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LEFT AND RIGHT SEMI-UNINORMS
ON A COMPLETE LATTICE

YONG SU, ZHUDENG WANG AND KEMING TANG

Uninorms are important generalizations of triangular norms and conorms, with a neutral element lying anywhere in the unit interval, and left (right) semi-uninorms are non-commutative and non-associative extensions of uninorms. In this paper, we firstly introduce the concepts of left and right semi-uninorms on a complete lattice and illustrate these notions by means of some examples. Then, we lay bare the formulas for calculating the upper and lower approximation left (right) semi-uninorms of a binary operation. Finally, we discuss the relations between the upper approximation left (right) semi-uninorms of a given binary operation and the lower approximation left (right) semi-uninorms of its dual operation.

Keywords: fuzzy connective, uninorm, left (right) semi-uninorm, upper (lower) approximation

Classification: 03B52, 03E72

1. INTRODUCTION

Uninorms, introduced by Yager and Rybalov [30], and studied by Fodor et al. [9], are special aggregation operators that have proven useful in many fields like fuzzy logic, expert systems, neural networks, aggregation, and fuzzy system modeling [10, 22, 27, 28, 29]. Uninorms are interesting because their structure is a special combination of \( t \)-norms and \( t \)-conorms [9]. It is well known that a uninorm \( U \) can be conjunctive or disjunctive whenever \( U(0, 1) = 0 \) or 1, respectively. This fact allows to use uninorms in defining fuzzy implications and coimplications [3, 19, 20].

There are real-life situations when truth functions can not be associative or commutative. By throwing away the commutativity from the axioms of uninorms, Mas et al. [17, 18] introduced the concepts of left and right uninorms on [0, 1], Wang and Fang [25, 26] studied the residual operators and the residual coimplicators of left (right) uninorms on a complete lattice. By removing the associativity and commutativity from the axioms of uninorms, Liu [15] introduced the concept of semi-uninorms on a complete lattice. In this paper, motivated by these generalizations, we will generalize the concepts of both left (right) uninorms and semi-uninorms, introduce a new concept, called the left (right) semi-uninorm, illustrate these notions by means of some examples and lay bare the formulas for calculating the upper and lower approximation left (right) semi-uninorms of a given binary operation on a complete lattice.
This paper is organized as follows. In section 2, we introduce the concepts of left and right semi-uninorms on a complete lattice and illustrate these concepts by means of some examples. In section 3, we give out the formulas for calculating the upper and lower approximation left (right) semi-uninorms of a binary operation. In section 4, we discuss the relations between the upper approximation left (right) semi-uninorms of a given binary operation and the lower approximation left (right) semi-uninorms of its dual operation.

The knowledge about lattices required in this paper can be found in [5].

Throughout this paper, unless otherwise stated, $L$ always represents any given complete lattice with maximal element $1$ and minimal element $0$; $J$ stands for any index set.

2. LEFT AND RIGHT SEMI-UNINORMS

Noting that the commutativity and associativity are not desired for aggregation operators in a lot of cases. In this section, based on [15, 17, 25, 26], we introduce the concepts of left and right semi-uninorms on a complete lattice and illustrate these notions by means of some examples.

**Definition 2.1.** A binary operation $U$ on $L$ is called a left (right) semi-uninorm if it satisfies the following two conditions:

1. (U1) there exists a left (right) neutral element, i.e., an element $e_L \in L$ ($e_R \in L$)
   satisfying $U(e_L, x) = x$ ($U(x, e_R) = x$) for all $x \in L$,
2. (U2) $U$ is non-decreasing in each variable.

For any left (right) semi-uninorm $U$ on $L$, $U$ is said to be left-conjunctive (right-conjunctive) if $U(0, 1) = 0$ ($U(1, 0) = 0$). $U$ is said to be conjunctive if both $U(0, 1) = 0$ and $U(1, 0) = 0$ since it satisfies the classical boundary conditions of AND. If $U(1, 0) = 1$ ($U(0, 1) = 1$), then we call $U$ left-disjunctive (right-disjunctive). We call $U$ disjunctive if both $U(1, 0) = 1$ and $U(0, 1) = 1$ by a similar reason.

If a left (right) semi-uninorm $U$ is associative, then $U$ is the left (right) uninorm (see [25, 26]).

If a left (right) semi-uninorm $U$ with left (right) neutral element $e_L$ ($e_R$) has a right (left) neutral element $e_R$ ($e_L$), then $e_L = U(e_L, e_R) = e_R$. Let $e = e_L = e_R$. Here, $U$ is the semi-uninorm (see [15]). In particular, if the neutral element $e = 1$, then the semi-uninorm $U$ becomes a $t$-seminorm (see [21]) or a semi-copula (see [4, 8]); if the neutral element $e = 0$, then the semi-uninorm $U$ becomes a $t$-seminorm (see [27]).

Clearly, $U(0, 0) = 0$ and $U(1, 1) = 1$ hold for any left (right) semi-uninorm $U$ on $L$. Moreover, the left (right) neutral elements need not to be unique. In fact, the projection operator given by $U(x, y) = x$ for all $x, y \in L$ is such that any element in $L$ is a right neutral element. But, left (right) neutral elements are all idempotent (see [2]) because $U(e_L, e_L) = e_L$ ($U(e_R, e_R) = e_R$) for any left (right) neutral element $e_L$ ($e_R$) of $U$. 
Definition 2.2. (Wang and Fang [26]) A binary operation $U$ on $L$ is called left (right) infinitely $\lor$-distributive if

$$U \left( \bigvee_{j \in J} x_j, y \right) = \bigvee_{j \in J} U(x_j, y) = \bigvee_{j \in J} U(x, y_j) \quad \forall x, y, x_j, y_j \in L;$$

left (right) infinitely $\land$-distributive if

$$U \left( \bigwedge_{j \in J} x_j, y \right) = \bigwedge_{j \in J} U(x_j, y) = \bigwedge_{j \in J} U(x, y_j) \quad \forall x, y, x_j, y_j \in L.$$ 

If a binary operation $U$ is left infinitely $\lor$-distributive ($\land$-distributive) and also right infinitely $\lor$-distributive ($\land$-distributive), then $U$ is said to be infinitely $\lor$-distributive ($\land$-distributive).

Noting that the least upper bound of the empty set is 0 and the greatest lower bound of the empty set is 1 (see [6]), we have that

$$U(0, y) = U \left( \bigvee_{j \in \emptyset} x_j, y \right) = 0 \quad U(x, 0) = U \left( x, \bigvee_{j \in \emptyset} y_j \right) = 0$$

for any $x, y \in L$ when $U$ is left (right) infinitely $\lor$-distributive and

$$U(1, y) = U \left( \bigwedge_{j \in \emptyset} x_j, y \right) = 1 \quad U(x, 1) = U \left( x, \bigwedge_{j \in \emptyset} y_j \right) = 1$$

for any $x, y \in L$ when $U$ is left (right) infinitely $\land$-distributive.

When $L = [0, 1]$, a binary function $f$ on $[0, 1]^2$ is infinitely sup-distributive if and only if, for any $x_0, y_0 \in [0, 1]$, $f(x, y_0)$ and $f(x_0, y)$ are left-continuous and increasing and $f(x, 0) = f(0, y) = 0$ for any $x, y \in [0, 1]$; and $f$ is infinitely inf-distributive if and only if, for any $x_0, y_0 \in [0, 1]$, $f(x, y_0)$ and $f(x_0, y)$ are right-continuous and increasing and $f(x, 1) = f(1, y) = 1$ for any $x, y \in [0, 1]$ (see [11]).

For the sake of convenience, we introduce the following symbols:

$\mathcal{U}_{sL}(L)$: the set of all left semi-uninorms with left neutral element $e_L$ on $L$;

$\mathcal{U}_{sR}(L)$: the set of all right semi-uninorms with right neutral element $e_R$ on $L$;

$\mathcal{U}_{s\lor L}(L)$: the set of all right infinitely $\lor$-distributive left semi-uninorms with left neutral element $e_L$ on $L$;

$\mathcal{U}_{s\lor R}(L)$: the set of all left infinitely $\lor$-distributive right semi-uninorms with right neutral element $e_R$ on $L$;

$\mathcal{U}_{s\land L}(L)$: the set of all right infinitely $\land$-distributive left semi-uninorms with left neutral element $e_L$ on $L$;

$\mathcal{U}_{s\land R}(L)$: the set of all left infinitely $\land$-distributive right semi-uninorms with right neutral element $e_R$ on $L$.
Now, we illustrate the notions of left (right) semi-uninorms by means of some examples.

**Example 2.3.** Let $L = \{0, a, b, c, d, 1\}$ be a lattice, where $0 < a < b < d < 1$, $0 < a < c < d < 1$, $b \land c = a$ and $b \lor c = d$. Define two binary operations $U_1, U_2$ on $L$ as follows:

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Obviously, $U_1$ and $U_2$ are neither commutative nor associative. It is easy to verify that $U_1$ is a conjunctive infinitely $\lor$-distributive semi-uninorm with the neutral element $b$ and $U_2$ is a disjunctive infinitely $\land$-distributive semi-uninorm with the neutral element $b$.

**Example 2.4.** Let $L = \{0, a, b, c, 1\}$ be a lattice, where $0 < a < b < 1$, $0 < a < c < 1$, $b \land c = a$ and $b \lor c = 1$. Define a binary operation $U$ on $L$ as follows:

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Clearly, $U$ is a conjunctive left semi-uninorm with two left neutral elements $b$ and $c$. But, $U$ has no right neutral element. It is easy to see that $U$ is neither commutative nor associative. Moreover, $U$ is neither left infinitely $\lor$-distributive ($\land$-distributive) nor right infinitely $\lor$-distributive ($\land$-distributive).

**Example 2.5.** Let $e_L \in L$,

$$U_{sW}^{e_L}(x, y) = \begin{cases} y & \text{if } x \geq e_L, \\ 0 & \text{otherwise,} \end{cases} \quad U_{sM}^{e_L}(x, y) = \begin{cases} y & \text{if } x \leq e_L, \\ 1 & \text{otherwise,} \end{cases}$$

$$U_{sW}^{e_L}(x, y) = \begin{cases} 1 & \text{if } y = 1, \\ y & \text{if } x \geq e_L, y \neq 1, \\ 0 & \text{otherwise,} \end{cases} \quad U_{sM}^{e_L}(x, y) = \begin{cases} 0 & \text{if } y = 0, \\ y & \text{if } x \leq e_L, y \neq 0, \\ 1 & \text{otherwise,} \end{cases}$$

where $x$ and $y$ are elements of $L$. Then $U_{sW}^{e_L}$ and $U_{sM}^{e_L}$ are, respectively, the smallest and greatest elements of $U_{sW}^{e_L}(L)$; $U_{sW}^{e_L}$ and $U_{sM}^{e_L}$ are, respectively, the smallest and greatest elements of $U_{sW}^{e_L}(L)$; $U_{sW}^{e_L}$ and $U_{sM}^{e_L}$ are, respectively, the smallest and greatest elements of $U_{sW}^{e_L}(L)$. \hfill \qed
Example 2.6. Let $e_R \in L$,

$$U_{sW}^{e_R}(x, y) = \begin{cases} 
    x & \text{if } y \geq e_R, \\
    0 & \text{otherwise},
\end{cases} \quad U_{sM}^{e_R}(x, y) = \begin{cases} 
    x & \text{if } y \leq e_R, \\
    1 & \text{otherwise},
\end{cases}$$

$$U_{sW}^{e_R*}(x, y) = \begin{cases} 
    1 & \text{if } x = 1, \\
    x & \text{if } y \geq e_R, \ x \neq 1, \\
    0 & \text{otherwise},
\end{cases} \quad U_{sM}^{e_R*}(x, y) = \begin{cases} 
    0 & \text{if } x = 0, \\
    x & \text{if } y \leq e_R, \ x \neq 0, \\
    1 & \text{otherwise},
\end{cases}$$

where $x$ and $y$ are elements of $L$. Then $U_{sW}^{e_R}$ and $U_{sM}^{e_R}$ are, respectively, the smallest and greatest elements of $U_{s}^{e_R}(L)$; $U_{sW}^{e_R*}$ and $U_{sM}^{e_R*}$ are, respectively, the smallest and greatest elements of $U_{s\vee}^{e_R}(L)$; $U_{sW}^{e_R*}$ and $U_{sM}^{e_R}$ are, respectively, the smallest and greatest elements of $U_{s\wedge}^{e_R}(L)$.

3. THE UPPER AND LOWER APPROXIMATION LEFT (RIGHT) SEMI-UNINORMS OF A BINARY OPERATION

Constructing logic operators is an interesting work. Recently, Jenei and Montagna [12, 13, 14] introduced several new types of constructions of left-continuous $t$-norms and Wang [24] laid bare the formulas for calculating the smallest pseudo-$t$-norm that is stronger than a binary operation. In this section, we continue the work in [24] and give out the formulas for calculating the upper and lower approximation left (right) semi-uninorms of a binary operation.

For any nonempty subfamily $\{T_j \mid j \in J\}$ of $L^{L \times L}$, the least upper bound $\bigvee_{j \in J} T_j$ and the greatest lower bound $\bigwedge_{j \in J} T_j$ of $T_j$’s, respectively, define by

$$\left( \bigvee_{j \in J} T_j \right)(x, y) = \bigvee_{j \in J} T_j(x, y) \quad \text{and} \quad \left( \bigwedge_{j \in J} T_j \right)(x, y) = \bigwedge_{j \in J} T_j(x, y) \ \forall x, y \in L.$$ 

It is easy to verify that $(L^{L \times L}, \leq, \bigvee, \bigwedge)$ is a complete lattice. Moreover, we have the following two theorems.

**Theorem 3.1.**

1. $U_{s}^{e_R}(L)$ is a complete sublattice of $L^{L \times L}$ with $U_{sW}^{e_R}$ and $U_{sM}^{e_R}$ as its minimal and maximal elements, respectively.

2. $U_{s\vee}^{e_R}(L)$ is a complete sublattice of $L^{L \times L}$ with $U_{sW}^{e_R}$ and $U_{sM}^{e_R}$ as its minimal and maximal elements, respectively.

**Theorem 3.2.**

1. $U_{s\wedge}^{e_R}(L)$ is a complete sublattice of $L^{L \times L}$ with $U_{sW}^{e_R*}$ and $U_{sM}^{e_R}$ as its minimal and maximal elements, respectively.
2. \( \mathcal{U}_{s_M}^{e_R}(L) \) is a complete sublattice of \( L^{L \times L} \) with \( U_{s_W}^{e_R} \) and \( U_{s_M}^{e_R} \) as its minimal and maximal elements, respectively.

3. \( \mathcal{U}_{s_V}^{e_L}(L) \) is a complete sublattice of \( L^{L \times L} \) with \( U_{s_L}^{e_L} \) and \( U_{s_M}^{e_L} \) as its minimal and maximal elements, respectively.

4. \( \mathcal{U}_{s_L}^{e_R}(L) \) is a complete sublattice of \( L^{L \times L} \) with \( U_{s_W}^{e_R} \) and \( U_{s_M}^{e_R} \) as its minimal and maximal elements, respectively.

**Proof.** We only prove that statement (1) holds.

Suppose that \( U_j \in \mathcal{U}_{s_L}^{e_L}(L) \) \( (j \in J) \) and \( J \neq \emptyset \). Then it follows from Theorem 3.1 that \( \bigwedge_{j \in J} U_j \in \mathcal{U}_{s_L}^{e_L}(L) \). Moreover, we have

\[
\left( \bigwedge_{j \in J} \bigwedge_{k \in K} U_j (x, y_k) \right) = \bigwedge_{j \in J} U_j (x, \bigwedge_{k \in K} y_k) = \bigwedge_{j \in J} U_j (x, y_k)
\]

where \( K \) is any index set, and \( x \) and \( y_k \) \( (k \in K) \) are any elements of \( L \). Hence, \( \bigwedge_{j \in J} U_j \in \mathcal{U}_{s_L}^{e_L}(L) \). Noting that fact \( U_{s_M}^{e_L} \in \{ U \in \mathcal{U}_{s_A}^{e_L}(L) \mid U \leq U \ \forall j \in J \} \), let \( U^* = \bigwedge \{ U \in \mathcal{U}_{s_A}^{e_L}(L) \mid U_j \leq U \ \forall j \in J \} \), then \( U^* \in \mathcal{U}_{s_A}^{e_L}(L) \) and \( U^* = \bigvee_{j \in J} U_j \). Thus, \( \mathcal{U}_{s_R}^{e_M}(L) \) is a complete sublattice of \( L^{L \times L} \) with \( U_{s_M}^{e_L} \) and \( U_{s_W}^{e_L} \) as its maximal and minimal elements, respectively.

For a binary operation \( A \) on \( L \), if there exists \( U \in \mathcal{U}_{s_L}^{e_L}(L) \) such that \( A \leq U \), then it follows from Theorem 3.1 that \( \bigwedge \{ U \mid A \leq U, U \in \mathcal{U}_{s_L}^{e_L}(L) \} \) is the smallest left semi-uninorm of \( A \) on \( L \), we call it the upper approximation left semi-uninorm of \( A \) and written as \( [A]_{s_L}^{e_L} \); if there exists \( U \in \mathcal{U}_{s_L}^{e_L}(L) \) such that \( U \leq A \), then \( \bigvee \{ U \mid U \leq A, U \in \mathcal{U}_{s_L}^{e_L}(L) \} \) is the largest left semi-uninorm that is weaker than \( A \) on \( L \), we call it the lower approximation left semi-uninorm of \( A \) and written as \( [A]_{s_L}^{e_R} \).

Similarly, we introduce the following symbols:

- \( [A]_{s_R}^{e_R} \) the upper approximation right semi-uninorm of \( A \);
- \( [A]_{s_R}^{e_L} \) the lower approximation right semi-uninorm of \( A \);
- \( [A]_{s_A}^{e_L} \) the right infinitely \( \wedge \)-distributive lower approximation left semi-uninorm of \( A \);
- \( [A]_{s_A}^{e_R} \) the left infinitely \( \wedge \)-distributive lower approximation right semi-uninorm of \( A \);
- \( [A]_{s_V}^{e_L} \) the right infinitely \( \vee \)-distributive upper approximation left semi-uninorm of \( A \);
- \( [A]_{s_V}^{e_R} \) the left infinitely \( \vee \)-distributive upper approximation right semi-uninorm of \( A \).

Now we consider how to construct the upper and lower approximation left (right) semi-uninorms of a binary operation.
**Definition 3.3.** Let $A \in L^{L \times L}$. Define the upper approximation $A_u$ and the lower approximation $A_l$ of $A$ as follows:

$$A_u(x, y) = \bigvee \{A(u, v) \mid u \leq x, v \leq y\}, \quad A_l(x, y) = \bigwedge \{A(u, v) \mid u \geq x, v \geq y\} \quad \forall x, y \in L.$$ 

**Theorem 3.4.** Let $A, B \in L^{L \times L}$. Then the following statements hold:

1. $A_l \leq A \leq A_u$.
2. $(A \lor B)_u = A_u \lor B_u$ and $(A \land B)_l = A_l \land B_l$.
3. $A_u$ and $A_l$ are non-decreasing in its each variable.
4. If $A$ is non-decreasing in its each variable, then $A_u = A_l = A$.

**Proof.** Clearly, statements (1) and (2) hold.

3. We only prove that $A_l$ is non-decreasing in its first variable.

If $x_1 \leq x_2$, then

$$\{A(u, v) \mid u \geq x_1, v \geq y\} \supseteq \{A(u, v) \mid u \geq x_2, v \geq y\}.$$ 

Thus $A_l(x_1, y) \leq A_l(x_2, y)$ for any $y \in L$ by Definition 3.3, i.e., $A_l$ is non-decreasing in its first variable.

4. If $A$ is non-decreasing in its each variable, then

$$A_l(x, y) = \bigwedge \{A(u, v) \mid u \geq x, v \geq y\} \geq \bigwedge \{A(x, y) \mid u \geq x, v \geq y\} = A(x, y) \quad \forall x, y \in L$$

and hence $A_l \geq A$. Thus, it follows from statement (1) that $A_l = A$.

Similarly, we can show that $A_u = A$. □

As usual, the upper or lower approximation of a binary operation is neither a left semi-uninorm nor a right semi-uninorm.

**Example 3.5.** Let

$$A(x, y) = \begin{cases} 
\frac{1}{4}y & \text{if } x \leq \frac{1}{2}, \\
1 & \text{otherwise.}
\end{cases}$$

Then $A \leq U_{sM}^{\left(\frac{1}{2}\right)}$ and $A_u = A$. Clearly, $A_u$ is not a left semi-uninorm. Let

$$U(x, y) = \begin{cases} 
\frac{1}{4}y & \text{if } x < \frac{1}{2}, \\
y & \text{if } x = \frac{1}{2}, \\
1 & \text{otherwise.}
\end{cases}$$

It is easy to see that $U$ is the upper approximation left semi-uninorm with left neutral element $\frac{1}{2}$ of $A$. 

The following two theorems give out the formulas for calculating the upper and lower approximation left (right) semi-uninorms of a binary operation.

**Theorem 3.6.** Let $A \in L^{L \times L}$ and $e_L \in L$.

1. If $A \leq U^{e_L}_{sM}$, then $[A]^{e_L}_s = U^{e_L}_{sW} \lor A_u$.
2. If $U^{e_L}_{sW} \leq A$, then $(A)^{e_L}_s = U^{e_L}_{sM} \land A_l$.
3. If $A \leq U^{e_L}_{sM}$ and $A$ is non-decreasing in its first variable and right infinitely $\lor$-distributive, then $[A]^{e_L}_s = U^{e_L}_{sW} \lor A$.
4. If $U^{e_L}_{sW} \leq A$ and $A$ is non-decreasing in its first variable and right infinitely $\land$-distributive, then $(A)^{e_L}_s = U^{e_L}_{sM} \land A$.

**Proof.** We only prove the statements (1) and (3) hold.

1. Let $U = U^{e_L}_{sW} \lor A_u$. Clearly, $U \geq A$ and $U^{e_L}_{sW} \leq U \leq U^{e_L}_{sM}$. Thus, $U(e_L, x) = x$ for all $x \in L$. By Theorem 3.4(3) and the monotonicity of $U^{e_L}_{sW}$, we see that $U$ is non-decreasing in its each variable. So, $U \in \mathcal{U}^{e_L}_s(L)$. If $A \leq U_1$ and $U_1 \in \mathcal{U}^{e_L}_s(L)$, then $U_1 = (U_1)_u \geq A_u$ and $U_1 \geq U^{e_L}_{sW} \lor A_u = U$. Therefore, $[A]^{e_L}_s = U^{e_L}_{sW} \lor A_u$.

3. Let $U^* = U^{e_L}_{sW} \lor A$. If $A$ is non-decreasing in its first variable and right infinitely $\lor$-distributive, then $A$ is non-decreasing in its each variable and so $A_u = A$. Noting that $U^{e_L}_{sW}$ and $A$ are all right infinitely $\lor$-distributive, we can see that $U^*$ is also right infinitely $\lor$-distributive. By the proof of statement (1), we have that $[A]^{e_L}_s = U^{e_L}_{sW} \lor A$.

In Theorem 3.6(3), $A(x, 0) = 0$ for any $x \in L$ when $A$ is right infinitely $\lor$-distributive. Thus, $A \leq U^{e_L}_{sM}$ can be replaced by $A \leq U^{e_L}_{sM}$. Similarly, $U^{e_L}_{sW} \leq A$ can be replaced by $U^{e_L}_{sW} \leq A$ in Theorem 3.6(4).

Analogous to Theorem 3.6, we have the following theorem.

**Theorem 3.7.** Let $A \in L^{L \times L}$ and $e_R \in L$.

1. If $A \leq U^{e_R}_{sM}$, then $[A]^{e_R}_s = U^{e_R}_{sW} \lor A_u$.
2. If $U^{e_R}_{sW} \leq A$, then $(A)^{e_R}_s = U^{e_R}_{sM} \land A_l$.
3. If $A \leq U^{e_R}_{sM}$ and $A$ is non-decreasing in its second variable and left infinitely $\lor$-distributive, then $[A]^{e_R}_s = U^{e_R}_{sW} \lor A$.
4. If $U^{e_R}_{sW} \leq A$ and $A$ is non-decreasing in its second variable and left infinitely $\land$-distributive, then $(A)^{e_R}_s = U^{e_R}_{sM} \land A$.

The following example shows that analogous to the above theorems may not hold for calculating the right (left) infinitely $\land$-distributive upper approximation left (right) semi-uninorm and the right (left) infinitely $\lor$-distributive lower approximation left (right) semi-uninorm of a binary operation.
Example 3.8. Let \( L = \{0, a, b, 1\} \) be a lattice, where \( 0 < a < 1, 0 < b < 1, a \lor b = 1 \) and \( a \land b = 0 \). Define two binary operations \( A \) and \( B \) on \( L \) as follows:

\[
\begin{array}{c|cccc}
   & 0 & a & b & 1 \\
\hline
A & 0 & 0 & 0 & 0 \\
   & a & 1 & a & 1 \\
   & b & 0 & 0 & 0 \\
   & 1 & a & 1 & a \\
\end{array}
\begin{array}{c|cccc}
   & 0 & a & b & 1 \\
\hline
B & 0 & 0 & b & 0 \\
   & a & 1 & 1 & 1 \\
   & b & 0 & b & 0 \\
   & 1 & a & 1 & 1 \\
\end{array}
\]

Clearly, \( A \leq U_{sM}^0, U_{sW}^1 \leq B, A \) is non-decreasing in its first variable and right infinitely \( \land \)-distributive, and \( B \) is non-decreasing in its first variable and right infinitely \( \lor \)-distributive. Let \( U_1 = U_{sW}^0 \lor A \) and \( U_2 = U_{sM}^1 \land B \). Then

\[
\begin{array}{c|cccc}
   & 0 & a & b & 1 \\
\hline
U_1 & 0 & a & b & 1 \\
   & a & 1 & 1 & 1 \\
   & b & 0 & a & b \\
   & 1 & a & 1 & 1 \\
\end{array}
\begin{array}{c|cccc}
   & 0 & a & b & 1 \\
\hline
U_2 & 0 & 0 & 0 & b \\
   & a & 0 & a & b \\
   & b & 0 & 0 & 0 \\
   & 1 & 0 & a & b \\
\end{array}
\]

It is easy to see that \( U_1 \) is not right infinitely \( \land \)-distributive and \( U_2 \) is not right infinitely \( \lor \)-distributive. This shows that \( U_1 \) is not the right infinitely \( \land \)-distributive upper approximation left semi-uninorm of \( A \) and \( U_2 \) is not the right infinitely \( \lor \)-distributive lower approximation left semi-uninorm of \( B \).

4. THE RELATIONS BETWEEN THE UPPER APPROXIMATION LEFT (RIGHT) SEMI-UNINORMS OF A GIVEN BINARY OPERATION AND LOWER APPROXIMATION LEFT (RIGHT) SEMI-UNINORMS OF ITS DUAL OPERATION

In section 3, we give out the formulas for calculating the upper and lower approximation left (right) semi-uninorms of a binary operation. In this section, we investigate the relations between the upper approximation left (right) semi-uninorm of a given binary operation and the lower approximation left (right) semi-uninorm of its dual operation.

We firstly review some basic concepts and properties which will be used in this section.

Definition 4.1. (Ma and Wu [16]) A mapping \( N : L \to L \) is called a negation if

(N1) \( N(0) = 1 \) and \( N(1) = 0 \),

(N2) \( x \leq y, x, y \in L \Rightarrow N(y) \leq N(x) \).

A negation \( N \) is called strong if it is an involution, i.e., \( N(N(x)) = x \) for any \( x \in L \).

Theorem 4.2. (Wang and Yu [23]) Let \( x_j \in L (j \in J) \). If \( N \) is a strong negation on \( L \), then

\[
N\left( \bigvee_{j \in J} x_j \right) = \bigwedge_{j \in J} N(x_j), \quad N\left( \bigwedge_{j \in J} x_j \right) = \bigvee_{j \in J} N(x_j).
\]
Definition 4.3. (De Baets [1]) Consider a strong negation $N$ on $L$. The $N$-dual operation of a binary operation $A$ on $L$ is the binary operation $A_N$ on $L$ defined by

$$A_N(x, y) = N^{-1}(A(N(x), N(y))) \quad \forall x, y \in L.$$ 

Note that $(A_N)^{-1} = (A_N)_N = A$ for any binary operation $A$ on $L$.

The following theorem about $N$-dual is easily verified.

Theorem 4.4. Let $A$, $B$ be two binary operations and $N$ a strong negation on $L$. Then the following statements hold:

1. $(A \land B)_N = A_N \lor B_N$ and $(A \lor B)_N = A_N \land B_N$.

2. If $A$ is left (right) infinitely $\lor$-distributive, then $A_N$ is left (right) infinitely $\land$-distributive.

3. If $A$ is left (right) infinitely $\land$-distributive, then $A_N$ is left (right) infinitely $\lor$-distributive.

4. If $A$ is increasing (decreasing) in its $i$th variable, then $A_N$ is increasing (decreasing) in its $i$th variable ($i = 1, 2$).

5. The $N$-dual operation of a left (right) semi-uninorm with a left (right) neutral element $e_L$ ($e_R$) is a left (right) semi-uninorm with a left (right) neutral element $N(e_L)$ ($N(e_R)$).

6. $(U^{c_L}_{sW})_N = U^{N(e_L)}_{sM}$, $(U^{c_L}_{sM})_N = U^{N(e_L)}_{sW}$, $(U^{c_R}_{sW})_N = U^{N(e_R)}_{sM}$, and $(U^{c_R}_{sM})_N = U^{N(e_R)}_{sW}$.

Theorem 4.5. If $A$ is a binary operation and $N$ a strong negation on $L$, then $(A_N)_u = (A_l)_N$ and $(A_N)_l = (A_u)_N$.

Proof. By Definition 4.3 and Theorem 4.2, we can see that

$$(A_N)_u(x, y) = \bigvee\{A_N(u, v) \mid u \leq x, v \leq y\}$$

$= \bigvee\{N^{-1}(A(N(u), N(v))) \mid u \leq x, v \leq y\}$$

$= N^{-1}\left(\bigwedge\{A(N(u), N(v)) \mid u \leq x, v \leq y\}\right)$

$= N^{-1}\left(\bigwedge\{A(u', v') \mid u' \geq N(x), v' \geq N(y)\}\right)$

$= N^{-1}(A_l(N(x), N(y))) = (A_l)_N(x, y) \quad \forall x, y \in L.$

Moreover, we have that $(A_u)_N = ((A_N)_u)_N = ((A_N)_l)_N = (A_N)_l$. \qed
Below, we investigate the relations between the upper approximation left (right) semi-uninorms of a given binary operation and lower approximation left (right) semi-uninorms of its dual operation.

**Theorem 4.6.** Let $A$, $N$ and $e_L$ be a binary operation, strong negation and fixed element on $L$, respectively. Then the following statements hold:

1. If $A \leq U_{s_M}^{e_L}$, then $[A]_s^{e_L} = ([A]_s N(e_L))_N$.
2. If $U_{s_W}^{e_L} \leq A$, then $(A)_s^{e_L} = ([A]_s N(e_L))_N$.
3. If $A \leq U_{s_M}^{e_L}$ and $A$ is non-decreasing in its first variable and right infinitely $\lor$-distributive, then $[A]_{s\lor}^{e_L} = ([A]_s N(e_L))_N$.
4. If $U_{s_W}^{e_L} \leq A$ and $A$ is non-decreasing in its first variable and right infinitely $\land$-distributive, then $(A)_{s\land}^{e_L} = ([A]_s N(e_L))_N$.

**Proof.** We only prove the statements (1) and (3) hold.

1. If $A \leq U_{s_M}^{e_L}$, then $[A]_s^{e_L} = U_{s_W}^{e_L} \lor A_u$ by Theorem 3.6 and $A_N \geq (U_{s_M}^{e_L})_N = U_{s_W}^{N(e_L)}$ by Theorem 4.4. Thus, $(A_N)_{s\lor}^{N(e_L)} = U_{s_M}^{N(e_L)} \land (A_N)_t$ by Theorem 3.6. Moreover, by virtue of Theorems 3.6, 4.4 and 4.5, we see that

\[
\begin{align*}
(A_N)_{s\lor}^{N(e_L)} &= (U_{s_M}^{N(e_L)} \land (A_N)_t)_N = (U_{s_M}^{N(e_L)} \land (A_u)_N)_N \\
&= (U_{s_M}^{N(e_L)})_N \lor ((A_u)_N)_N = U_{s_W}^{e_L} \lor A_u = [A]_s^{e_L}.
\end{align*}
\]

3. If $A \leq U_{s_M}^{e_L}$ and $A$ is non-decreasing in its first variable and right infinitely $\lor$-distributive, then $A_u = A$ by Theorem 3.4(4), $[A]_{s\lor}^{e_L} = U_{s_W}^{e_L} \lor A$ by Theorem 3.6, $A_N \geq (U_{s_M}^{e_L})_N = U_{s_W}^{N(e_L)}$ and $A_N$ is non-decreasing in its first variable and right infinitely $\land$-distributive by Theorem 4.4. Thus, $(A_N)_{s\land}^{N(e_L)} = U_{s_M}^{N(e_L)} \land A_N$ by Theorem 3.6. Moreover, we see that $[A]_{s\land}^{e_L} = ([A]_s N(e_L))_N$ by the proof of statement (1).

Analogous to Theorem 4.6, we have the following theorem.

**Theorem 4.7.** Let $A$, $N$ and $e_R$ be a binary operation, strong negation and fixed element on $L$, respectively. Then the following statements hold:

1. If $A \leq U_{s_M}^{e_R}$, then $[A]_s^{e_R} = ([A]_s N(e_R))_N$.
2. If $U_{s_W}^{e_R} \leq A$, then $(A)_{s}^{e_R} = ([A]_s N(e_R))_N$.
3. If $A \leq U_{s_M}^{e_R}$ and $A$ is non-decreasing in its second variable and left infinitely $\lor$-distributive, then $[A]_{s\lor}^{e_R} = ([A]_s N(e_R))_N$.
4. If $U_{s_W}^{e_R} \leq A$ and $A$ is non-decreasing in its second variable and left infinitely $\land$-distributive, then $(A)_{s\land}^{e_R} = ([A]_s N(e_R))_N$.
5. CONCLUSIONS AND FUTURE WORKS

Uninorms are important generalizations of triangular norms and conorms, with a neutral element lying anywhere in the unit interval. Noting that the associative binary operators are often used to generate $n$-ary aggregation operators and the commutativity is not desired for these aggregation operators in a lot of cases, Mas et al. [17, 18] introduced the concepts of left and right uninorms on $[0, 1]$ by eliminating the commutativity from the axioms of uninorm, Wang and Fang [23, 26] studied the residual operations and the residual coimplications of left (right) uninorms on a complete lattice, and Liu [15] discussed the concept of semi-uninorms on a complete lattice by removing the associativity and commutativity from the axioms of uninorms. In this paper, motivated by these generalizations, we introduce the concepts of left and right semi-uninorms on a complete lattice, lay bare the formulas for calculating the upper and lower approximation left (right) semi-uninorms of a binary operation, and discuss the relations between the upper approximation left (right) semi-uninorms of a given binary operation and the lower approximation left (right) semi-uninorms of its dual operation.

In a forthcoming paper, we will investigate the relationships among left (right) semi-uninorms, implications and coimplications on a complete lattice.

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