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Applications of Mathematics, Vol. 59 (2014), No. 1, 1--13

Persistent URL: http://dml.cz/dmlcz/143592

VARIATIONAL CHARACTERIZATION OF EIGENVALUES OF A NON-SYMMETRIC EIGENVALUE PROBLEM GOVERNING ELASTOACOUSTIC VIBRATIONS

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(Received November 21, 2012)

Cordially dedicated to Ivo Marek on the occasion of his 80th birthday

Abstract. Small amplitude vibrations of an elastic structure completely filled by a fluid are considered. Describing the structure by displacements and the fluid by its pressure field one arrives at a non-selfadjoint eigenvalue problem. Taking advantage of a Rayleigh functional we prove that its eigenvalues can be characterized by variational principles of Rayleigh, minmax and maxmin type.

Keywords: eigenvalue problem; fluid-solid vibration; variational characterization; minmax principle; maxmin principle

MSC 2010: 35P05, 47A75, 65N25

1. Introduction

In this note we deal with the free vibrations of an elastic structure coupled with a fluid. Such multi-physics problems are encountered in a wide variety of applications, such as the analysis of acoustic simulations of passenger car bodies, response of piping systems and liquid or gas storage tanks, and simulation of mechanical vibrations of ships and off-shore constructions, to name just a few. Here we restrict ourselves to the elastoacoustic vibration problem, which consists of determining the small amplitude vibration modes of an elastic structure coupled with an internal inviscid, homogeneous, compressible fluid, where we neglect gravity effects.

The interaction between the structure and the fluid can significantly affect the response of the whole system and has to be taken into account properly. Different formulations have been proposed to solve this problem. One of them, the pure displacement formulation [3], has an attractive feature: it leads to a simple symmetric eigenvalue problem. However, it suffers from the presence of zero-frequency spurious circulation modes with no physical meaning, and after discretization by standard finite elements, these modes correspond to non-zero eigenfrequencies commingled with physical ones.

In order to remove the problem with non-physical modes, a potential description consists of modeling the fluid by the pressure field p and the structure by the displacement field u (cf. [7], [8], [12]). Thus one arrives at a non-symmetric variational formulation of the problem and a Rayleigh-Ritz projection (by finite elements, e.g.) yields a linear but non-symmetric matrix eigenvalue problem. This formulation has the advantage that it is smaller than the one from the pure displacement model since it introduces only one unknown per node to describe the fluid, but it seems to be undesirable because eigensolvers for non-symmetric matrices such as Arnoldi's method require much higher cost than symmetric eigensolvers, both in terms of storage and computation.

Symmetric models of coupled fluid-structure vibration problems without spurious solutions have been achieved by describing the structure-acoustic system by a three field formulation complementing the structural displacement and the fluid pressure with the fluid velocity potential [9], [11] or the fluid displacement potential [10], [14]. Finite element approximations based on this type of modeling are favored today, since one obtains symmetric matrix eigenvalue problems and hence variational characterizations of eigenvalues allow for using standard spectral approximation theory (see Babuška and Osborne [2]) to obtain convergence results for eigenvalues and eigenvectors for Galerkin type projection methods (cf. [1], [5], [6], [13]).

In this note we consider the elastoacoustic vibration problem describing the fluid by its pressure field and the structure by its displacement field. We prove that although the resulting eigenvalue problem is non-symmetric it shares many important properties with the symmetric model: taking advantage of a Rayleigh functional (which generalizes the Rayleigh quotient for linear problems) its eigenvalues allow for the variational characterizations known from the linear theory. Namely, they can be characterized by Rayleigh's principle, and are minmax and maxmin values of the Rayleigh functional.

The paper is organized as follows. Chapter 2 introduces the fluid-solid interaction problem and its variational formulation, and Chapter 3 collects properties of the problem, in particular a relation between the left and right eigenfunctions corresponding to the same eigenvalue, which motivates the definition of a Rayleigh functional. In Chapter 4 we prove variational characterizations of the eigenvalues generalizing Rayleigh's principle and the minmax and maxmin characterizations known for self-adjoint problems. The paper closes with concluding remarks in Chapter 5 about finite element discretizations and its efficient numerical solution.

2. Model problem

We consider the free vibrations of an elastic structure completely filled with a homogeneous, inviscid, and compressible fluid neglecting gravity effects. The fluid and the solid occupy Lipschitz domains $\Omega_f \subset \mathbb{R}^d$ and $\Omega_s \subset \mathbb{R}^d$, respectively, which we assume non-overlapping, $\Omega_f \cap \Omega_s = \emptyset$.

The boundary shall be divided by

$$\partial\Omega_s = \Gamma_D \cup \Gamma_I \text{ and } \partial\Omega_f = \Gamma_N \cup \Gamma_I$$

into pairwise disjoint parts $\Gamma_D, \Gamma_N, \Gamma_I$, where Γ_D and Γ_N are Dirichlet and Neumann type boundaries and Γ_I is a common interface which is responsible for the coupling effect. The linear-elastic solid is modeled by its displacement function $u\colon \Omega_s \to \mathbb{R}^d$, $d = 2, 3$. The compressible, inviscid, and homogeneous fluid is described by the relative pressure $p\colon \Omega_f \to \mathbb{R}$. This yields a formulation as a system of homogeneous time-independent partial differential equations

$$\begin{aligned}
\operatorname{Div}\sigma(u) + \omega^2 \varrho_s u &= 0 \text{ in } \Omega_s,\\
\nabla^2 p + \frac{\omega^2}{c^2} p &= 0 \text{ in } \Omega_f,\\
u &= 0 \text{ on } \Gamma_D,\\
\nabla p \cdot n_f &= 0 \text{ on } \Gamma_N,\\
\sigma(u)n - pn &= 0 \text{ on } \Gamma_I,\\
\omega^2 \varrho_f u \cdot n + \nabla p \cdot n &= 0 \text{ on } \Gamma_I,
\end{aligned}$$

where ω is the eigenfrequency of vibrations, σ is the stress tensor of the solid, n_f is the unit normal vector on Γ_N, and n denotes the unit normal vector on Γ_I oriented towards the solid part. The interface boundary conditions are a consequence of an equilibrium of acceleration and force densities at the contact interface. We assume that the fluid density $\varrho_f > 0$ is constant in Ω_f and that the solid density $\varrho_s\colon \Omega_s \to \mathbb{R}$ satisfies $0 < C_1 < \varrho_s < C_2$ where C_1 and C_2 (as in the whole paper) denote positive generic constants.

The variational form can be obtained separately for the solid and the fluid. For a bounded domain $D \subset \mathbb{R}^d$, appropriate function spaces are given by the Sobolev spaces

$$H^k(\Omega) := \{u \in L^2(\Omega)\colon\ D^\alpha u \in L^2(\Omega) \text{ for } |\alpha| \leqslant k\}, \quad k = 0, 1, 2, \ldots$$

endowed with the scalar product

$$\langle u, v\rangle_k := \sum_{|\alpha|\leqslant k} \langle D^\alpha u, D^\alpha v\rangle_{L^2(\Omega)},$$

where the derivatives are meant in the weak sense. To take into account homogeneous Dirichlet boundary conditions we introduce the space $H^k_\Gamma(\Omega)$ for $\Gamma \subset \partial\Omega$ as the completion of $C^\infty_\Gamma(\Omega)$ in $H^k(\Omega)$, where $C^\infty_\Gamma(\Omega)$ denotes the space of infinitely times differentiable functions u on Ω with $u = 0$ in a neighborhood of Γ.

To rewrite the problem in a variational formulation, we define bilinear forms

$$\begin{aligned}
a_s &\colon H^1_{\Gamma_D}(\Omega_s)^d \times H^1_{\Gamma_D}(\Omega_s)^d \to \mathbb{R}, & a_s(v,u) &= \int_{\Omega_s} \sigma(u) : \nabla v \,\mathrm{d}x,\\
c &\colon H^1_{\Gamma_D}(\Omega_s)^d \times H^1(\Omega_f) \to \mathbb{R}, & c(v,p) &= \int_{\Gamma_I} -pn\cdot v \,\mathrm{d}s,\\
a_f &\colon H^1(\Omega_f) \times H^1(\Omega_f) \to \mathbb{R}, & a_f(q,p) &= \int_{\Omega_f} \frac{1}{\varrho_f}\nabla p\cdot\nabla q \,\mathrm{d}x,\\
b_s &\colon H^1_{\Gamma_D}(\Omega_s)^d \times H^1_{\Gamma_D}(\Omega_s)^d \to \mathbb{R}, & b_s(v,u) &= \int_{\Omega_s} \varrho_s uv \,\mathrm{d}x,\\
b_f &\colon H^1(\Omega_f) \times H^1(\Omega_f) \to \mathbb{R}, & b_f(q,p) &= \int_{\Omega_f} \frac{1}{\varrho_f c^2} pq \,\mathrm{d}x,
\end{aligned}$$

where $A : B = \sum_{ij} a_{ij}b_{ij}$ denotes the scalar matrix product. Then we obtain the following problem.

Find $\lambda \in \mathbb{C}$ and nonzero $(u,p) \in H^1_{\Gamma_D}(\Omega_s)^d \times H^1(\Omega_f)$ such that

$$a_s(v,u) + c(v,p) = \lambda b_s(v,u) \tag{2.1a}$$

and

$$a_f(q,p) = \lambda(-c(u,q) + b_f(q,p)) \tag{2.1b}$$

for all $(v,q) \in H^1_{\Gamma_D}(\Omega_s)^d \times H^1(\Omega_f)$.

We can immediately formulate the adjoint eigenvalue problem:

Find $\lambda \in \mathbb{C}$ and nonzero $(u,p) \in H^1_{\Gamma_D}(\Omega_s)^d \times H^1(\Omega_f)$ such that

$$a_s(u,v) = \lambda(b_s(u,v) - c(v,p)) \tag{2.2a}$$

and

$$c(u,q) + a_f(p,q) = \lambda b_f(p,q) \tag{2.2b}$$

for all $(v,q) \in H^1_{\Gamma_D}(\Omega_s)^d \times H^1(\Omega_f)$.

Note that a_s and a_f are continuous in $H^1_{\Gamma_D}(\Omega_s)^d$ and $H^1(\Omega_f)$. For the linearized strain tensor ε in the solid we assume that the strain-stress relationship satisfies

$$\sigma(v)\colon \nabla v \geqslant C_1 \varepsilon(v)\colon \varepsilon(v)$$

for some constant $C_1 > 0$, such that Korn's second inequality implies that a_s is a coercive bilinear form.

Problem (2.1) can be written in operator notation. The aim is to find $\lambda \in \mathbb{C}$ and nonzero $(u,p) \in H^1_{\Gamma_D}(\Omega_s)^d \times H^1(\Omega_f)$ so that

$$\mathcal{K}_s u + \mathcal{C} p = \lambda \mathcal{M}_s u \tag{2.3a}$$

$$\mathcal{K}_f p = \lambda(-\mathcal{C}' u + \mathcal{M}_f p), \tag{2.3b}$$

where the operators are defined corresponding to the variational formulation in (2.1).

3. Properties

Some elementary properties of the fluid-solid interaction eigenvalue problem can be given as follows.

Lemma 3.1.

(i) *The eigenvalue problem and its adjoint problem have a zero eigenvalue with corresponding one dimensional eigenspaces* (u_0, p_0) *and* $(0, p_0)$, *where* $p_0 \equiv 1$ *and* u_0 *is determined in the proof.*

(ii) *A function* (u,p) *is an eigensolution of the right eigenvalue problem corresponding to an eigenvalue* $\lambda \neq 0$ *if and only if* $(\lambda u, p)$ *is an eigensolution of the adjoint eigenvalue problem corresponding to the same eigenvalue.*

(iii) *Eigenfunctions* (u_1, p_1) *and* (u_2, p_2) *of* (2.1) *corresponding to distinct eigenvalues* $\lambda_1 \neq \lambda_2$ *are orthogonal with respect to the inner product*

$$\langle (u,p),(v,q)\rangle := a_s(u,v) + b_f(p,q).$$

(iv) *Assume that* (u_1, p_1) *is an eigensolution of* (2.1) *and* $(\hat{u}_2, \hat{p}_2)$ *an eigensolution of the adjoint problem* (2.2) *corresponding to the eigenvalues* λ_1 *and* λ_2, *respectively.*

If $\lambda_1 \neq \lambda_2$ *then it holds that*

$$a_s(\hat{u}_2, u_1) + c(\hat{u}_2, p_1) + a_f(\hat{p}_2, p_1) = b_s(\hat{u}_2, u_1) - c(u_1, \hat{p}_2) + b_f(p_1, \hat{p}_2) = 0.$$

If $\lambda_1 = \lambda_2$ *and* $(\hat{u}_2, \hat{p}_2) = (\lambda_1 u_1, p_1)$ *then it holds that*

$$a_s(\hat{u}_2, u_1) + c(\hat{u}_2, p_1) + a_f(\hat{p}_2, p_1) \geqslant 0$$

and

$$b_s(\hat{u}_2, u_1) - c(u_1, \hat{p}_2) + b_f(p_1, \hat{p}_2) > 0.$$

(v) *The eigenvalue problem* (2.1) *has only real non-negative eigenvalues.*

Proof. (i) The eigenspace corresponding to $\lambda = 0$ is obtained by the variational equation $a_f(p, q) = 0$ for all $q \in H^1(\Omega_f)$ which is solved by all constant functions p_0. This yields a variational equality $a_s(v, u) = -c(v, p_0)$ for the solid part which has a unique solution u_0 as $a_s(\cdot, \cdot)$ is coercive by Korn's second inequality.

The adjoint eigenvalue problem is solved by the eigenfunction $(0, p_0)$.

(ii) Let (u, p) be an eigensolution. Equations (2.1a) multiplied by λ and (2.1b) constitute the adjoint eigenproblem. Conversely, assume that $(\hat{u}, \hat{p})$ is an adjoint eigensolution. Then we obtain (2.1) when we divide (2.2a) by λ and consider (2.2b).

(iii) Consider (2.1) for the eigenpairs $(\lambda_1, (u_1, p_1))$ with $(v, q) = (u_2, p_2)$ and $(\lambda_2, (u_2, p_2))$ with $(v, q) = (u_1, p_1)$. Then

$$\begin{aligned}
&(\lambda_1 - \lambda_2)(a_s(u_1, u_2) + b_f(p_1, p_2)) \\
&\quad = \lambda_1 a_s(u_1, u_2) - \lambda_2 a_s(u_1, u_2) + \lambda_1 b_f(p_1, p_2) - \lambda_2 b_f(p_1, p_2) \\
&\quad = \lambda_1 \lambda_2 b_s(u_1, u_2) - \lambda_1 c(u_1, p_2) - \lambda_2 \lambda_1 b_s(u_2, u_1) + \lambda_2 c(u_2, p_1) \\
&\qquad + a_f(p_2, p_1) + \lambda_1 c(u_1, p_2) - a_f(p_1, p_2) - \lambda_2 c(u_2, p_1) = 0.
\end{aligned}$$

(iv) Using $(v, q) := (\hat{u}_2, \hat{p}_2)$ as a test function in problem (2.1), we obtain

$$a_s(\hat{u}_2, u_1) + c(\hat{u}_2, p_1) + a_f(\hat{p}_2, p_1) = \lambda_1 \big(b_s(\hat{u}_2, u_1) - c(u_1, \hat{p}_2) + b_f(\hat{p}_2, p_1)\big),$$

and $(v, q) := (u_1, p_1)$ in the adjoint problem (2.2) yields

$$a_s(\hat{u}_2, u_1) + c(\hat{u}_2, p_1) + a_f(\hat{p}_2, p_1) = \lambda_2 \big(b_s(\hat{u}_2, u_1) - c(u_1, \hat{p}_2) + b_f(\hat{p}_2, p_1)\big),$$

from which we get statement (iv).

(v) This is a consequence of (iv) where $\lambda_1 = \lambda_2$ and $(\hat{u}_2, \hat{p}_2) = (\lambda_1 u_1, p_1)$. □

We will make use of the operator notation to show that the fluid-solid eigenvalue problem has an infinite countable number of eigenvalues. We can set up the self-adjoint problem to find $\lambda \in \mathbb{R}$ and nonzero $(u, p) \in H^1_{\Gamma_D}(\Omega_s)^d \times H^1(\Omega_f)$ such that

$$\mathcal{K}_s \mathcal{M}_s^{-1} \mathcal{K}_s u + \mathcal{K}_s \mathcal{M}_s^{-1} \mathcal{C} p = \lambda \mathcal{K}_s u, \tag{3.1a}$$

$$\mathcal{C}' \mathcal{M}_s^{-1} \mathcal{K}_s u + (\mathcal{K}_f + \mathcal{C}' \mathcal{M}_s^{-1} \mathcal{C}) p = \lambda \mathcal{M}_f p. \tag{3.1b}$$

Problems (2.3) and (3.1) share the same eigenvalues and eigenfunctions. For a given eigensolution $(\lambda,(u,p))$ of (2.3), apply the operator $\mathcal{C}'\mathcal{M}_s^{-1}$ to (2.3a), add the resulting equation to (2.3b) to obtain (3.1b), and apply $\mathcal{K}_s\mathcal{M}_s^{-1}$ to (2.3a) to obtain (3.1a). The converse implication follows by undoing these transformations.

To obtain H^0-coercive operators on the left hand side, we substitute $\mathcal{K}_s^{1/2}u$ by u and obtain

$$\mathcal{K}_s^{1/2}\mathcal{M}_s^{-1}\mathcal{K}_s^{1/2\prime}u + \mathcal{K}_s^{1/2}\mathcal{M}_s^{-1}\mathcal{C}p = \lambda u, \tag{3.2a}$$

$$\mathcal{C}'\mathcal{M}_s^{-1}\mathcal{K}_s^{1/2\prime}u + (\mathcal{K}_f + \mathcal{C}'\mathcal{M}_s^{-1}\mathcal{C})p = \lambda\mathcal{M}_f p \tag{3.2b}$$

which is equivalent to (2.3) and (3.1) up to the transformation of the solid eigenfunction.

In order to apply the spectral theory for compact self-adjoint operators on Hilbert spaces to fluid-solid eigenvalue problems we shift the eigenvalues of (3.2) by $\tau > 0$ and obtain the eigenproblem to find $\lambda \in \mathbb{R}$ and nonzero $(u,p) \in H^1_{\Gamma_D}(\Omega_s)^d \times H^1(\Omega_f)$ such that

$$\mathcal{K}(u,p) = \lambda\mathcal{M}(u,p), \tag{3.3}$$

where the operators $\mathcal{K}\colon H^1_{\Gamma_D}(\Omega_s)^d \times H^1(\Omega_f) \to H^1_{\Gamma_D}(\Omega_s)^d \times H^1(\Omega_f)$ and $\mathcal{M}\colon H^0(\Omega_s)^d \times H^0(\Omega_f) \to H^0(\Omega_s)^d \times H^0(\Omega_f)$ are given by

$$\begin{aligned}\mathcal{M}(u,p) &= (u, \mathcal{M}_f p),\\ \mathcal{K}(u,p) &= (\mathcal{K}_s^{1/2}\mathcal{M}_s^{-1}\mathcal{K}_s^{1/2\prime}u + \mathcal{K}_s^{1/2}\mathcal{M}_s^{-1}\mathcal{C}p, \mathcal{C}'\mathcal{M}_s^{-1}\mathcal{K}_s^{1/2\prime}u + (\mathcal{K}_f + \mathcal{C}'\mathcal{M}_s^{-1}\mathcal{C})p)\\ &\quad + \tau\mathcal{M}(u,p).\end{aligned}$$

The Riesz-Schauder theory for compact self-adjoint operators implies that (3.3) and (2.3) have an infinite countable number of eigenvalues $\lambda_k \in \mathbb{R}$ converging to ∞ with the corresponding finite dimensional eigenspaces $E_k \subset H^1_{\Gamma_D}(\Omega_s)^d \times H^1(\Omega_f)$.

4. Variational characterizations

Lemma 3.1 states the relationship between the eigenfunctions of problem (2.1) and the adjoint problem (2.2). The adjoint eigenfunction $(\lambda u, p)$ can be used as a test function in equation (2.1) so that we obtain

$$\lambda a_s(u,u) + \lambda c(u,p) + a_f(p,p) = \lambda^2 b_s(u,u) - \lambda c(u,p) + \lambda b_f(p,p)$$

for any eigensolution $(\lambda,(u,p))$, i.e. it is a zero of the function

$$g(\lambda,(u,p)) := \lambda^2 b_s(u,u) + \lambda(b_f(p,p) - a_s(u,u) - 2c(u,p)) - a_f(p,p). \tag{4.1}$$

If $b_s(u,u) > 0$, this equation is quadratic in λ and the question arises which of its roots is the eigenvalue λ of (2.1).

Lemma 4.1. *Let* (u,p) *be an eigenfunction of problem* (2.1)*. Then the maximal root of* $g(\lambda,(u,p))$ *is an eigenvalue of* (2.1) *corresponding to* (u,p)*.*

Proof. If $a_f(p,p) > 0$, we have $g(0,(u,p)) < 0$, and (4.1) has exactly one positive solution λ, which is the eigenvalue of (2.1) corresponding to (u,p).

If $a_f(p,p) = 0$, the quadratic equation (4.1) has two solutions, say $\lambda_2 \leqslant \lambda_1$. We consider a sequence of perturbations of problem (2.1) such that its solutions $(\lambda^{(j)},(u_j,p_j))$ close to $(\lambda,(u,p))$ satisfy $a_f(p_j,p_j) > 0$ and $a_f(p_j,p_j) \to 0$ as $j \to \infty$. Then the corresponding perturbed quadratic equations (4.1) have two roots $\lambda_2^{(j)} < 0 < \lambda^{(j)}$, $\lambda_1 = \lim_{j\to\infty} \lambda^{(j)} \geqslant 0$ is an eigenvalue of problem (2.1), and $\lim_{j\to\infty} \lambda_2^{(j)} = \lambda_2 \leqslant 0$. □

Hence, in any case if (u,p) is an eigenfunction of problem (2.1), the maximal root of (4.1) is the nonnegative eigenvalue of (2.1) corresponding to (u,p). This suggests to introduce an eigenvalue approximation for some general nonzero $(u,p) \in H^1_{\Gamma_D}(\Omega_s)^d \times H^1(\Omega_f)$ by g and we define the nonlinear Rayleigh functional as the maximal root of $g(\cdot,(u,p))$.

Definition 4.1. The functional $r\colon\ H^1_{\Gamma_D}(\Omega_s)^d \times H^1(\Omega_f) \setminus \{0\} \to \mathbb{R}$, where any nonzero $(u,p) \in H^1_{\Gamma_D}(\Omega_s)^d \times H^1(\Omega_f)$ is mapped to the maximal root of $g(\cdot,(u,p))$ is called the nonlinear Rayleigh functional, i.e.,

$$r(u,p) = \begin{cases} \Delta + \sqrt{\Delta^2 + \dfrac{a_f(p,p)}{b_s(u,u)}} & \text{if } b_s(u,u) \neq 0, \\ \dfrac{a_f(p,p)}{b_f(p,p)} & \text{if } b_s(u,u) = 0, \end{cases}$$

where

$$\Delta = \frac{1}{2}\frac{-b_f(p,p) + a_s(u,u) + 2c(u,p)}{b_s(u,u)}.$$

Although fluid-solid eigenvalue problems are not self-adjoint, one obtains variational characterizations using the nonlinear Rayleigh functional. These results generalize variational principles known from the linear self-adjoint case.

First we prove a lemma relating a function spanned by certain eigenfunctions to its nonlinear Rayleigh functional.

Lemma 4.2. *Let $I = \mathbb{N}$ or $I = \{1, \ldots, m\}$ be an index set, $(u_i, p_i)_{i \in I}$ linearly independent eigenfunctions of* (2.1) *corresponding to distinct eigenvalues $(\lambda_i)_{i \in I}$ enumerated in ascendending order, $\lambda_i < \lambda_j$ if $i < j$, and let*

$$(u, p) = \sum_{i \in I} (u_i, p_i).$$

(i) *It holds for any $j \in I$ that*

$$g(\lambda_j, (u, p)) = g(\lambda_j, (u, p) - (u_j, p_j)).$$

(ii) *It holds that*

$$\lambda_1 \leqslant r(u, p) \leqslant \sup_{i \in I} \lambda_i. \tag{4.2}$$

P r o o f. (i) From (2.2) we obtain for any $j \in I$ that

$$\begin{aligned} g(\lambda_j, (u, p)) &= \lambda_j^2 b_s(u, u) + \lambda_j (b_f(p, p) - a_s(u, u) - 2c(u, p)) - a_f(p, p) \\ &= \sum_{k,l} (\lambda_j^2 b_s(u_k, u_l) + \lambda_j (b_f(p_k, p_l) - a_s(u_k, u_l) - 2c(u_k, p_l)) - a_f(p_k, p_l)) \\ &= \sum_{k,l} (\lambda_j (\lambda_j - \lambda_l) b_s(u_k, u_l) + (\lambda_j - \lambda_l)(b_f(p_k, p_l) - c(u_k, p_l))) \\ &= \sum_{k,l \neq j} (\lambda_j (\lambda_j - \lambda_l) b_s(u_k, u_l) + (\lambda_j - \lambda_l)(b_f(p_k, p_l) - c(u_k, p_l))). \end{aligned}$$

From the orthogonality relation in Lemma 3.1 (iv) we obtain for every $l \neq j$ with $\hat{u}_j = \lambda_j u_j$

$$\lambda_j b_s(u_j, u_l) - c(u_j, p_l) + b_f(p_j, p_l) = 0,$$

and hence,

$$\begin{aligned} g(\lambda_j, (u, p)) = &\sum_{k,l \neq j} \lambda_j (\lambda_j - \lambda_l) b_s(u_k - u_j, u_l) \\ &+ (\lambda_j - \lambda_l)(b_f(p_k - p_j, p_l) - c(u_k - u_j, p_l)) \\ = &\, g(\lambda_j, (u, p) - (u_j, p_j)). \end{aligned}$$

(ii) For $m = 1$, we have $r(u_1, p_1) = \lambda_1$ by construction of r.
Assume that (4.2) is true for some m. Then it follows from

$$g(\cdot, (u_2, p_2) + \ldots + (u_{m+1}, p_{m+1})) \leqslant 0 \quad \text{in } [0, \lambda_2]$$

that

$$g(\lambda_1, (u_1, p_1) + \ldots + (u_{m+1}, p_{m+1})) = g(\lambda_1, (u_2, p_2) + \ldots + (u_{m+1}, p_{m+1})) \leqslant 0,$$

and from

$$g(\cdot, (u_1, p_1) + \ldots + (u_m, p_m)) \geqslant 0 \quad \text{in } [\lambda_m, \infty)$$

that

$$g(\lambda_{m+1}, (u_1, p_1) + \ldots + (u_{m+1}, p_{m+1})) = g(\lambda_{m+1}, (u_1, p_1) + \ldots + (u_m, p_m)) \geqslant 0.$$

This implies $\lambda_1 \leqslant r(u, p) \leqslant \lambda_{m+1}$. □

These results can be used to prove the variational principles known from the self-adjoint case.

Theorem 4.1. *Let* $\lambda_1 \leqslant \lambda_2 \leqslant \ldots$ *be the eigenvalues of* (2.1) *in ascending order and* $(u_1, p_1), (u_2, p_2), \ldots$ *the corresponding eigenfunctions. Then*

(i) *(Rayleigh's principle)*

$$\lambda_k = \min\{r(u, p)\colon\ a_s(u, u_j) + b_f(p, p_j) = 0,\ j = 1, \ldots, k-1\}, \tag{4.3}$$

(ii) *(*minmax *characterization)*

$$\lambda_k = \min_{\substack{S_k \subset H^1_{\Gamma_D}(\Omega_s)^d \times H^1(\Omega_f) \\ \dim S_k = k}} \ \max_{0 \neq (u,p) \in S_k} r(u, p), \tag{4.4}$$

(iii) *(*maxmin *characterization)*

$$\lambda_k = \max_{\substack{S_{k-1} \subset H^1_{\Gamma_D}(\Omega_s)^d \times H^1(\Omega_f) \\ \dim S_{k-1} = k-1}} \ \min_{0 \neq (u,p) \in S^{\perp}_{k-1}} r(u, p), \tag{4.5}$$

where

$$S^{\perp} := \{(u, p) \in H^1_{\Gamma_D}(\Omega_s)^d \times H^1(\Omega_f)\colon\ a_s(u, v) + b_f(p, q) = 0 \ for\ (v, q) \in S\}.$$

P r o o f. (i) We obtain the inequality

$$\lambda_k \leqslant \min_{(u,p) \in \operatorname{span}\{(u_i, p_i)\}_{i \in \mathbb{N}}} \{r(u, p)\colon\ a_s(u, u_j) + b_f(p, p_j) = 0,\ j = 1, \ldots, k-1\}$$

from (4.2) by $I = (k, k+1, \dots)$ and the corresponding equality by choosing $(u, p) = (u_k, p_k)$. The restriction $(u, p) \in \operatorname{span}\{(u_i, p_i)\}_{i\in\mathbb{N}}$ can be removed as r is continuous and the direct sum of the eigenspaces is dense in $H^1_{\Gamma_D}(\Omega_s)^d \times H^1(\Omega_f)$.

(ii) Let $S' = \operatorname{span}((u_k, p_k), (u_{k+1}, p_{k+1}), \dots)$. Then $S' \cap S_k \neq \{0\}$ for any k-dimensional subspace $S_k \subset \operatorname{span}((u_1, p_1), (u_2, p_2), \dots)$. Therefore, there exists a nonzero function $(u, p) \in S' \cap S_k$ such that

$$r(u, p) \geqslant \lambda_k \quad \text{for any } k\text{-dimensional subspace } S_k.$$

Hence,

$$\sup_{0\neq(u,p)\in S_k} r(u, p) \geqslant \lambda_k$$

for any k-dimensional subspace S_k. The supremum is attained because r is continuous and S_k closed, and we obtain

$$\inf_{\substack{S_k\subset H^1_{\Gamma_D}(\Omega_s)^d\times H^1(\Omega_f)\\ \dim S_k=k}} \max_{0\neq(u,p)\in S_k} r(u, p) \geqslant \lambda_k.$$

Choosing $S_k = \operatorname{span}((u_1, p_1), (u_2, p_2), \dots, (u_k, p_k))$, we have

$$\min_{\substack{S_k\subset H^1_{\Gamma_D}(\Omega_s)^d\times H^1(\Omega_f)\\ \dim S_k=k}} \max_{0\neq(u,p)\in S_k} r(u, p) = \lambda_k.$$

(iii) Let $V_k := \operatorname{span}\{(u_j, p_j) \colon j = 1, \dots, k\}$. Then $V_k \cap S^{\perp}_{k-1} \neq \emptyset$ for every $(k-1)$-dimensional subspace S_{k-1}, and it follows from (4.2) that $r(u, p) \leqslant \lambda_k$ for every $(u, p) \in V_k \cap S^{\perp}_{k-1}$, i.e.,

$$\min_{0\neq(u,p)\in S^{\perp}_{k-1}} r(u, p) \leqslant \lambda_k.$$

Choosing $S_{k-1} := \operatorname{span}\{(u_j, p_j) \colon j = 1, \dots, k-1\}$, we get (4.5). □

5. Conclusions

For the non-selfadjoint elastoacoustic vibration problem describing the fluid by its pressure field and the structure by its displacement field we have proved variational characterizations of its eigenvalues generalizing Rayleigh's principle as well as minmax and maxmin characterizations. Discretizing the elastoacoustic problem with finite elements where the triangulation obeys the geometric partition into the fluid and the structure domain one obtains a non-symmetric matrix eigenvalue problem which inherits the variational properties and the eigenvalues of which are upper bounds of the eigenvalues of the original problem. The standard spectral approximation theory applies [15] to proving convergence results for Galerkin type methods. For the matrix eigenvalue problem the Rayleigh functional iteration is cubically convergent as is the Rayleigh quotient iteration for linear symmetric problems, and based on this structure preserving iterative projection methods of Jacobi-Davidson type and nonlinear Arnoldi type can be defined [17], [18]. The automated multi-level sub-structuring method (AMLS) introduced by Bennighof [4] for linear eigenvalue problems in structural analysis can be generalized to the non-symmetric elastoacoustic problem, and an a priori error bound can be proved using the minmax characterization [16].

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