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Xing-Cai Zhou; Jin-Guan Lin

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COMPLETE q-ORDER MOMENT CONVERGENCE OF MOVING AVERAGE PROCESSES UNDER φ-MIXING ASSUMPTIONS

Xing-Cai Zhou, Tongling, Jin-Guan Lin, Nanjing

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Abstract. Let $\{Y_i, -\infty < i < \infty\}$ be a doubly infinite sequence of identically distributed φ-mixing random variables, and $\{a_i, -\infty < i < \infty\}$ an absolutely summable sequence of real numbers. We prove the complete q-order moment convergence for the partial sums of moving average processes $\Big\{X_n = \sum\limits_{i=-\infty}^{\infty} a_i Y_{i+n}, n \geqslant 1\Big\}$ based on the sequence $\{Y_i, -\infty < i < \infty\}$ of φ-mixing random variables under some suitable conditions. These results generalize and complement earlier results.

Keywords: moving average; φ-mixing; complete convergence; q-order moment

MSC 2010: 60F15, 60G50

1. Introduction

We assume that $\{Y_i, -\infty < i < \infty\}$ is a doubly infinite sequence of identically distributed random variables. Let $\{a_i, -\infty < i < \infty\}$ be an absolutely summable sequence of real numbers, and

$$X_n = \sum_{i=-\infty}^{\infty} a_i Y_{i+n}, \quad n \geqslant 1. \tag{1.1}$$

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We denote $S_n = \sum\limits_{k=1}^{n} X_k$, $n \geqslant 1$, the sequence of partial sums. The limiting behavior of the moving average process $\{X_n, n \geqslant 1\}$ has been extensively investigated when the random variables $\{Y_i, -\infty < i < \infty\}$ form a sequence of independent random variables. For example, Ibragimov [8] established the central limit theorem, Burton and Dehling [2] obtained a large deviation principle, and Li et al. [11] obtained the complete convergence. Motivated by applications to sequential analysis of time series and to the renewal theory, the complete convergence was extended to weakly dependent (φ-mixing and ϱ-mixing) sequences by a lot of authors (cf. Lai [10], Shao [15], [16], and Peligrad [14]). Hsu and Robbins [7] first discussed the concept of complete convergence, and proved complete convergence for the partial sums of a sequence of i.i.d random variables as follows:

Theorem A. *Suppose $\{X_n, n \geqslant 1\}$ is a sequence of i.i.d random variables. If $EX_1 = 0$ and $E|X_1|^2 < \infty$, then $\sum\limits_{n=1}^{\infty} P\{|S_n| \geqslant \varepsilon n\} < \infty$ for all $\varepsilon > 0$.*

The following theorem due to [11] extended the above result for moving average processes.

Theorem B. *Suppose $\{X_n, n \geqslant 1\}$ is the moving average process based on a sequence $\{Y_i, -\infty < i < \infty\}$ of i.i.d random variables with $EY_1 = 0$ and $E|Y_1|^2 < \infty$. Then $\sum\limits_{n=1}^{\infty} P\{|S_n| \geqslant \varepsilon n\} < \infty$ for all $\varepsilon > 0$.*

We know that if $\{Y_i, -\infty < i < \infty\}$ is a sequence of i.i.d random variables, the moving average random variables $\{X_n, n \geqslant 1\}$ are dependent, which is called weak dependence. Recently, some limiting results on complete convergence for moving average processes based on the dependent sequences have been obtained. For example, Yu and Wang [17] and Baek et al. [1] under the negative dependence assumption, Chen et al. [3] under the negative association random variables, and Zhang [18] under the φ-mixing assumption. Chen et al. [4] further improved the result of [18].

When $\{X_k, k \geqslant 1\}$ is a sequence of i.i.d random variables with mean zeros and positive finite variances, Chow [6] obtained the following result on complete moment convergence:

Theorem C. *Suppose that $\{X_n, n \geqslant 1\}$ is a sequence of i.i.d random variables with $EX_1 = 0$. For $1 \leqslant p < 2$ and $r > p$, if $E\{|X_1|^r + |X_1| \log(1 + |X_1|)\} < \infty$, then*

$$\sum_{n=1}^{\infty} n^{r/p-2-1/p} E\{|S_n| - \varepsilon n^{1/p}\}_+ < \infty$$

for any $\varepsilon > 0$, where and in the following $x_+ = \max\{0, x\}$ and $x_+^q = (x_+)^q$.

Li and Zhang [13] showed that this kind of result also holds for moving average processes under negatively associated random variables, and Kim and Ko [9] to moving average processes under φ-mixing assumptions. Then, Zhou [19] improved the result of [9].

Recently, Li and Spătaru [12] obtained the refinement of complete convergence

$$\int_{\varepsilon}^{\infty} \sum_{n=1}^{\infty} n^{r-2} P\left\{\left|\sum_{k=1}^{n} X_n - nb\right| > x^{1/q} n^{1/p}\right\} \mathrm{d}x < \infty \quad \text{for all } \varepsilon > 0,$$

under some suitable moment conditions for a sequence of i.i.d random variables $\{X_n, n \geqslant 1\}$, where $0 < p < 2$, $r \geqslant 1$, $q > 0$, and $b = EX$ if $rp \geqslant 1$ and $b = 0$ if $0 < rp < 1$. Chen and Wang [5] pointed out that the two concepts of the refinement of complete convergence and the complete q-order moment convergence are equivalent, because they showed that for all $\varepsilon > 0$

$$\int_{\varepsilon}^{\infty} \sum_{n=1}^{\infty} a_n P\{|Z_n| > x^{1/q} b_n\} \, \mathrm{d}x < \infty$$

and

$$\sum_{n=1}^{\infty} a_n E\{b_n^{-1}|Z_n| - \varepsilon\}_+^q < \infty$$

are equivalent for any $a_n > 0$, $b_n > 0$, $q > 0$, and any sequence of random variables $\{Z_n, n \geqslant 1\}$.

As we know, the complete moment convergence can describe the convergence rate of a sequence of random variables more exactly than the complete convergence. In this paper, we shall show complete q-order moment convergence of moving average processes based on the φ-mixing sequence. Our results extend the results of Chen et al. [4] on complete convergence to the complete q-order moment convergence, and improve the results of Zhou [19] under more optimal moment conditions. Section 2 states the main results and some technical lemmas. Proofs of the main results are provided in Section 3.

2. The main results and some lemmas

Recall that a sequence $\{Y_i, -\infty < i < \infty\}$ is said to be φ-mixing if the mixing coefficient satisfies

$$\varphi(m) = \sup_{k \geqslant 1} \sup\{|P(B|A) - P(B)|, A \in \mathcal{F}_{-\infty}^{k}, P(A) \neq 0, B \in \mathcal{F}_{k+m}^{\infty}\} \to 0$$

as $m \to \infty$, where $\mathcal{F}_n^m = \sigma(Y_i, n \leqslant i \leqslant m)$, $-\infty \leqslant n \leqslant m \leqslant \infty$.

In the following, C will represent a positive constant although its value may change from one appearance to the next, and $[x]$ indicates the maximum integer not larger than x.

Now we state our main results and some lemmas. The proofs of main results will be given in the next section.

Theorem 2.1. *Let $q > 0$, $1 \leqslant p < 2$, $r > 1$. Suppose that $\{X_n, n \geqslant 1\}$ is a moving average process based on a sequence $\{Y_i, -\infty < i < \infty\}$ of identically distributed φ-mixing random variables. If $EY_1 = 0$ and*

$$\begin{aligned} E|Y_1|^{rp} &< \infty, \quad \text{if } q < rp, \\ E|Y_1|^{rp}\log(1+|Y_1|) &< \infty, \quad \text{if } q = rp, \\ E|Y_1|^{q} &< \infty, \quad \text{if } q > rp, \end{aligned}$$

then

$$\sum_{n=1}^{\infty} n^{r-2-q/p} E\{\max_{1\leqslant k\leqslant n} |S_k| - \varepsilon n^{1/p}\}_+^q < \infty \quad \text{for all } \varepsilon > 0, \tag{2.1}$$

and

$$\sum_{n=1}^{\infty} n^{r-2} E\{\sup_{k\geqslant n} k^{-1/p}|S_k| - \varepsilon\}_+^q < \infty \quad \text{for all } \varepsilon > 0. \tag{2.2}$$

When $r = 1$, we get the following theorems.

Theorem 2.2. *Let $1 \leqslant p < 2$ and $q > 0$. Assume that $\sum\limits_{i=-\infty}^{\infty} |a_i|^\theta < \infty$, where θ belongs to $(0,1)$ if $p = 1$ and $\theta = 1$ if $1 < p < 2$. Suppose that $\{X_n, n \geqslant 1\}$ is a moving average process based on a sequence $\{Y_i, -\infty < i < \infty\}$ of identically distributed φ-mixing random variables with $\sum\limits_{m=1}^{\infty} \varphi^{1/2}(m) < \infty$. If $EY_1 = 0$ and*

$$\begin{aligned} E|Y_1|^{p} &< \infty, \quad \text{if } q < p, \\ E|Y_1|^{p}\log(1+|Y_1|) &< \infty, \quad \text{if } q = p, \\ E|Y_1|^{q} &< \infty, \quad \text{if } q > p, \end{aligned}$$

then

$$\sum_{n=1}^{\infty} n^{-1-q/p} E\{\max_{1\leqslant k\leqslant n} |S_k| - \varepsilon n^{1/p}\}_+^q < \infty \quad \text{for all } \varepsilon > 0. \tag{2.3}$$

Theorem 2.3. *Under the conditions of Theorem 2.2, if* $EY_1 = 0$ *and*

$$\begin{aligned} E|Y_1|^p \log(1+|Y_1|) &< \infty, \quad \text{if } q < p, \\ E|Y_1|^p \log^2(1+|Y_1|) &< \infty, \quad \text{if } q = p, \\ E|Y_1|^q &< \infty, \quad \text{if } q > p, \end{aligned}$$

then

$$\sum_{n=1}^{\infty} n^{-1-q/p} \log n E\{\max_{1 \leqslant k \leqslant n} |S_k| - \varepsilon n^{1/p}\}_+^q < \infty \quad \text{for all } \varepsilon > 0, \tag{2.4}$$

and

$$\sum_{n=1}^{\infty} n^{-1} E\{\sup_{k \geqslant n} k^{-1/p} |S_k| - \varepsilon\}_+^q < \infty \quad \text{for all } \varepsilon > 0. \tag{2.5}$$

Remark 2.1. Theorems 2.1–2.3 provide complete q-order moment convergence statements for the maximums and supremums of partial sums of moving average processes based on a sequence of φ-mixing random variables. When $r > 1$, Theorem 2.1 provides the results without any mixing rate. Theorems 2.2 and 2.3 cover the case where $r = 1$, which require the mixing rate to satisfy $\sum\limits_{m=1}^{\infty} \varphi^{1/2}(m) < \infty$.

Remark 2.2. Theorems 2.1 and 2.2 extend respectively Theorems 1 and 2 of Chen et al. [4] on complete convergence to complete q-order moment convergence, and improve Theorems 2.1 and 2.2 of Zhou [19] under more optimal moment conditions, if we ignore some insignificant details connected with slowly varying functions. Theorem 2.3 is similar to Theorem 2.2, but it is a new result.

The following lemmas will be useful. In the first two lemmas we assume that $\{Y_n, n \geqslant 1\}$ is a φ-mixing sequence and $S_k(n) = \sum\limits_{i=k+1}^{k+n} Y_i$, $n \geqslant 1$, $k \geqslant 0$.

Lemma 2.1 (Shao [15]). *Let* $EY_i = 0$ *and* $EY_i^2 < \infty$ *for all* $i \geqslant 1$. *Then for all* $n \geqslant 1$ *and* $k \geqslant 0$ *we have*

$$ES_k^2(n) \leqslant 8000 n \exp\left\{6 \sum_{i=1}^{[\log n]} \varphi^{1/2}(2^i)\right\} \max_{k+1 \leqslant i \leqslant k+n} EY_i^2.$$

Lemma 2.2 (Shao [15]). *Suppose that there exists an array $\{C_{k,n}, k \geqslant 0, n \geqslant 1\}$ of positive numbers such that $\max\limits_{1 \leqslant i \leqslant n} ES_k^2(i) \leqslant C_{k,n}$ for every $k \geqslant 0$ and $n \geqslant 1$. Then for any $s \geqslant 2$ there exists $C = C(s, \varphi(\cdot))$ such that for any $k \geqslant 0$ and $n \geqslant 1$*

$$E \max_{1 \leqslant i \leqslant n} |S_k(i)|^s \leqslant C\Big(C_{k,n}^{s/2} + E\Big(\max_{k < i \leqslant k+n} |Y_i|^s\Big)\Big).$$

Lemma 2.3. *Let $1 \leqslant p < 2$, $r > 1$. Suppose that $\{X_n, n \geqslant 1\}$ is a moving average process based on a sequence $\{Y_i, -\infty < i < \infty\}$ of identically distributed φ-mixing random variables. If $EY_1 = 0$ and $E|Y_1|^{rp} < \infty$, then*

$$\sum_{n=1}^{\infty} n^{r-2} P\big\{\max_{1 \leqslant k \leqslant n} |S_k| \geqslant \varepsilon n^{1/p}\big\} < \infty \quad \text{for all } \varepsilon > 0.$$

Lemma 2.4. *Let $1 \leqslant p < 2$. Assume that $\sum\limits_{i=-\infty}^{\infty} |a_i|^{\theta} < \infty$, where θ belongs to $(0,1)$ if $p = 1$ and $\theta = 1$ if $1 < p < 2$. Suppose that $\{X_n, n \geqslant 1\}$ is a moving average process based on a sequence $\{Y_i, -\infty < i < \infty\}$ of identically distributed φ-mixing random variables with $\sum\limits_{m=1}^{\infty} \varphi^{1/2}(m) < \infty$. If $EY_1 = 0$ and $E|Y_1|^p < \infty$, then*

$$\sum_{n=1}^{\infty} \frac{1}{n} P\{\max_{1 \leqslant k \leqslant n} |S_k| \geqslant \varepsilon n^{1/p}\} < \infty \quad \text{for all } \varepsilon > 0.$$

Remark 2.3. The proofs of Lemmas 2.3 and 2.4 are similar to those of Theorems 1 and 2 of Chen et al. [4] and so they are omitted.

3. Proof of main results

Proof of Theorem 2.1. First, we prove (2.1). Let $Y_{xj} = Y_j I[|Y_j| \leqslant x^{1/q}] - EY_j I[|Y_j| \leqslant x^{1/q}]$, and $l(n) = n^{r-2-q/p}$. Recall that

$$\sum_{k=1}^{n} X_k = \sum_{k=1}^{n} \sum_{i=-\infty}^{\infty} a_i Y_{i+k} = \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_j$$

and since $\sum\limits_{i=-\infty}^{\infty} |a_i| < \infty$, for $x > n^{q/p}$ we have

$$\begin{aligned}
&\max_{1\leqslant k\leqslant n} x^{-1/q}\Big| E\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_j I[|Y_j| \leqslant x^{1/q}]\Big| \\
&\qquad = \max_{1\leqslant k\leqslant n} x^{-1/q}\Big| E\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_j I[|Y_j| > x^{1/q}]\Big| \ (EY_j = 0) \\
&\qquad \leqslant x^{-1/q} \sum_{i=-\infty}^{\infty} |a_i| \sum_{j=i+1}^{i+n} E|Y_j| I[|Y_j| > x^{1/q}] \\
&\qquad \leqslant x^{-1/q} n \sum_{i=-\infty}^{\infty} |a_i| E|Y_1| I[|Y_1| > x^{1/q}] \\
&\qquad \leqslant C x^{-1/q} x^{p/q} E|Y_1| I[|Y_1| > x^{1/q}] \\
&\qquad \leqslant CE|Y_1|^p I[|Y_1| > x^{1/q}] \to 0, \text{ as } x \to \infty.
\end{aligned}$$

Hence, for x large enough one gets

$$x^{-1/q}\Big| E\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_j I[|Y_j| \leqslant x^{1/q}]\Big| < \varepsilon/4. \tag{3.1}$$

We have

$$\begin{aligned}
&\sum_{n=1}^{\infty} l(n) E\{\max_{1\leqslant k\leqslant n} |S_k| - \varepsilon n^{1/p}\}_+^q \\
&\qquad = \sum_{n=1}^{\infty} l(n) \int_0^{\infty} P\{\max_{1\leqslant k\leqslant n} |S_k| - \varepsilon n^{1/p} > x^{1/q}\}\,\mathrm{d}x \\
&\qquad = \sum_{n=1}^{\infty} l(n) \int_0^{n^{q/p}} P\{\max_{1\leqslant k\leqslant n} |S_k| > \varepsilon n^{1/p} + x^{1/q}\}\,\mathrm{d}x \\
&\qquad\quad + \sum_{n=1}^{\infty} l(n) \int_{n^{q/p}}^{\infty} P\{\max_{1\leqslant k\leqslant n} |S_k| > \varepsilon n^{1/p} + x^{1/q}\}\,\mathrm{d}x \\
&\qquad \leqslant \sum_{n=1}^{\infty} n^{q/p} l(n) P\{\max_{1\leqslant k\leqslant n} |S_k| > \varepsilon n^{1/p}\} \\
&\qquad\quad + \sum_{n=1}^{\infty} l(n) \int_{n^{q/p}}^{\infty} P\{\max_{1\leqslant k\leqslant n} |S_k| > x^{1/q}\}\,\mathrm{d}x =: I_1 + I_2.
\end{aligned} \tag{3.2}$$

For I_1, by Lemma 2.3 we have

$$I_1 = \sum_{n=1}^{\infty} n^{r-2} P\{\max_{1\leqslant k\leqslant n} |S_k| > \varepsilon n^{1/p}\} < \infty. \tag{3.3}$$

For I_2, from (3.1) we have

$$I_2 \leqslant C\sum_{n=1}^{\infty} l(n)\int_{n^{q/p}}^{\infty} P\Big\{\max_{1\leqslant k\leqslant n}\Big|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_j I[|Y_j| > x^{1/q}]\Big| \geqslant x^{1/q}/2\Big\}\,\mathrm{d}x \tag{3.4}$$
$$+ C\sum_{n=1}^{\infty} l(n)\int_{n^{q/p}}^{\infty} P\Big\{\max_{1\leqslant k\leqslant n}\Big|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_{xj}\Big| \geqslant x^{1/q}/4\Big\}\,\mathrm{d}x$$
$$=: I_{21} + I_{22}.$$

For I_{21}, by Markov's inequality and the mean-value theorem we have

$$\begin{aligned} I_{21} &\leqslant C\sum_{n=1}^{\infty} l(n)\int_{n^{q/p}}^{\infty} x^{-1/q} E\max_{1\leqslant k\leqslant n}\Big|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_j I[|Y_j| > x^{1/q}]\Big|\,\mathrm{d}x \\ &\leqslant C\sum_{n=1}^{\infty} l(n)\int_{n^{q/p}}^{\infty} n x^{-1/q} E|Y_1| I[|Y_1| > x^{1/q}]\,\mathrm{d}x \\ &= C\sum_{n=1}^{\infty} n l(n)\sum_{m=n}^{\infty}\int_{m^{q/p}}^{(m+1)^{q/p}} x^{-1/q} E|Y_1| I[|Y_1| > x^{1/q}]\,\mathrm{d}x \\ &\leqslant C\sum_{n=1}^{\infty} n l(n)\sum_{m=n}^{\infty} m^{q/p-1/p-1} E|Y_1| I[|Y_1| > m^{1/p}] \\ &= C\sum_{m=1}^{\infty} m^{q/p-1/p-1} E|Y_1| I[|Y_1| > m^{1/p}]\sum_{n=1}^{m} n^{r-q/p-1} \\ &\leqslant \begin{cases} C\sum_{m=1}^{\infty} m^{r-1/p-1} E|Y_1| I[|Y_1| > m^{1/p}], & \text{if } q < rp, \\ C\sum_{m=1}^{\infty} m^{r-1/p-1}\log(1+m) E|Y_1| I[|Y_1| > m^{1/p}], & \text{if } q = rp, \\ C\sum_{m=1}^{\infty} m^{q/p-1/p-1} E|Y_1| I[|Y_1| > m^{1/p}], & \text{if } q > rp, \end{cases} \\ &\leqslant \begin{cases} CE|Y_1|^{rp} < \infty, & \text{if } q < rp, \\ CE|Y_1|^{rp}\log(1+|Y_1|) < \infty, & \text{if } q = rp, \\ CE|Y_1|^{q} < \infty, & \text{if } q > rp. \end{cases} \end{aligned} \tag{3.5}$$

For I_{22}, by Markov's and Hölder's inequalities, the mean-value theorem, Lemmas 2.1 and 2.2, one gets that for any $s \geqslant 2$

$$\begin{aligned} I_{22} &\leqslant C\sum_{n=1}^{\infty} l(n)\int_{n^{q/p}}^{\infty} x^{-s/q} E\max_{1\leqslant k\leqslant n}\Big|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_{xj}\Big|^{s}\,\mathrm{d}x \\ &\leqslant C\sum_{n=1}^{\infty} l(n)\int_{n^{q/p}}^{\infty} x^{-s/q} E\Big[\sum_{i=-\infty}^{\infty}(|a_i|^{1-1/s})\Big(|a_i|^{1/s}\max_{1\leqslant k\leqslant n}\Big|\sum_{j=i+1}^{i+k} Y_{xj}\Big|\Big)\Big]^{s}\,\mathrm{d}x \end{aligned} \tag{3.6}$$

$$
\begin{aligned}
&\leqslant C\sum_{n=1}^{\infty} l(n)\int_{n^{q/p}}^{\infty} x^{-s/q}\bigg(\sum_{i=-\infty}^{\infty}|a_i|\bigg)^{s-1}\bigg(\sum_{i=-\infty}^{\infty}|a_i|E\max_{1\leqslant k\leqslant n}\bigg|\sum_{j=i+1}^{i+k}Y_{xj}\bigg|^{s}\bigg)\,\mathrm{d}x\\
&\leqslant C\sum_{n=1}^{\infty} l(n)\int_{n^{q/p}}^{\infty} x^{-s/q}\\
&\qquad\qquad \times\sum_{i=-\infty}^{\infty}|a_i|\bigg(\max_{1\leqslant k\leqslant n} k\exp\bigg\{6\sum_{j=1}^{[\log k]}\varphi^{1/2}(2^j)\bigg\}\max_{i+1\leqslant j\leqslant i+k} EY_{xj}^2\bigg)^{s/2}\,\mathrm{d}x\\
&\quad + C\sum_{n=1}^{\infty} l(n)\int_{n^{q/p}}^{\infty} x^{-s/q}\sum_{i=-\infty}^{\infty}|a_i|E\max_{i<j\leqslant i+n}|Y_{xj}|^s\,\mathrm{d}x\\
&\leqslant C\sum_{n=1}^{\infty} l(n)\int_{n^{q/p}}^{\infty} x^{-s/q}\\
&\qquad\qquad \times\sum_{i=-\infty}^{\infty}|a_i|\bigg(n\exp\bigg\{6\sum_{j=1}^{[\log n]}\varphi^{1/2}(2^j)\bigg\}\max_{i+1\leqslant j\leqslant i+n} EY_{x1}^2\bigg)^{s/2}\,\mathrm{d}x\\
&\quad + C\sum_{n=1}^{\infty} l(n)\int_{n^{q/p}}^{\infty} x^{-s/q}\sum_{i=-\infty}^{\infty}n|a_i|E|Y_{x1}|^s\,\mathrm{d}x\\
&\leqslant C\sum_{n=1}^{\infty} l(n)\int_{n^{q/p}}^{\infty} x^{-s/q}\bigg(n\exp\bigg\{6\sum_{j=1}^{[\log n]}\varphi^{1/2}(2^j)\bigg\}E|Y_1|^2I[|Y_1|\leqslant x^{1/q}]\bigg)^{s/2}\,\mathrm{d}x\\
&\quad + C\sum_{n=1}^{\infty} l(n)\int_{n^{q/p}}^{\infty} nx^{-s/q}E|Y_1|^sI[|Y_1|\leqslant x^{1/q}]\,\mathrm{d}x =: I_{221}+I_{222}.
\end{aligned}
$$

Note that $\varphi(m)\to 0$ as $m\to\infty$, hence $\sum\limits_{j=1}^{[\log n]}\varphi^{1/2}(2^j)=o(\log n)$. Therefore, for any $\lambda>0$ and $t>0$, $\exp\Big\{\lambda\sum\limits_{j=1}^{[\log n]}\varphi^{1/2}(2^j)\Big\}=o(n^t)$.

For I_{221}, we consider the following two cases.

If $rp<2$, take $s>2$ and let $u=st/2$. We have that $r-(r-1)s/2<1$. Then take $t>0$ small enough such that $u>0$ is so small that $r-(r-1)s/2+u<1$. By the mean-value theorem we have

$$
\begin{aligned}
(3.7)\quad I_{221}&\leqslant C\sum_{n=1}^{\infty} n^{(1+t)s/2}l(n)\int_{n^{q/p}}^{\infty} x^{-s/q}(E|Y_1|^2I[|Y_1|\leqslant x^{1/q}])^{s/2}\,\mathrm{d}x\\
&= C\sum_{n=1}^{\infty} n^{s/2+u}l(n)\sum_{m=n}^{\infty}\int_{m^{q/p}}^{(m+1)^{q/p}} x^{-s/q}(E|Y_1|^2I[|Y_1|\leqslant x^{1/q}])^{s/2}\,\mathrm{d}x\\
&\leqslant C\sum_{n=1}^{\infty} n^{s/2+u}l(n)\sum_{m=n}^{\infty} m^{q/p-s/p-1}(E|Y_1|^2I[|Y_1|\leqslant (m+1)^{1/p}])^{s/2}
\end{aligned}
$$

$$
\begin{aligned}
&= C\sum_{m=1}^{\infty} m^{q/p-s/p-1}(E|Y_1|^2 I[|Y_1| \leqslant (m+1)^{1/p}])^{s/2}\sum_{n=1}^{m} n^{s/2+u} l(n)\\
&\leqslant C\sum_{m=1}^{\infty} m^{r+s/2-s/p+u-2}(E|Y_1|^{rp}|Y_1|^{2-rp} I[|Y_1| \leqslant (m+1)^{1/p}])^{s/2}\\
&\leqslant C\sum_{m=1}^{\infty} m^{r-(r-1)s/2+u-2}(E|Y_1|^{rp} I[|Y_1| \leqslant (m+1)^{1/p}])^{s/2}\\
&\leqslant C\sum_{m=1}^{\infty} m^{r-(r-1)s/2+u-2} < \infty.
\end{aligned}
$$

If $rp \geqslant 2$, take $s > \max(2p(r-1)/(2-p), q)$ and let $u = st/2$. We have that $r - s/p + s/2 < 1$. Then, take $t > 0$ small enough such that $u > 0$ is so small that we have $r - s/p + s/2 + u < 1$. In this case, we note that $E|Y_1|^2 < \infty$. Therefore, one gets

$$
\begin{aligned}
I_{221} &\leqslant C\sum_{n=1}^{\infty} n^{(t+1)s/2} l(n)\int_{n^{q/p}}^{\infty} x^{-s/q}(E|Y_1|^2 I[|Y_1| \leqslant x^{1/q}])^{s/2}\,\mathrm{d}x\\
&= C\sum_{n=1}^{\infty} n^{r-s/p+s/2+u-2} < \infty.
\end{aligned}
\tag{3.8}
$$

So, by (3.7) and (3.8) we get

$$
I_{221} < \infty. \tag{3.9}
$$

For I_{222}, by the mean-value theorem we have

$$
\begin{aligned}
I_{222} &= C\sum_{n=1}^{\infty} nl(n)\sum_{m=n}^{\infty}\int_{m^{q/p}}^{(m+1)^{q/p}} x^{-s/q} E|Y_1|^s I[|Y_1| \leqslant x^{1/q}]\,\mathrm{d}x\\
&\leqslant C\sum_{n=1}^{\infty} nl(n)\sum_{m=n}^{\infty} m^{q/p-s/p-1} E|Y_1|^s I[|Y_1| \leqslant (m+1)^{1/p}]\\
&= C\sum_{m=1}^{\infty} m^{q/p-s/p-1} E|Y_1|^s I[|Y_1| \leqslant (m+1)^{1/p}]\sum_{n=1}^{m} n^{r-q/p-1}\\
&\leqslant \begin{cases}
C\sum\limits_{m=1}^{\infty} m^{r-s/p-1} E|Y_1|^s I[|Y_1| \leqslant (m+1)^{1/p}], & \text{if } q < rp,\\
C\sum\limits_{m=1}^{\infty} m^{q/p-s/p-1}\log(1+m) E|Y_1|^s I[|Y_1| \leqslant (m+1)^{1/p}], & \text{if } q = rp,\\
C\sum\limits_{m=1}^{\infty} m^{q/p-s/p-1} E|Y_1|^s I[|Y_1| \leqslant (m+1)^{1/p}], & \text{if } q > rp,
\end{cases}
\end{aligned}
\tag{3.10}
$$

$$\leqslant \begin{cases} CE|Y_1|^{rp} < \infty, & \text{if } q < rp, \\ CE|Y_1|^{rp}\log(1+|Y_1|) < \infty, & \text{if } q = rp, \\ CE|Y_1|^{q} < \infty, & \text{if } q > rp. \end{cases}$$

Thus, (2.1) follows from (3.2)–(3.6), (3.9), and (3.10).

Now, we prove (2.2). We have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2} E\{\sup_{k\geqslant n} |k^{-\frac{1}{p}} S_k| - \varepsilon\}_+^q \\
&\quad = \sum_{n=1}^{\infty} n^{r-2} \int_0^{\infty} P\{\sup_{k\geqslant n} |k^{-\frac{1}{p}} S_k| > \varepsilon + t^{1/q}\}\,\mathrm{d}t \\
&\quad = \sum_{i=1}^{\infty} \sum_{n=2^{i-1}}^{2^i-1} n^{r-2} \int_0^{\infty} P\{\sup_{k\geqslant n} |k^{-\frac{1}{p}} S_k| > \varepsilon + t^{1/q}\}\,\mathrm{d}t \\
&\quad \leqslant C\sum_{i=1}^{\infty} \int_0^{\infty} P\{\sup_{k\geqslant 2^{i-1}} |k^{-\frac{1}{p}} S_k| > \varepsilon + t^{1/q}\}\,\mathrm{d}t \sum_{n=2^{i-1}}^{2^i-1} 2^{i(r-2)} \\
&\quad \leqslant C\sum_{i=1}^{\infty} 2^{i(r-1)} \int_0^{\infty} P\Big\{\sup_{k\geqslant 2^{i-1}} \Big|k^{-\frac{1}{p}} \sum_{j=1}^{k} X_j\Big| > \varepsilon + t^{1/q}\Big\}\,\mathrm{d}t \\
&\quad \leqslant C\sum_{i=1}^{\infty} 2^{i(r-1)} \sum_{l=i}^{\infty} \int_0^{\infty} P\Big\{\max_{2^{l-1}\leqslant k<2^l} \Big|k^{-\frac{1}{p}} \sum_{j=1}^{k} X_j\Big| > \varepsilon + t^{1/q}\Big\}\,\mathrm{d}t \\
&\quad \leqslant C\sum_{l=1}^{\infty} \int_0^{\infty} P\Big\{\max_{2^{l-1}\leqslant k<2^l} \Big|k^{-\frac{1}{p}} \sum_{j=1}^{k} X_j\Big| > \varepsilon + t^{1/q}\Big\}\,\mathrm{d}t \sum_{i=1}^{l} 2^{i(r-1)} \\
&\quad \leqslant C\sum_{l=1}^{\infty} 2^{l(r-1)} \int_0^{\infty} P\Big\{\max_{2^{l-1}\leqslant k<2^l} \Big|\sum_{j=1}^{k} X_j\Big| > (\varepsilon + t^{1/q})2^{(l-1)/p}\Big\}\,\mathrm{d}t \\
&\qquad\qquad (\text{letting } y = 2^{(l-1)q/p}t) \\
&\quad \leqslant C\sum_{l=1}^{\infty} 2^{l(r-1-q/p)} \int_0^{\infty} P\Big\{\max_{1\leqslant k<2^l} \Big|\sum_{j=1}^{k} X_j\Big| > 2^{(l-1)/p}\varepsilon + y^{1/q}\Big\}\,\mathrm{d}y \\
&\quad \leqslant C\sum_{n=1}^{\infty} n^{r-2-q/p} \int_0^{\infty} P\Big\{\max_{1\leqslant k\leqslant n} \Big|\sum_{j=1}^{k} X_j\Big| > n^{1/p}2^{-1/p}\varepsilon + y^{1/q}\Big\}\,\mathrm{d}y \\
&\qquad\qquad (\text{letting } \varepsilon_0 = 2^{-1/p}\varepsilon) \\
&\quad = \sum_{n=1}^{\infty} n^{r-2-q/p} E\Big\{\max_{1\leqslant k\leqslant n} \Big|\sum_{j=1}^{k} X_j\Big| - \varepsilon_0 n^{1/p}\Big\}_+^q < \infty.
\end{aligned}$$

Thus, (2.2) holds. □

Proof of Theorem 2.2. Here, let $l(n)=n^{-1-q/p}$. Similarly to the proof of (3.1) and (3.2) we have

$$\begin{aligned}(3.11)\qquad &\sum_{n=1}^{\infty} l(n)E\Big\{\max_{1\leqslant k\leqslant n}\Big|\sum_{j=1}^{k}X_j\Big|-\varepsilon n^{1/p}\Big\}_+^q\\ &=\sum_{n=1}^{\infty} n^{q/p}l(n)P\Big\{\max_{1\leqslant k\leqslant n}\Big|\sum_{j=1}^{k}X_j\Big|>\varepsilon n^{1/p}\Big\}\\ &\quad+\sum_{n=1}^{\infty} l(n)\int_{n^{q/p}}^{\infty}P\Big\{\max_{1\leqslant k\leqslant n}\Big|\sum_{j=1}^{k}X_j\Big|>x^{1/q}\Big\}\,\mathrm{d}x=:J_1+J_2.\end{aligned}$$

For J_1, by Lemma 2.4 we have

$$(3.12)\qquad J_1=\sum_{n=1}^{\infty}\frac{1}{n}P\Big\{\max_{1\leqslant k\leqslant n}\Big|\sum_{j=1}^{k}X_j\Big|>\varepsilon n^{1/p}\Big\}<\infty.$$

For J_2, similarly to the proof of (3.4) we have

(3.13)

$$\begin{aligned}J_2&\leqslant C\sum_{n=1}^{\infty} l(n)\int_{n^{q/p}}^{\infty}P\Big\{\max_{1\leqslant k\leqslant n}\Big|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+k}Y_jI[|Y_j|>x^{1/q}]\Big|\geqslant x^{1/q}/2\Big\}\,\mathrm{d}x\\ &\quad+C\sum_{n=1}^{\infty} l(n)\int_{n^{q/p}}^{\infty}P\Big\{\max_{1\leqslant k\leqslant n}\Big|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+k}Y_{xj}\Big|\geqslant x^{1/q}/4\Big\}\,\mathrm{d}x=:J_{21}+J_{22}.\end{aligned}$$

For J_{21}, by Markov's and C_r inequalities we have

$$\begin{aligned}(3.14)\qquad J_{21}&\leqslant C\sum_{n=1}^{\infty} l(n)\int_{n^{q/p}}^{\infty}x^{-\theta/q}E\max_{1\leqslant k\leqslant n}\Big|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+k}Y_jI[|Y_j|>x^{1/q}]\Big|^{\theta}\,\mathrm{d}x\\ &\leqslant C\sum_{n=1}^{\infty} nl(n)\int_{n^{q/p}}^{\infty}x^{-\theta/q}E|Y_1|^{\theta}I[|Y_1|>x^{1/q}]\,\mathrm{d}x\\ &=C\sum_{n=1}^{\infty} nl(n)\sum_{m=n}^{\infty}\int_{m^{q/p}}^{(m+1)^{q/p}}x^{-\theta/q}E|Y_1|^{\theta}I[|Y_1|>x^{1/q}]\,\mathrm{d}x\\ &\leqslant C\sum_{n=1}^{\infty} nl(n)\sum_{m=n}^{\infty}m^{q/p-\theta/p-1}E|Y_1|^{\theta}I[|Y_1|>m^{1/p}]\\ &=C\sum_{m=1}^{\infty}m^{q/p-\theta/p-1}E|Y_1|^{\theta}I[|Y_1|>m^{1/p}]\sum_{n=1}^{m}n^{-q/p}\end{aligned}$$

$$\leqslant \begin{cases} C\sum\limits_{m=1}^{\infty} m^{-\theta/p} E|Y_1|^{\theta} I[|Y_1| > m^{1/p}], & \text{if } q < p, \\ C\sum\limits_{m=1}^{\infty} m^{q/p-\theta/p-1} \log(1+m) E|Y_1|^{\theta} I[|Y_1| > m^{1/p}], & \text{if } q = p, \\ C\sum\limits_{m=1}^{\infty} m^{q/p-\theta/p-1} E|Y_1|^{\theta} I[|Y_1| > m^{1/p}], & \text{if } q > p, \end{cases}$$

$$\leqslant \begin{cases} CE|Y_1|^p < \infty, & \text{if } q < p, \\ CE|Y_1|^p \log(1+|Y_1|) < \infty, & \text{if } q = p, \\ CE|Y_1|^q < \infty, & \text{if } q > p. \end{cases}$$

For J_{22}, similarly to the proof of I_{22}, taking $s = 2$ in I_{22} and noting that $\sum\limits_{m=1}^{\infty} \varphi^{1/2}(m) < \infty$, we have

(3.15)

$$\begin{aligned} J_{22} &\leqslant C\sum_{n=1}^{\infty} l(n) \int_{n^{q/p}}^{\infty} x^{-2/q} E \max_{1\leqslant k\leqslant n} \left| \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_{xj} \right|^2 \mathrm{d}x \\ &\leqslant C\sum_{n=1}^{\infty} l(n) \int_{n^{q/p}}^{\infty} x^{-2/q} \left(n \exp\left\{ 6 \sum_{j=1}^{[\log n]} \varphi^{1/2}(2^j) \right\} E|Y_1|^2 I[|Y_1| \leqslant x^{1/q}] \right) \mathrm{d}x \\ &\quad + C\sum_{n=1}^{\infty} l(n) \int_{n^{q/p}}^{\infty} n x^{-2/q} E|Y_1|^2 I[|Y_1| \leqslant x^{1/q}] \,\mathrm{d}x \\ &\leqslant C\sum_{n=1}^{\infty} n l(n) \int_{n^{1/p}}^{\infty} x^{-2/q} E|Y_1|^2 I[|Y_1| \leqslant x^{1/q}] \,\mathrm{d}x \\ &= C\sum_{n=1}^{\infty} n l(n) \sum_{m=n}^{\infty} \int_{m^{q/p}}^{(m+1)^{q/p}} x^{-2/q} E|Y_1|^2 I[|Y_1| \leqslant x^{1/q}] \,\mathrm{d}x \\ &\leqslant C\sum_{n=1}^{\infty} n l(n) \sum_{m=n}^{\infty} m^{q/p-2/p-1} E|Y_1|^2 I[|Y_1| \leqslant (m+1)^p] \\ &= C\sum_{m=1}^{\infty} m^{q/p-2/p-1} E|Y_1|^2 I[|Y_1| \leqslant (m+1)^p] \sum_{n=1}^{m} n^{-q/p} \\ &\leqslant \begin{cases} C\sum\limits_{m=1}^{\infty} m^{-2/p} E|Y_1|^2 I[|Y_1| > m^{1/p}], & \text{if } q < p, \\ C\sum\limits_{m=1}^{\infty} m^{q/p-2/p-1} \log(1+m) E|Y_1|^2 I[|Y_1| > m^{1/p}], & \text{if } q = p, \\ C\sum\limits_{m=1}^{\infty} m^{q/p-2/p-1} E|Y_1|^2 I[|Y_1| > m^{1/p}], & \text{if } q > p, \end{cases} \\ &\leqslant \begin{cases} CE|Y_1|^p < \infty, & \text{if } q < p, \\ CE|Y_1|^p \log(1+|Y_1|) < \infty, & \text{if } q = p, \\ CE|Y_1|^q < \infty, & \text{if } q > p. \end{cases} \end{aligned}$$

From (3.13)–(3.15) one gets

$$J_2 < \infty. \tag{3.16}$$

So, (2.3) holds due to (3.11), (3.12), and (3.16). □

P r o o f of Theorem 2.3. We omit the proof of the theorem, since it is similar to that of Theorem 2.2. □

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Authors' addresses: *Xing-cai Zhou*, School of Mathematics and Computer, Tongling University, Tongling, Anhui, 244000, P. R. China and Department of Mathematics, Southeast University, Nanjing, 210096, P. R. China, e-mail: `xczhou@nuaa.edu.cn`; *Jin-guan Lin*, Department of Mathematics, Southeast University, Nanjing, 210096, P. R. China, e-mail: `jglin@seu.edu.cn`.