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## Paratopological (topological) groups with certain networks

CHUAN LIU

*Abstract.* In this paper, we discuss certain networks on paratopological (or topological) groups and give positive or negative answers to the questions in [13]. We also prove that a non-locally compact,  $k$ -gentle paratopological group is metrizable if its remainder (in the Hausdorff compactification) is a Fréchet-Urysohn space with a point-countable  $cs^*$ -network, which improves some theorems in [Liu C., *Metrizability of paratopological (semitopological) groups*, Topology Appl. **159** (2012), 1415–1420], [Liu C., Lin S., *Generalized metric spaces with algebraic structures*, Topology Appl. **157** (2010), 1966–1974].

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*Classification:* 54E20, 54E35, 54H11

### 1. Introduction

Recall that a *topological group*  $G$  is a group  $G$  with a (Hausdorff) topology such that the product map of  $G \times G$  into  $G$  is jointly continuous and the inverse map of  $G$  onto itself associating  $x^{-1}$  with arbitrary  $x \in G$  is continuous. A *paratopological group*  $G$  is a group  $G$  with a topology such that the product map of  $G \times G$  into  $G$  is jointly continuous. A *semitopological group* is a group with a topology such that the product map of  $G \times G$  into  $G$  is separately continuous. A *quasitopological group* is a semitopological group and the inverse map is continuous.

Let  $X$  be a topological space and  $F$  is a subset of  $X$ ,  $F$  is called a *sequential neighborhood* of  $x$  in  $X$  if every sequence converging to  $x$  is eventually in  $F$ .  $F$  is a *sequentially open* subset of  $X$  if  $F$  is a sequential neighborhood of  $x$  for each  $x \in F$ .

**Definition 1.1.** Let  $\mathcal{P} = \bigcup_{x \in X} \mathcal{P}_x$  be a cover of a space  $X$  such that for each  $x \in X$ , (a) if  $U, V \in \mathcal{P}_x$ , then  $W \subset U \cap V$  for some  $W \in \mathcal{P}_x$ ; (b) the family  $\mathcal{P}_x$  is a network of  $x$  in  $X$ , i.e.,  $x \in \bigcap \mathcal{P}_x$ , and if  $x \in U$  with  $U$  open in  $X$ , then  $P \subset U$  for some  $P \in \mathcal{P}_x$ .

(1) The family  $\mathcal{P}$  is called a *sn-network* (*sequential neighborhood network*) for  $X$  [12] if each element of  $\mathcal{P}_x$  is sequential neighborhood of  $x$  for all  $x \in X$ .  $X$  is called *snf-countable* if  $X$  has a *sn-network*  $\mathcal{P}$  such that each  $\mathcal{P}_x$  is countable.

(2) The family  $\mathcal{P}$  is called a *so-network* (*sequentially open network*) [12] for  $X$  if each element of  $\mathcal{P}_x$  is a sequentially open neighborhood of  $X$ .  $X$  is called *sof-countable* if  $X$  has a *so-network*  $\mathcal{P}$  such that each  $\mathcal{P}_x$  is countable.

(3) Fix  $x \in X$ ,  $\mathcal{P}_x$  is said to be a *strong so-network* at  $x$  if  $\mathcal{P}_x$  is a so-network at  $x$ , and for any sequential open set  $W$  with  $x \in W$ , there is a  $P \in \mathcal{P}_x$  such that  $x \in P \subset W$ .

(4) The family  $\mathcal{P}$  is called a *weak base* [1] for  $X$  if for every  $A \subset X$ , the set  $A$  is open in  $X$  whenever for each  $x \in A$  there exists  $P \in \mathcal{P}_x$  such that  $P \subset A$ .  $X$  is called *weakly first-countable* if for each  $x \in X$ ,  $\mathcal{P}_x$  is countable.

We can see that first-countable  $\rightarrow$  sof-countable  $\rightarrow$  snf-countable; first-countable  $\rightarrow$  weakly first-countable  $\rightarrow$  snf-countable. A sequential, snf-countable (sof-countable) space is weakly first-countable (first-countable).

In this paper, we consider the following questions.

**Question 1.2** ([13, Question 4.1]). *Let  $G$  be a snf-countable semitopological group or quasitopological group. Is  $G$  sof-countable?*

**Question 1.3** ([13, Question 4.3]). *Let  $G$  be a topological group. Is  $\sigma G$  a topological group?*

**Question 1.4** ([13, Question 4.5]). *Is every snf-countable topological group an  $\aleph$ -space?*

**Question 1.5** ([13, Question 4.6]). *Does every snf-countable  $\omega$ -narrow topological group have a countable sn-network?*

**Question 1.6** ([13, Question 4.12]). *Let  $G$  be a paratopological group with a  $G_\delta$ -diagonal. If  $G$  is a  $wM$ -space, is it metrizable?*

We shall give positive answers to Question 1.6 (when  $G$  is regular) and negative answers to Questions 1.2, 1.4, 1.5. Ordman and Smith-Thomas [18] gave an example that the sequential coreflection of a topological group is not a topological group, it implies the answer of Question 1.3 is negative, we present another example for Question 1.3 and give a sufficient and necessary condition for  $\sigma G$  to be a topological group in terms of strong so-network.

By a remainder of a space  $X$  we mean the subspace  $bX \setminus X$  of a Hausdorff compactification  $bX$  of  $X$ . Arhangel'skii [2] proved that if the remainder of a Hausdorff compactification of a non-locally compact topological group  $G$  has a point-countable base, then  $G$  and  $bG$  are separable and metrizable. It is natural to ask if Arhangel'skii's result is still valid for a paratopological group. The author [15] proved that Arhangel'skii theorem is valid for a  $k$ -gentle paratopological group. We could improve the above result by replacing "point-countable base" with "Fréchet-Urysohn space with a point-countable  $cs^*$ -network".

All spaces are Hausdorff unless stated otherwise. The notations  $\mathbb{N}, \mathbb{Q}, \mathbb{R}$  denote natural numbers, rational numbers and real numbers respectively. The letter  $e$  denotes the neutral element of a group.  $F(X)$  is a free group on  $X$ . Readers may refer to [2], [7], [10] for notations and terminology not explicitly given here.

## 2. Main results

Let  $X$  be a topological space, a function  $d : X \times X \rightarrow \mathbb{R}^+$  is a *symmetric* on the set  $X$  if for  $x, y \in X$

- (1)  $d(x, y) = 0$  if and only if  $x = y$ ,
- (2)  $d(x, y) = d(y, x)$ .

A space  $X$  is said to be *symmetrizable* if there is a symmetric  $d$  on  $X$  satisfying the following condition:  $U \subset X$  is open if and only if for each  $x \in U$ , there exists  $\epsilon > 0$  with  $B(x, \epsilon) \subset U$ . Here  $B(x, \epsilon) = \{y \in x : d(x, y) \in \epsilon\}$ .

*Example 2.1.* There is a separable, snf-countable quasitopological group that is not sof-countable.

PROOF: Let  $G = \mathbb{R}^2$  with usual addition “+”, then  $(G, +)$  is a group. Define  $d : G \times G \rightarrow \mathbb{R}^+ \cup \{0\}$  as follows:

$$d((x, y), (x', y')) = \begin{cases} |x - x'|, & x \neq x', y = y'; \\ |y - y'|, & x = x', y \neq y'; \\ 0, & x = x', y = y'; \\ 1, & \text{otherwise.} \end{cases}$$

It is easy to check that  $d(x, y)$  is a symmetric and  $(G, +)$  is a separable, quasitopological group.  $G$  is weakly first-countable, in fact, for each  $x \in G$ , let  $\mathcal{P}_x = \{B(x, 1/n) : n \in \mathbb{N}\}$ , where  $B(x, 1/n) = \{y \in G : d(x, y) < 1/n\}$ .

It is easy to see that  $(0,0)$  is a cluster point of  $\{(r_1, r_2) : r_1, r_2 \in \mathbb{Q}^+\}$ , where  $\mathbb{Q}^+ = \{r \in \mathbb{Q} : r > 0\}$ . If  $G$  is first-countable, then there is a sequence  $\{s_n : n \in \mathbb{N}\} \subset \{(r_1, r_2) : r_1, r_2 \in \mathbb{Q}^+\}$  such that  $s_n \rightarrow (0, 0)$ .  $d(s_n, (0, 0)) \rightarrow 0$  by [10, Lemma 9.3]. This is a contradiction since  $d(s_n, (0, 0)) = 1$ . Hence  $G$  is not first-countable. Therefore,  $G$  is not sof-countable since a sof-countable sequential space is first-countable. □

The proof of the following proposition is based on the idea in [13].

**Proposition 2.2.** *Let  $G$  be a paratopological group satisfying the condition (w): for any two sn-networks  $\{U_\alpha(e) : \alpha \in \Gamma\}$ ,  $\{V_\beta(e) : \beta \in \Gamma\}$  at  $e$  and for any  $\alpha \in \Gamma$ , there exists  $\beta \in \Gamma$  such that  $V_\beta(e) \subset U_\alpha(e)$ . Then there is a so-network  $\{W_\alpha(e) : \alpha \in \Gamma\}$  at  $e$  and for each  $\alpha \in \Gamma$ , there exists  $\beta \in \Gamma$  such that  $W_\beta(e)W_\beta(e) \subset W_\alpha(e)$ .*

PROOF: Since  $G$  is a paratopological group,  $\{U_\alpha(e)U_\alpha(e) : \alpha \in \Gamma\}$  is still a sn-network at  $e$ . Let  $W_\alpha(e) = \{x \in U_\alpha(e) : xU_\beta(e) \subset U_\alpha(e) \text{ for some } \beta \in \Gamma\} \subset U_\alpha(e)$ . So  $e \in W_\alpha(e)$  for each  $\alpha$ , then  $\{W_\alpha(e) : \alpha \in \Gamma\}$  is a network at  $e$  and satisfies the condition (a) in Definition 1.1, in fact, for any  $W_\alpha(e), W_\beta(e)$ , let  $U_\gamma(e) \subset U_\alpha(e) \cap U_\beta(e)$ ,  $W_\gamma(e) = \{x \in U_\gamma(e) : xU_\delta(e) \subset U_\gamma(e)\}$ , then  $W_\gamma(e) \subset W_\alpha(e) \cap W_\beta(e)$ . We prove that each  $W_\alpha(e)$  is sequentially open. For  $y \in W_\alpha(e)$  and  $\{y_n\}$  is a sequence converging to  $y$ ,  $yU_\beta(e) \subset U_\alpha(e)$ . By the condition (w), we choose  $\gamma \in \Gamma$  such that  $U_\gamma(e)U_\gamma(e) \subset U_\beta(e)$ .  $(yU_\gamma(e))U_\gamma(e) \subset yU_\beta(e) \subset U_\alpha(e)$ , which implies  $yU_\gamma(e) \subset W_\alpha(e)$ . Since  $yU_\gamma(e)$  is a sequential neighborhood of  $y$ ,

then  $\{y_n\}$  is eventually in  $yU_\gamma(e)$ , hence  $\{y_n\}$  is eventually in  $W_\alpha(e)$  and  $W_\alpha(e)$  is sequentially open. For  $\alpha \in \Gamma$ , choose  $\beta \in \Gamma$  so that  $U_\beta(e)U_\beta(e) \subset U_\alpha(e)$ . For  $y, z \in W_\beta(e) = \{x \in U_\beta(e) : xU_\gamma(e) \subset U_\beta(e) \text{ for some } \gamma \in \Gamma\} \subset U_\beta(e)$  we have  $yU_\gamma(e) \subset U_\beta(e)$ ,  $zU_\gamma(e) \subset U_\beta(e)$ , then  $yzU_\gamma(e) \subset yU_\gamma(e)zU_\gamma(e) \subset U_\beta(e)U_\beta(e) \subset U_\alpha(e)$ , that implies  $yz \in W_\alpha(e)$ , and hence  $W_\beta(e)W_\beta(e) \subset W_\alpha(e)$ .  $\square$

**Lemma 2.3.** *Let  $\{U_n : n \in \mathbb{N}\}$  be a decreasing countable network at  $x$  and  $W$  be sequential neighborhood of  $x$ , then there exists  $n_0 \in \mathbb{N}$  such that  $U_{n_0} \subset W$ .*

PROOF: Suppose not,  $U_n \setminus W \neq \emptyset$  and pick  $x_n \in U_n \setminus W$ . Then  $x_n \rightarrow x$  and  $\{x_n\} \cap W = \emptyset$ . This is a contradiction since  $W$  is a sequential neighborhood of  $x$ .  $\square$

Note that if  $G$  is snf-countable, we may assume  $G$  has a decreasing countable sn-network. By Lemma 2.3, a snf-countable paratopological group satisfies the condition (w).

**Corollary 2.4** ([13, Theorem 3.4]). *Every snf-countable paratopological group  $G$  is sof-countable.*

Since a weakly first-countable space is a sequential snf-countable space and a sequential sof-countable space is first-countable, we have the following.

**Corollary 2.5.** *Let  $G$  be a weakly first-countable paratopological group. Then  $G$  is first-countable.*

**Definition 2.6.** Let  $(X, \tau)$  be a space. A *sequential closure topology*  $\sigma_\tau$  [8] on  $X$  is defined as follows:  $O \in \sigma_\tau$  if and only if  $O$  is a sequentially open subset in  $(X, \tau)$ . The topological space  $(X, \sigma_\tau)$  is denoted by  $\sigma X$ .

Obviously,  $\sigma X$  is a sequential space for any space  $X$ . If  $G$  is a topological group, it is easy to see that  $\sigma G$  is a quasitopological group.

**Theorem 2.7.** *Let  $G$  be a paratopological group. Then  $\sigma G$  is a paratopological group if and only if  $G$  has a strong so-network  $\mathcal{P}_e$  at  $e$  satisfying the condition (\*): for each  $P_1 \in \mathcal{P}_e$ , there is a  $P_2 \in \mathcal{P}_e$  such that  $P_2P_2 \subset P_1$ .*

PROOF: Necessity: Let  $\{V_\alpha(e) : \alpha \in \Gamma\}$  be the local base at  $e$  in  $\sigma G$ , and let  $W$  be a sequentially open neighborhood of  $G$  with  $e \in W$ , then  $W$  is open in  $\sigma G$ , there is a  $V_\beta(e) \in \{V_\alpha(e) : \alpha \in \Gamma\}$  such that  $V_\beta(e) \subset W$ . Since  $\{V_\alpha(e) : \alpha \in \Gamma\}$  is a so-network at  $e$  in  $G$ , then  $\{V_\alpha(e) : \alpha \in \Gamma\}$  is a strong so-network at  $e$  in  $G$ . Since  $\sigma G$  is a paratopological group and  $\{V_\alpha(e) : \alpha \in \Gamma\}$  is the local base at  $e$ , it is easy to see that the condition (\*) is satisfied.

Sufficiency: Suppose  $G$  has a strong so-network  $\{V_\alpha(e) : \alpha \in \Gamma\}$  at  $e$  such that for each  $V_\alpha(e)$ ,  $V_\beta(e)V_\beta(e) \subset V_\alpha(e)$ . Fix  $a, b \in G$ , and let  $U$  be an open neighborhood (in  $\sigma G$ ) of  $ab$ . Since  $(ab)^{-1}U$  is a sequentially open neighborhood of  $e$  in  $G$ , there is a  $V \in \{V_\alpha(e) : \alpha \in \Gamma\}$  such that  $V \subset (ab)^{-1}U$ , then  $abV \subset U$ . Let  $W, W' \in \{V_\alpha(e) : \alpha \in \Gamma\}$  such that  $WW \subset V$ ,  $W' \subset W$  and  $W'b \subset bW$  (note that  $e \in bWb^{-1}$  is sequentially open in  $G$ ). Then  $aW', bW'$  are open neighborhoods of  $a, b$  in  $\sigma G$  respectively,  $aW'bW' \subset abWW' \subset abWW \subset abV \subset U$ .  $\square$

**Corollary 2.8.** *Let  $G$  be a topological group. Then  $\sigma G$  is a topological group if and only if  $G$  has a strong so-network  $\mathcal{P}_e$  at  $e$  satisfying the condition (\*): for each  $P_1 \in \mathcal{P}_e$ , there is a  $P_2 \in \mathcal{P}_e$  such that  $P_2 P_2 \subset P_1$ .*

**Corollary 2.9** ([13, Theorem 4.4]). *Let  $G$  be a snf-countable topological group. Then  $\sigma G$  is a topological group.*

PROOF: By Proposition 2.2,  $G$  has a countable so-network  $\mathcal{P}_e$  at  $e$  satisfying the condition (\*): for each  $P_1 \in \mathcal{P}_e$ , there is a  $P_2 \in \mathcal{P}_e$  such that  $P_2 P_2 \subset P_1$ . We also can see that  $\mathcal{P}_e$  is a strong so-network at  $e$  by Lemma 2.3. Then  $\sigma G$  is a topological group by Corollary 2.8.  $\square$

**Proposition 2.10.** *Let  $F(X)$  be a free topological group on a sequential space  $X$ . Then  $\sigma F(X)$  is a topological group if and only if  $F(X)$  is a sequential space.*

PROOF: Sufficiency is obvious.

Necessity: Suppose  $F(X)$  is not sequential, then the topology on  $\sigma F(X)$  is strictly finer than the topology on  $F(X)$  and the topology on  $X$  as a subspace of  $\sigma F(X)$  is compatible with the original topology on  $X$  (note that  $X$  is sequential). However, the topology on  $F(X)$  is the finest group topology on  $F(X)$  that generates on  $X$  its original topology [6, Corollary 7.1.8]. Hence  $\sigma F(X)$  is not a topological group.  $\square$

Remark: Usually, the sequential coreflection of a topological group need not to be a topological group. Let  $S_{\omega_1}$  be the space obtained by identifying all limit points of the topological sum of  $\omega_1$  convergent sequences. Then  $S_{\omega_1}$  is Fréchet-Urysohn. Let  $F(S_{\omega_1})$  be the free topological group on  $S_{\omega_1}$ , by [6, Theorem 7.1.13 (b)],  $F(S_{\omega_1})$  contains a closed copy of  $S_{\omega_1} \times S_{\omega_1}$ . Since  $S_{\omega_1} \times S_{\omega_1}$  is not a sequential space [9], then  $F(S_{\omega_1})$  is not a sequential space, hence its sequential coreflection  $\sigma F(S_{\omega_1})$  is not a topological group by Proposition 2.10.

A subset  $B$  of a paratopological group  $G$  is called  $\omega$ -narrow in  $G$  if, for each neighborhood  $U$  of the neutral element of  $G$ , there is a countable subset  $F$  of  $G$  such that  $B \subset FU \cap UF$ .

Let  $X = \prod_{i \in I} X_i$  be the product of spaces  $X_i$ , with  $i \in I$ . A standard base of the  $\omega$ -box topology on  $X$  consists of the  $\omega$ -cubes  $B = \prod_{i \in I} B_i$ , where each  $B_i$  is open in  $X_i$  (and, clearly, the number of indices  $i \in I$  with  $B_i \neq X_i$  is countable).

*Example 2.11.* There is a Lindelöf (hence,  $\omega$ -narrow), snf-countable, zero-dimensional topological group  $G$  such that  $G^n$  is topologically isomorphic to  $G$ ,  $w(G) = c$  and  $G$  does not have a  $\sigma$ -locally finite network.

PROOF: Let  $D = \{0, 1\}$  be the discrete topological group with operation “addition”. In the product group  $\prod D^c$ , consider the subgroup  $G = \sigma \prod D^c = \{x \in \prod D^c : |supp(x)| < \omega\}$ , where  $supp(x)$  denotes the set  $\{\alpha \in \omega_1 : x(\alpha) \neq 0\}$ . Endow  $G$  with  $\omega$ -box topology  $\mathcal{T}$ . Then  $(G, +, \mathcal{T})$  is a zero-dimensional topological group. It is proved in [6, Example 4.4.11] that  $G$  is a Lindelöf topological  $P$ -group,  $G^n$  is topologically isomorphic to  $G$  and  $w(G) = c$ .

**Claim.** Every countable subset of  $G$  does not have a cluster point.

Suppose not, then there is a countable subset  $A$  of  $G$  such that  $a \in \overline{A \setminus \{a\}}$  for some  $a \in G$ . Put  $J = \cup\{supp(x) : x \in A \setminus \{a\}\}$ , then  $J$  is a countable subset of  $\omega_1$ . Let  $V = \Pi Y_i \cap G$ , where  $p(Y_i) = D$  if  $i \notin J \cup supp(a)$ ;  $p(Y_i) = \{1\}$  if  $i \in supp(a)$ ;  $p(Y_i) = \{0\}$  if  $i \in J \setminus supp(a)$ .  $V$  is an open neighborhood of  $a$  since  $V = \Pi Y_i$  is open in  $\Pi D^c$  that is endowed with  $\omega$ -box topology. It is easy to see that  $V \cap A = \emptyset$ . This is a contradiction.

1)  $G$  is snf-countable.

By Claim, there is no non-trivial convergent sequence in  $G$ ,  $\{x\}$  is a sequential neighborhood of  $x \in G$ , hence  $G$  is snf-countable.

2)  $G$  does not have  $\sigma$ -locally finite network.

Suppose that  $G$  has a  $\sigma$ -locally finite network. Since  $G$  is a Lindelöf space,  $G$  is a cosmic space (i.e.  $G$  has a countable network). Hence  $G$  is hereditarily separable. This is a contradiction since  $|G| > \omega$  and every countable subset of  $G$  is discrete by Claim. □

Remark: The topological group  $G$  in Example 2.11 is neither an  $\aleph$ -space nor a cosmic space (i.e. a space with countable network). Hence the answers for Questions 1.4, 1.5 are negative. However, the group  $G$  in Example 2.11 is not separable. Note that a separable topological group is  $\omega$ -narrow [6, Corollary 3.4.8], it is natural to ask if there is a Lindelöf, separable, snf-countable topological group that is not a  $\sigma$ -space.

In what follows, we construct a Lindelöf, separable, snf-countable topological group that is not a  $\sigma$ -space.

Simon [19] proved the following:

**Theorem 2.12.** *There is a countable dense subset  $A$  of  $\Pi D^c$  such that  $|\overline{H}| = 2^c$  for any infinite subset  $H \subset A$ .*

The following proposition comes from a discussion with Arhangel'skii.

**Proposition 2.13.** *There is a Lindelöf, separable space  $Y$  satisfying the following:*

- (1)  $Y$  is not a  $\sigma$ -space (i.e. a space having no  $\sigma$ -locally finite network);
- (2) every compact subset of  $Y$  is finite;
- (3)  $Y^n$  is Lindelöf for each  $n \in \mathbb{N}$ .

PROOF: Let  $A(\Pi D^c) = X \cup X_1$  be the Alexandroff duplicate of  $X = \Pi D^c$ , where  $X_1$  is a copy of  $X$ , and let  $G$  be the Lindelöf topological group of Example 2.11. Since  $G$  is zero-dimensional and  $w(G) = c$ , then  $G$  is homeomorphic to a subspace of  $X = \Pi D^c$  by [7, Theorem 6.2.16]. By Theorem 2.12, we can choose a countable dense subset  $A$  of  $X$  such that  $|\overline{H}| = 2^c$  for any infinite subset  $H \subset A$ . Let  $A_1 \subset X_1$  be a copy of  $A$ , and let  $Y = G \cup A_1$ . Note that  $G$  is a Lindelöf space that is not a  $\sigma$ -space and  $A_1$  is countable, then  $Y$  is a Lindelöf, separable space that is not a  $\sigma$ -space. We prove each compact subset of  $Y$  is finite. Let  $K$  be

a compact subset of  $Y$ , then  $K \cap G$  is compact in  $Y$  since  $G$  is a closed subset of  $Y$ . By Claim in Example 2.11,  $K \cap G$  is finite. If  $K \cap A_1$  is infinite, by Theorem 2.12,  $|\overline{K \cap A_1}| = 2^c$ .  $\overline{K \cap A_1} \subset K \subset Y$ , then  $\overline{K \cap A_1} \cap G \subset K \cap G$  is an infinite compact subset of  $G$ . This is a contradiction. So  $K \cap A_1$  is finite, therefore  $K$  is finite.

Note that  $G^n$  is Lindelöf for each  $n$  and  $A_1$  is countable, it is easy to see that  $Y^n$  is a union of countably many Lindelöf subspaces, hence  $Y^n$  is Lindelöf for each  $n$ . □

**Theorem 2.14.** *There is a Lindelöf, separable, snf-countable topological group that is not a  $\sigma$ -space.*

PROOF: Let  $Y$  be the space in Proposition 2.13, and let  $F(Y)$  be the free topological group on  $Y$ . Since  $Y$  is separable and  $Y^n$  is Lindelöf for each  $n$ ,  $F(Y)$  is also Lindelöf and separable by [6, Corollary 7.1.18, Theorem 7.1.13].  $F(Y)$  is not a  $\sigma$ -space since  $Y$  is not a  $\sigma$ -space. We prove that each compact subset of  $F(Y)$  is finite. Let  $K$  be a compact subset of  $F(Y)$ . Since  $Y$  is Dieudonné-complete, by [5, Corollary 1.8], there exist a compact  $Z \subset Y$  and  $n \in \mathbb{N}$  such that  $K$  is a continuous image of a subspace in  $Z^n$ .  $Z$  is finite since each compact subset of  $Y$  is finite,  $Z^n$  is also finite, hence  $K$  is finite and  $F(Y)$  is snf-countable. □

A space  $X$  is a  $q$ -space if  $X$  has a  $g$ -function satisfying: for  $x \in X$ , if  $x_n \in g(n, x)$ , then  $\{x_n\}$  has a cluster point in  $X$ . A space  $X$  is a  $wM$ -space if there exists a sequence  $(\mathcal{U}_n)$  of open covers of  $X$  such that if  $x_n \in st^2(x, \mathcal{U}_n)$  for each  $n \in \mathbb{N}$ , then the set  $\{x_n : n \in \mathbb{N}\}$  has a cluster point in  $X$ .

**Theorem 2.15.** *Let  $G$  be a regular paratopological group in which each singleton is a  $G_\delta$ -set. If  $G$  is a  $wM$ -space, then  $G$  is metrizable.*

PROOF: Since  $G$  is a  $wM$ -space, then  $G$  is a  $q$ -space. Moreover,  $G$  is first-countable since a regular  $q$ -space in which each singleton is a  $G_\delta$ -set is first-countable [17], hence  $G$  has a regular  $G_\delta$ -diagonal [14]. Therefore,  $G$  is metrizable since a  $wM$ -space with a regular  $G_\delta$ -diagonal is metrizable [20]. □

Remark: Theorem 2.15 gives a positive answer to Question 1.6 when  $G$  is regular and  $T_1$ . But the author doesn't know if we can replace "paratopological group" with "semitopological group" in Theorem 2.15.

Next, we discuss remainder of a paratopological group in its Hausdorff compactification. Arhangel'skii [2] proved the following.

**Theorem 2.16** ([2]). *Let  $G$  be a non-locally compact topological group and the remainder  $Y = bG \setminus G$  have a point-countable base. Then  $G$  and  $bG$  are separable and metrizable.*

Let  $f : X \rightarrow Y$  be a map. The map  $f$  is called  $k$ -gentle [4] if for each compact subset  $F$  of  $X$  the image  $f(F)$  is also compact. A paratopological group is called  $k$ -gentle if the inverse map  $x \rightarrow x^{-1}$  is  $k$ -gentle. Liu and Lin [16] improved the result by replacing "point-countable base" with "pseudo open  $s$ -image of a space with a point-countable base". On the other hand, the author also proved the following theorem on  $k$ -gentle, paratopological group.



**Theorem 2.17** ([15]). *Let  $G$  be a non-locally compact,  $k$ -gentle paratopological group and the remainder  $Y = bG \setminus G$  have a point-countable base. Then  $G$  and  $bG$  are separable and metrizable.*

Next, we are able to improve both theorems in [15], [16].

A family  $\mathcal{S}$  of subsets of a space  $X$  is said to be a  $\kappa$ -sensor [3] at  $x \in X$  if, for each open neighborhood  $O(x)$  of  $x$  and each open set  $U$  such that  $x \in \overline{U}$ , there exists  $P \in \mathcal{S}$  satisfying the following conditions:  $P \subset O(x)$  and  $x \in \overline{U \cap P}$ .

If there exists a countable  $\kappa$ -sensor at  $x$ , the space is said to be *countably  $\kappa$ -sensitive* at  $x$  [3].

The family  $\mathcal{P}$  is called a *cs\*-network* for  $X$  [12] if, whenever  $x \in X$  and a sequence  $S$  converges to  $x \in U$  with  $U$  open, there exists  $P \in \mathcal{P}$  such that  $x \in P \subset U$  and  $P$  contains a subsequence of  $S$ .

Tanaka [21] proved that a space  $X$  is a pseudo open  $s$ -image of a space with a point-countable base if and only if  $X$  is a Fréchet-Urysohn space with a point-countable  $cs^*$ -network.

**Lemma 2.18.** *Let  $X$  be a Fréchet-Urysohn space with a point-countable  $cs^*$ -network. Then  $X$  is of countably  $\kappa$ -sensitive at each  $x \in X$ .*

PROOF:  $X$  is a Fréchet-Urysohn space with a point-countable  $cs^*$ -network  $\mathcal{P}$ . Fix  $x \in X$ , an open neighborhood  $O(x)$  of  $x$  and an open set  $U$  with  $x \in \overline{U}$ . Let  $\mathcal{P}_x = \{P \in \mathcal{P} : x \in P\}$ ,  $|\mathcal{P}_x| \leq \omega$ . Since  $X$  is Fréchet-Urysohn, there is a sequence  $S \subset U$  converging to  $x$ .  $\mathcal{P}_x$  is a  $cs^*$ -network at  $x$ , then there is  $P \in \mathcal{P}_x$  such that  $x \in P \subset O(x)$  and  $P$  contains a subsequence  $S_1$  of  $S$ .  $x \in \overline{S_1} \subset \overline{P \cap U}$ . Hence  $\mathcal{P}_x$  is a countable  $\kappa$ -sensor at  $x$ .  $\square$

**Theorem 2.19.** *Let  $G$  be a non-locally compact,  $k$ -gentle paratopological group. If the remainder  $Y = bG \setminus G$  is a pseudo open  $s$ -image of a space with a point-countable base, then  $G$  and  $bG$  are separable and metrizable.*

PROOF: By [4, Theorem 4.4],  $Y$  is either Lindelöf or pseudocompact. If  $Y$  is Lindelöf, then  $G$  is a topological group [4, Corollary 4.5]. Hence  $G$  and  $bG$  are separable and metrizable by [16, Theorem 5.2].

If  $Y$  is pseudocompact, by Lemma 2.18,  $Y$  is of countably  $\kappa$ -sensitive at each  $x \in Y$ . Then  $Y$  is first-countable by [3, Theorem 1.5].  $Y$  has a point-countable base since a first-countable, quotient  $s$ -image of a space with a point-countable base has a point-countable base [11]. Therefore,  $G$  and  $bG$  are separable and metrizable by Theorem 2.17.  $\square$

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