

$$(2.20) \quad g(\text{grad}^M \varphi^\alpha, \text{grad}^M \varphi^\beta) = 0,$$

for all $\alpha \neq \beta = 1, \dots, n$. It follows from (2.18) and (2.20) that the f -harmonic morphism φ is horizontally weakly conformal map

$$(2.21) \quad g(\text{grad}^M \varphi^\alpha, \text{grad}^M \varphi^\beta) = \lambda^2 \delta_{\alpha\beta},$$

for all $\alpha, \beta = 1, \dots, n$. For every C^2 -function $v : V \rightarrow \mathbb{R}$ defined on an open subset V of N , we have

$$(2.22) \quad \begin{aligned} \Delta_f^M(v \circ \varphi) &= f_{v \circ \varphi} \Delta^M(v \circ \varphi) + dv(d\varphi(\text{grad}^M f_{v \circ \varphi})) - e(v \circ \varphi)(f')_{v \circ \varphi} \\ &= f_{v \circ \varphi} dv(\tau(\varphi)) + f_{v \circ \varphi} \text{trace}_g \nabla dv(d\varphi, d\varphi) \\ &\quad + dv(d\varphi(\text{grad}^M f_{v \circ \varphi})) - e(v \circ \varphi)(f')_{v \circ \varphi}. \end{aligned}$$

Since φ is horizontally weakly conformal map, we obtain

$$(2.23) \quad \begin{aligned} \Delta_f^M(v \circ \varphi) &= f_{v \circ \varphi} dv(\tau(\varphi)) + f_{v \circ \varphi} \lambda^2 (\Delta^N v) \circ \varphi \\ &\quad + dv(d\varphi(\text{grad}^M f_{v \circ \varphi})) - e(v \circ \varphi)(f')_{v \circ \varphi}. \end{aligned}$$

By choosing v to be a harmonic function and since φ is an f -harmonic morphism, we conclude that

$$f_{v \circ \varphi} dv(\tau(\varphi)) + dv(d\varphi(\text{grad}^M f_{v \circ \varphi})) - e(v \circ \varphi)(f')_{v \circ \varphi} = 0,$$

i.e. in any local coordinates (y^α) on N , we have

$$f_{\varphi^\alpha} \tau(\varphi)^\alpha + g(\text{grad}^M f_{\varphi^\alpha}, \text{grad}^M \varphi^\alpha) - \frac{1}{2} \lambda^2 (f')_{\varphi^\alpha} (h^{\alpha\alpha} \circ \varphi) = 0,$$

for all $\alpha = 1, \dots, n$.

Thus, we obtain the implication (1) \implies (2). Furthermore, the implication (2) \implies (3) follows from the formula (2.23). The implication (3) \implies (1) is trivial. \square

Example 2.1. The identity map $Id : (\mathbb{R}^m, \langle \cdot, \cdot \rangle_{\mathbb{R}^m}) \longrightarrow (\mathbb{R}^m, \langle \cdot, \cdot \rangle_{\mathbb{R}^m})$ is f -harmonic morphism if f satisfies the system of differential equation

$$(2.24) \quad \frac{\partial f}{\partial x^i} + \frac{1}{2} \frac{\partial f}{\partial t} = 0,$$

for all $i = 1, \dots, m$, where $f \in C^\infty(\mathbb{R}^m \times \mathbb{R})$ is a smooth positive function. Let $F \in C^\infty(\mathbb{R}^m)$ be a smooth positive function, then the function of the type $f(x^1, \dots, x^m, t) = F(t - \frac{1}{2}x^1, \dots, t - \frac{1}{2}x^m)$, satisfies the system of differential equation (2.24).

If $f(x, t) = 1$ for all $(x, t) \in M \times \mathbb{R}$, the condition (2.6) is equivalent to the condition $\tau(\varphi) = 0$ i.e. φ is harmonic. We arrive at the following corollary.

Corollary 2.1 ([6], [8]). *A smooth map $\varphi : M \longrightarrow N$ between Riemannian manifolds is a harmonic morphism if and only if $\varphi : M \longrightarrow N$ is both harmonic and horizontally weakly conformal.*

If $f(x, t) = f_1(x)$ for all $(x, t) \in M \times \mathbb{R}$, where $f_1 \in C^\infty(M)$ is a smooth positive function, the condition (2.6) is equivalent to the condition $f_1 \tau(\varphi) + d\varphi(\text{grad}^M f_1) = 0$ i.e. φ is f_1 -harmonic. We arrive at the following corollary.

Corollary 2.2 ([10]). *A smooth map $\varphi : M \longrightarrow N$ between Riemannian manifolds is a f_1 -harmonic morphism if and only if $\varphi : M \longrightarrow N$ is both f_1 -harmonic and horizontally weakly conformal with $f_1 \in C^\infty(M)$ being a smooth positive function.*

Let $f : M \times \mathbb{R} \longrightarrow (0, +\infty)$, $(x, t) \longmapsto f(x, t)$ be a smooth function.

Corollary 2.3. *Let $\varphi : M \longrightarrow N$ be an f -harmonic morphism between Riemannian manifolds with dilation λ_1 and $\psi : N \longrightarrow P$ a harmonic morphism between Riemannian manifolds with dilation λ_2 . Then the composition $\psi \circ \varphi : M \longrightarrow P$ is an f -harmonic morphism with dilation $\lambda_1(\lambda_2 \circ \varphi)$.*

PROOF: This follows from the fact that

$$\Delta_f^M(v \circ \varphi) = f_{v \circ \varphi} \lambda_1^2(\Delta^N v) \circ \varphi,$$

for every smooth function $v : V \longrightarrow \mathbb{R}$ defined on an open subset V of N , and

$$\Delta^N(u \circ \psi) = \lambda_2^2(\Delta^P u) \circ \psi,$$

for every smooth function $u : U \longrightarrow \mathbb{R}$ defined on an open subset U of P . So that

$$\begin{aligned} \Delta_f^M(u \circ \psi \circ \varphi) &= f_{u \circ \psi \circ \varphi} \lambda_1^2(\Delta^N(u \circ \psi)) \circ \varphi \\ &= f_{u \circ \psi \circ \varphi} \lambda_1^2(\lambda_2 \circ \varphi)^2(\Delta^P u) \circ \psi \circ \varphi. \end{aligned}$$

□

Corollary 2.4. *Let $\varphi : (M, g) \longrightarrow (N, h)$ be a smooth map of two Riemannian manifolds. If $f(x, t) = f_1(x) f_2(t)$ for all $(x, t) \in M \times \mathbb{R}$, where $f_1 \in C^\infty(M)$ is a smooth positive function and $f_2 \in C^\infty(\mathbb{R})$ is a smooth positive function. Then, the following are equivalent:*

- (1) φ is an f -harmonic morphism;
- (2) φ is a horizontally weakly conformal with dilation λ satisfying

$$(2.25) \quad (f_2 \circ \varphi^\alpha) \tau_{f_1}(\varphi)^\alpha + \frac{1}{2} \lambda^2 f_1(f_2' \circ \varphi^\alpha)(h^{\alpha\alpha} \circ \varphi) = 0,$$

for all $\alpha = 1, \dots, n$ and in any local coordinates (y^α) on N .

PROOF: By Theorem 2.1, the map $\varphi : (M, g) \longrightarrow (N, h)$ is f -harmonic morphism if and only if $\varphi : (M, g) \longrightarrow (N, h)$ is a horizontally weakly conformal with dilation λ satisfying the condition

$$f_{\varphi^\alpha} \tau(\varphi)^\alpha + g(\text{grad}^M f_{\varphi^\alpha}, \text{grad}^M \varphi^\alpha) - \frac{1}{2} \lambda^2 (f')_{\varphi^\alpha} (h^{\alpha\alpha} \circ \varphi) = 0,$$

for all $\alpha = 1, \dots, n$, and in any local coordinates (y^α) on N , i.e.

$$(2.26) \quad \begin{aligned} & f_1(f_2 \circ \varphi^\alpha) \tau(\varphi)^\alpha + f_1 g(\text{grad}^M (f_2 \circ \varphi^\alpha), \text{grad}^M \varphi^\alpha) \\ & + (f_2 \circ \varphi^\alpha) g(\text{grad}^M f_1, \text{grad}^M \varphi^\alpha) - \frac{1}{2} \lambda^2 f_1(f_2' \circ \varphi^\alpha)(h^{\alpha\alpha} \circ \varphi) = 0, \end{aligned}$$

because $f_{\varphi^\alpha} = f_1(f_2 \circ \varphi^\alpha)$.

Let $\tau_{f_1}(\varphi) = f_1 \tau(\varphi) + d\varphi(\text{grad}^M f_1)$ be the f_1 -tension field of φ , then one has

$$(2.27) \quad \tau_{f_1}(\varphi)^\alpha = f_1 \tau(\varphi)^\alpha + g(\text{grad}^M f_1, \text{grad}^M \varphi^\alpha).$$

By (2.26) and (2.27), we obtain

$$(2.28) \quad \begin{aligned} & (f_2 \circ \varphi^\alpha) \tau_{f_1}(\varphi)^\alpha + f_1 g(\text{grad}^M (f_2 \circ \varphi^\alpha), \text{grad}^M \varphi^\alpha) \\ & - \frac{1}{2} \lambda^2 f_1(f_2' \circ \varphi^\alpha)(h^{\alpha\alpha} \circ \varphi) = 0, \end{aligned}$$

the second term on the left-hand side of (2.28) is

$$\begin{aligned} f_1 g(\text{grad}^M (f_2 \circ \varphi^\alpha), \text{grad}^M \varphi^\alpha) &= f_1 (f_2' \circ \varphi^\alpha) g(\text{grad}^M \varphi^\alpha, \text{grad}^M \varphi^\alpha) \\ &= \lambda^2 f_1 (f_2' \circ \varphi^\alpha) (h^{\alpha\alpha} \circ \varphi). \end{aligned}$$

□

In the case where $f_2 = 1$, we recover the result obtained by Y.L. Ou [10] of f_1 -harmonic morphisms (in the sense of A. Lichnerowicz [9] and N. Course [3]).

Proposition 2.1. *Let (M, g) be a Riemannian manifold. A smooth map*

$$\varphi : (M, g) \longrightarrow (\mathbb{R}^n, \langle \cdot, \cdot \rangle_{\mathbb{R}^n}), \quad x \longmapsto (\varphi^1(x), \dots, \varphi^n(x))$$

is an f -harmonic morphism if and only if its components φ^α are f -harmonic functions whose gradients are orthogonal and of the same norm at each point.

PROOF: Let us notice that the condition (2.6) of Theorem 2.1 becomes

$$f_{\varphi^\alpha} \Delta^M \varphi^\alpha + g(\text{grad}^M f_{\varphi^\alpha}, \text{grad}^M \varphi^\alpha) - e(\varphi^\alpha)(f')_{\varphi^\alpha} = 0,$$

for all $\alpha = 1, \dots, n$, i.e. the functions φ^α are f -harmonic. \square

Proposition 2.2. *Let $\varphi : (M, g) \longrightarrow (\mathbb{R}^n, \langle \cdot, \cdot \rangle_{\mathbb{R}^n})$ be a harmonic morphism of two Riemannian manifolds. Then $\varphi : (M, g) \longrightarrow (\mathbb{R}^n, \langle \cdot, \cdot \rangle_{\mathbb{R}^n})$ is f -harmonic morphism with $f(x, t) = f_1(x) e^{t+c}$ for all $(x, t) \in M \times \mathbb{R}$ and $f_1 \in C^\infty(M)$ being a smooth positive function defined by the components of φ as follows*

$$f_1 = e^{-\frac{1}{2}(\varphi^1 + \dots + \varphi^n)},$$

where $c \in \mathbb{R}_+$.

PROOF: The map $\varphi : (M, g) \longrightarrow (\mathbb{R}^n, \langle \cdot, \cdot \rangle_{\mathbb{R}^n})$ where $\varphi = (\varphi^1, \dots, \varphi^n)$ is harmonic morphism if and only if it is harmonic horizontally and weakly conformal with dilation λ . Let $f_1 = e^{-\frac{1}{2}(\varphi^1 + \dots + \varphi^n)}$, so that

$$\tau_{f_1}(\varphi)^\alpha = f_1 \tau(\varphi)^\alpha + g(\text{grad}^M f_1, \text{grad}^M \varphi^\alpha) = g(\text{grad}^M f_1, \text{grad}^M \varphi^\alpha),$$

because φ is harmonic. One has

$$\begin{aligned} \text{grad}^M f_1 &= -\frac{1}{2} e^{-\frac{1}{2}(\varphi^1 + \dots + \varphi^n)} (\text{grad}^M \varphi^1 + \dots + \text{grad}^M \varphi^n) \\ &= -\frac{1}{2} f_1 (\text{grad}^M \varphi^1 + \dots + \text{grad}^M \varphi^n). \end{aligned}$$

So we get

$$\tau_{f_1}(\varphi)^\alpha = -\frac{1}{2} f_1 \left(g(\text{grad}^M \varphi^1, \text{grad}^M \varphi^\alpha) + \dots + g(\text{grad}^M \varphi^n, \text{grad}^M \varphi^\alpha) \right).$$

Since φ is horizontally and weakly conformal with dilation λ , we obtain

$$(2.29) \quad \tau_{f_1}(\varphi)^\alpha = -\frac{1}{2} \lambda^2 f_1 (\langle \cdot, \cdot \rangle_{\mathbb{R}^n})^{\alpha\alpha} \circ \varphi = -\frac{1}{2} \lambda^2 f_1.$$

Let $f(x, t) = f_1(x) e^{t+c}$ for all $(x, t) \in M \times \mathbb{R}$, where $c \in \mathbb{R}_+$. Then the condition (2.25) is equivalent to (2.29). Finally, by Corollary 2.4 the map φ is f -harmonic morphism. \square

Example 2.2. Let (M, g) be a Riemannian manifold, $\gamma : M \longrightarrow (0, \infty)$ be a smooth function and let $M \times_{\gamma^2} \mathbb{R}^n$ be the warped product equipped with the Riemannian metric $G_\gamma = g + \gamma^2 \langle \cdot, \cdot \rangle_{\mathbb{R}^n}$. The natural projection

$$\pi_2 : (M \times_{\gamma^2} \mathbb{R}^n, G_\gamma) \longrightarrow (\mathbb{R}^n, \langle \cdot, \cdot \rangle_{\mathbb{R}^n}),$$

is harmonic morphism ([2]). According to Proposition 2.2 the natural projection π_2 is f -harmonic morphism with

$$f(x, y_1, \dots, y_n, t) = e^{-\frac{1}{2}(y^1 + \dots + y^n) + t + c}, \quad c \in \mathbb{R}_+$$

for all $(x, y_1, \dots, y_n, t) \in M \times \mathbb{R}^n \times \mathbb{R}$.

Example 2.3. Let $H^m = (\mathbb{R}^{m-1} \times \mathbb{R}_+^*, \frac{1}{x_m^2} \langle \cdot, \cdot \rangle_{\mathbb{R}^m})$. The projection

$$\pi_1 : H^m \longrightarrow (\mathbb{R}^{m-1}, \langle \cdot, \cdot \rangle_{\mathbb{R}^{m-1}}), \quad (x_1, \dots, x_{m-1}, x_m) \longmapsto a(x_1, \dots, x_{m-1}),$$

where $a \in \mathbb{R} \setminus \{0\}$ is harmonic morphism ([2]). According to Proposition 2.2 the projection π_1 is f -harmonic morphism with

$$f(x_1, \dots, x_{m-1}, x_m, t) = e^{-\frac{a}{2}(x_1 + \dots + x_{m-1}) + t + c}, \quad c \in \mathbb{R}_+$$

for all $(x_1, \dots, x_{m-1}, x_m, t) \in H^m \times \mathbb{R}$.

Example 2.4. (1) Let $\varphi : (\mathbb{R}^2 \setminus \{0\}, \langle \cdot, \cdot \rangle_{\mathbb{R}^2}) \longrightarrow (\mathbb{R}^2 \setminus \{0\}, \langle \cdot, \cdot \rangle_{\mathbb{R}^2})$ be defined by

$$\varphi(x, y) = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right).$$

Then φ is a horizontally and weakly conformal map with dilation $\lambda(x, y) = \frac{1}{x^2 + y^2}$, and φ is f -harmonic morphism with

$$f(x, y, t) = F\left(2t - \frac{x + y}{x^2 + y^2}\right),$$

where $F : \mathbb{R} \longrightarrow (0, \infty)$ is a smooth function. Indeed, we have

$$\varphi^1(x, y) = \frac{x}{x^2 + y^2}, \quad \varphi^2(x, y) = \frac{y}{x^2 + y^2}, \quad f_{\varphi^1}(x, y) = F\left(\frac{x - y}{x^2 + y^2}\right),$$

$$f_{\varphi^2}(x, y) = F\left(\frac{y - x}{x^2 + y^2}\right), \quad \Delta^{\mathbb{R}^2} \varphi^1 = \Delta^{\mathbb{R}^2} \varphi^2 = 0,$$

$$\text{grad}^{\mathbb{R}^2} \varphi^1 = \left(\frac{y^2 - x^2}{(x^2 + y^2)^2}, -\frac{2xy}{(x^2 + y^2)^2} \right),$$

$$\text{grad}^{\mathbb{R}^2} \varphi^2 = \left(-\frac{2xy}{(x^2 + y^2)^2}, \frac{x^2 - y^2}{(x^2 + y^2)^2} \right),$$

$$\text{grad}^{\mathbb{R}^2} f_{\varphi^1} = F'\left(\frac{x - y}{x^2 + y^2}\right) \left(\frac{-x^2 + y^2 + 2xy}{(x^2 + y^2)^2}, -\frac{x^2 - y^2 + 2xy}{(x^2 + y^2)^2} \right),$$

$$\text{grad}^{\mathbb{R}^2} f_{\varphi^2} = F'\left(\frac{y - x}{x^2 + y^2}\right) \left(\frac{x^2 - y^2 - 2xy}{(x^2 + y^2)^2}, \frac{x^2 - y^2 + 2xy}{(x^2 + y^2)^2} \right),$$

$$\langle \text{grad}^{\mathbb{R}^2} \varphi^1, \text{grad}^{\mathbb{R}^2} f_{\varphi^1} \rangle_{\mathbb{R}^2} = \frac{F' \left(\frac{x-y}{x^2+y^2} \right)}{(x^2+y^2)^2},$$

$$\langle \text{grad}^{\mathbb{R}^2} \varphi^2, \text{grad}^{\mathbb{R}^2} f_{\varphi^2} \rangle_{\mathbb{R}^2} = \frac{F' \left(\frac{y-x}{x^2+y^2} \right)}{(x^2+y^2)^2},$$

$$e(\varphi^1) = e(\varphi^2) = \frac{1}{2(x^2+y^2)^2}, \quad (f')_{\varphi^1} = 2F' \left(\frac{x-y}{x^2+y^2} \right), \quad (f')_{\varphi^2} = 2F' \left(\frac{y-x}{x^2+y^2} \right).$$

By (2.3) the functions φ^1 and φ^2 are f -harmonic and by Proposition 2.1 the map φ is f -harmonic morphism. With the same method we find that:

(2) Let $\psi : (\mathbb{R}^3 \setminus \{0\}, \langle \cdot, \cdot \rangle_{\mathbb{R}^3}) \longrightarrow (\mathbb{R}^3 \setminus \{0\}, \langle \cdot, \cdot \rangle_{\mathbb{R}^3})$ be defined by

$$\psi(x, y, z) = \left(\frac{x}{x^2+y^2+z^2}, \frac{y}{x^2+y^2+z^2}, \frac{z}{x^2+y^2+z^2} \right).$$

Then ψ is f -harmonic morphism with

$$f(x, y, z, t) = \frac{F \left(2t - \frac{x+y+z}{x^2+y^2+z^2} \right)}{x^2+y^2+z^2},$$

where $F : \mathbb{R} \longrightarrow (0, \infty)$ is a smooth function. Here ψ is a horizontally and weakly conformal map with dilation $\lambda(x, y, z) = \frac{1}{x^2+y^2+z^2}$.

Remark 2.1. Using Proposition 2.1, we can construct many examples for f -harmonic morphisms (in a generalized sense).

Proposition 2.2 remains true for the map $\varphi : (M, g) \longrightarrow (N, h)$, where N is an open subsets of \mathbb{R}^n and $h = e^{\alpha(y)} \langle \cdot, \cdot \rangle_{\mathbb{R}^n}$ is a metric conformally equivalent to the standard inner product on \mathbb{R}^n .

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LABORATORY OF GEOMETRY, ANALYSIS, CONTROLE AND APPLICATIONS, SAIDA UNIVERSITY, ALGERIA

E-mail: Ahmedcherif29@Hotmail.fr

DEPARTMENT OF MATHEMATICS, RELIZANE UNIVERSITY, ALGERIA

E-mail: Djaamustapha@Live.com

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