Czechoslovak Mathematical Journal

Behrouz Sadeghi; Kamal Bahmanpour; Jafar A'zami

Artinian cofinite modules over complete Noetherian local rings

*Czechoslovak Mathematical Journal*, Vol. 63 (2013), No. 4, 877–885

Persistent URL: http://dml.cz/dmlcz/143604

**Czechoslovak Mathematical Journal, 63 (138) (2013), 877–885**

# ARTINIAN COFINITE MODULES OVER COMPLETE NOETHERIAN LOCAL RINGS

Behrouz Sadeghi, Marand, Kamal Bahmanpour, Jafar A'zami, Ardabil

(Received May 30, 2012)

*Abstract.* Let $(R,\mathfrak{m})$ be a complete Noetherian local ring, $I$ an ideal of $R$ and $M$ a nonzero Artinian $R$-module. In this paper it is shown that if $\mathfrak{p}$ is a prime ideal of $R$ such that $\dim R/\mathfrak{p} = 1$ and $(0 :_M \mathfrak{p})$ is not finitely generated and for each $i \geqslant 2$ the $R$-module $\operatorname{Ext}^i_R(M, R/\mathfrak{p})$ is of finite length, then the $R$-module $\operatorname{Ext}^1_R(M, R/\mathfrak{p})$ is not of finite length. Using this result, it is shown that for all finitely generated $R$-modules $N$ with $\operatorname{Supp}(N) \subseteq V(I)$ and for all integers $i \geqslant 0$, the $R$-modules $\operatorname{Ext}^i_R(N, M)$ are of finite length, if and only if, for all finitely generated $R$-modules $N$ with $\operatorname{Supp}(N) \subseteq V(I)$ and for all integers $i \geqslant 0$, the $R$-modules $\operatorname{Ext}^i_R(M, N)$ are of finite length.

*Keywords*: Artinian module; cofinite module; Krull dimension; local cohomology

*MSC 2010*: 13E10, 13D45, 14B15

## 1. Introduction

Throughout this paper, let $R$ denote a commutative Noetherian local ring (with identity) and $I$ an ideal of $R$. For an $R$-module $M$, the $i^{\rm th}$ local cohomology module of $M$ with respect to $I$ is defined as

$$H^i_I(M) = \varinjlim_{n \geqslant 1} \operatorname{Ext}^i_R(R/I^n, M).$$

We refer the reader to [7] or [4] for more details about local cohomology. In [8], Hartshorne defined an $R$-module $L$ to be $I$-*cofinite* if $\operatorname{Supp}(L) \subseteq V(I)$ and $\operatorname{Ext}^i_R(R/I, L)$ is a finitely generated module for all $i$. The concept of cofinite modules have been studied by several authors; see, for example, Hartshorne [8],

This research of the authors was supported by a grant from the Marand branch, Islamic Azad University.

Huneke and Koh [9], Delfino [5], Delfino and Marley [6], Yoshida [16], Bahmanpour and Naghipour [2], Abazari and Bahmanpour [1], Kawasaki [11], [12], Bahmanpour, Naghipour and Sedghi [3], Melkersson [15], [14]. More recently, using the main result of [3], in [10] Irani and Bahmanpour have proved that for any ideal $I$ of a Noetherian ring $R$ and any $I$-cofinite $R$-module $M$ of dimension $d \leqslant 1$, the $R$-modules $\operatorname{Ext}^i_R(M, N)$ are finitely generated, for all integers $i \geqslant 0$ and all finitely generated $R$-modules $N$ with support in $V(I)$. The main goal of this paper is to verify the converse of this result. In this direction as the main result of this paper we shall prove the following theorem:

**Theorem 1.1.** *Let $(R, \mathfrak{m})$ be a complete Noetherian local ring and $I$ an ideal of $R$. Let $M$ be an Artinian $R$-module. Then the following are equivalent:*

(i) *For all finitely generated $R$-modules $N$ with $\operatorname{Supp}(N) \subseteq V(I)$ and for all integers $i \geqslant 0$, the $R$-modules $\operatorname{Ext}^i_R(N, M)$ are of finite length.*

(ii) *For all finitely generated $R$-modules $N$ with $\operatorname{Supp}(N) \subseteq V(I)$ and for all integers $i \geqslant 0$, the $R$-modules $\operatorname{Ext}^i_R(M, N)$ are of finite length.*

One of our tools for proving Theorem 1.1 is the following:

**Theorem 1.2.** *Let $(R, \mathfrak{m})$ be a complete Noetherian local ring and $M$ a nonzero Artinian $R$-module. Let $\mathfrak{p}$ be a prime ideal of $R$ such that $\dim R/\mathfrak{p} = 1$ and $(0 :_M \mathfrak{p})$ is not finitely generated. If for all $i \geqslant 2$ the $R$-module $\operatorname{Ext}^i_R(M, R/\mathfrak{p})$ is of finite length, then the $R$-module $\operatorname{Ext}^1_R(M, R/\mathfrak{p})$ is not of finite length.*

Throughout this paper, $R$ will always be a commutative Noetherian ring with nonzero identity and $I$ will be an ideal of $R$. Recall that, for each $R$-module $M$, all integers $j \geqslant 0$ and all prime ideals $\mathfrak{p}$ of $R$, the $j^{\text{th}}$ Bass number of $M$ with respect to $\mathfrak{p}$ is defined as $\mu^j(\mathfrak{p}, M) = \dim_{k(\mathfrak{p})} \operatorname{Ext}^j_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p})$, where $k(\mathfrak{p}) := R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. For an Artinian $R$-module $A$ we denote by $\operatorname{Att}_R(A)$ the set of attached prime ideals of $A$. For any ideal $\mathfrak{a}$ of $R$ we denote $\{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} \supseteq \mathfrak{a}\}$ by $V(\mathfrak{a})$. We denote the support of each $R$-module $M$ by $\operatorname{Supp}(M)$. Also, for each $R$-module $M$ we denote by $\operatorname{Ass}_R(M)$ the set of associated prime ideals of $M$. Moreover, for each $R$-module $M$ we denote by $\operatorname{Ann}_R(M)$ the annihilator of $M$ in $R$. Finally, for each $R$-module $M$ we denote by $E_R(M)$ the injective envelope (or injective hull) of $M$.

## 2. The results

To prove the main results of this paper, we need the following lemmas.

**Lemma 2.1.** *Let $R$ be a Noetherian ring and $\mathfrak{p}$ a prime ideal of $R$. Let $M \neq 0$ be an arbitrary $R$-module such that $0 \neq \mu_R^0(\mathfrak{p}, M) = n < \infty$. Then there exists an exact sequence*

$$0 \to \bigoplus_{i=1}^{n} R/\mathfrak{p} \to M.$$

Proof. Let $E := E_R(M)$. Then we may assume $E = \Big(\bigoplus_{i=1}^{n} E_R(R/\mathfrak{p})\Big) \bigoplus E'$ for some injective $R$-module $E'$. Now for each $1 \leqslant i \leqslant n$, let $L_i = \Big(\bigoplus_{j=1}^{n} L_{i,j}\Big) \oplus 0$, where $L_{i,i} = R/\mathfrak{p}$ and $L_{i,j} = 0$ for each $j \in \{1, \ldots, n\} \setminus \{i\}$. Then as $L_i$ is a submodule of $E$ and $E$ is an essential extension of $M$, it follows from the definition that for each $1 \leqslant i \leqslant n$ we have $L_i \cap M \neq 0$. Therefore $\emptyset \neq \operatorname{Ass}_R(L_i \cap M) \subseteq \operatorname{Ass}_R(L_i) = \{\mathfrak{p}\}$ and hence $\operatorname{Ass}_R(L_i \cap M) = \{\mathfrak{p}\}$. Therefore the $R$-module $L_i \cap M$ has a submodule $L'_i$ such that $L'_i \cong R/\mathfrak{p}$. Now it is easy to see that $L'_1 + \ldots + L'_n \cong \bigoplus_{i=1}^{n} R/\mathfrak{p}$ and obviously $L'_1 + \ldots + L'_n$ is a submodule of $M$. This completes the proof. □

**Lemma 2.2.** *Let $R$ be a Noetherian ring, $I$ a proper ideal of $R$ and $A$ a nonzero Artinian $I$-cofinite $R$-module. Then for each nonzero finitely generated $R$-module $N$ with support in $V(I)$, the $R$-modules $\operatorname{Ext}_R^i(A, N)$ have finite length for all integers $i \geqslant 0$.*

Proof. See [10, Theorem 2.3]. □

**Corollary 2.3.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $A$ a nonzero Artinian $R$-module. Then for each nonzero $R$-module $N$ of finite length, the $R$-modules $\operatorname{Ext}_R^i(A, N)$ have finite length for all integers $i \geqslant 0$.*

Proof. Since each Artinian $R$-module is $\mathfrak{m}$-cofinite the assertion follows immediately from Lemma 2.2. □

The following theorem is our main tool for the proof of the main result of this paper.

**Theorem 2.4.** *Let $(R, \mathfrak{m})$ be a complete Noetherian local ring and $M$ a nonzero Artinian $R$-module. Let $\mathfrak{p}$ be a prime ideal of $R$ such that $\dim R/\mathfrak{p} = 1$ and $(0 :_M \mathfrak{p})$ is not finitely generated. If for each $i \geqslant 2$ the $R$-module $\operatorname{Ext}^i_R(M, R/\mathfrak{p})$ is of finite length, then the $R$-module $\operatorname{Ext}^1_R(M, R/\mathfrak{p})$ is not of finite length.*

Proof. Let $\lambda(\mathfrak{p}, M) = \dim_{R/\mathfrak{m}}((\mathfrak{p} + 0 :_R M)/(0 :_R M) \otimes_R R/\mathfrak{m})$. We prove the assertion by induction on $\lambda(\mathfrak{p}, M)$. Let $\lambda(\mathfrak{p}, M) = 0$. In this case $\mathfrak{p} \subseteq 0 :_R M$ and so $\mathfrak{p}M = 0$. Consider the following exact sequence:

$$0 \to R/\mathfrak{p} \to E_R(R/\mathfrak{p}) \to T \to 0. \tag{2.4.1}$$

Since $M$ is Artinian it follows from the definition that $\operatorname{Supp}(M) \subseteq \{\mathfrak{m}\}$ and so $\operatorname{Hom}_R(M, R/\mathfrak{p}) = \operatorname{Hom}_R(M, E_R(R/\mathfrak{p})) = \operatorname{Hom}_R(M, T/\Gamma_\mathfrak{m}(T)) = 0$. Now from the exact sequence

$$0 \to \Gamma_\mathfrak{m}(T) \to T \to T/\Gamma_\mathfrak{m}(T) \to 0, \tag{2.4.2}$$

we conclude that $\operatorname{Hom}_R(M, T) \simeq \operatorname{Hom}_R(M, \Gamma_\mathfrak{m}(T))$. Since the $R$-module $E_R(R/\mathfrak{p})$ is injective from the exact sequence (2.4.1) we have

$$\operatorname{Ext}^1_R(M, R/\mathfrak{p}) \simeq \operatorname{Hom}_R(M, T) \simeq \operatorname{Hom}_R(M, \Gamma_\mathfrak{m}(T)).$$

On the other hand,

$$\Gamma_\mathfrak{m}(E_R(R/\mathfrak{p})) = H^1_\mathfrak{m}(E_R(R/\mathfrak{p})) = 0.$$

Therefore $\Gamma_\mathfrak{m}(T) \simeq H^1_\mathfrak{m}(R/\mathfrak{p})$ and consequently

$$\operatorname{Ext}^1_R(M, R/\mathfrak{p}) \simeq \operatorname{Hom}_R(M, H^1_\mathfrak{m}(R/\mathfrak{p})).$$

So it is enough to show that the $R$-module $\operatorname{Hom}_R(M, H^1_\mathfrak{m}(R/\mathfrak{p}))$ is not of finite length. Suppose that $\operatorname{Hom}_R(M, H^1_\mathfrak{m}(R/\mathfrak{p}))$ is of finite length. Set $L := \operatorname{Hom}_R(H^1_\mathfrak{m}(R/\mathfrak{p}), E_R(R/\mathfrak{m}))$. In view of [4, Theorem 7.1.3] the $R$-module $H^1_\mathfrak{m}(R/\mathfrak{p})$ is Artinian. Since $R$ is complete, it follows that $L$ is a finitely generated $R$-module. Moreover, by [4, Theorem 7.3.2] we have

$$\operatorname{Att}_R(H^1_\mathfrak{m}(R/\mathfrak{p})) = \{\mathfrak{p}\}.$$

Now, as $R$ is a complete local ring, it follows from [4, Exercise 10.2.15(iii)] that

$$\{\mathfrak{p}\} = \operatorname{Att}_R(H^1_\mathfrak{m}(R/\mathfrak{p})) = \operatorname{Att}_R(0 :_{H^1_\mathfrak{m}(R/\mathfrak{p})} 0) = \operatorname{Ass}_R(L/0L) = \operatorname{Ass}_R(L).$$

Therefore, $L$ is a finitely generated $R$-module such that $\mathfrak{p} \in \operatorname{Ass}_R(L)$ and $\mathfrak{p}L = 0$. Let $\mu^0(\mathfrak{p}, L) = n$. By Lemma 2.1 there is an exact sequence

$$0 \to \bigoplus_{i=1}^{n} R/\mathfrak{p} \to L \to B \to 0, \tag{2.4.3}$$

which implies the following exact sequence:

$$0 \to \bigoplus_{i=1}^{n} R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \to L_{\mathfrak{p}} \to B_{\mathfrak{p}} \to 0. \tag{2.4.4}$$

From the assumption $\mathfrak{p}L = 0$ we conclude that

$$\begin{aligned}\mu^0(\mathfrak{p}, L) &= \dim_{R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}} \operatorname{Hom}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, L_{\mathfrak{p}}) \\ &= \dim_{R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}}(L_{\mathfrak{p}}) = n = \dim_{R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}}\left(\bigoplus_{i=1}^{n} R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}\right).\end{aligned}$$

Therefore from the exact sequence (2.4.4) we conclude that $B_{\mathfrak{p}} = 0$. Hence, we have $\operatorname{Supp}(B) \subseteq V(\mathfrak{p}) \setminus \{\mathfrak{p}\} \subseteq V(\mathfrak{m}) = \{\mathfrak{m}\}$ and $\mathfrak{p}B = 0$. Since $R$ is complete, applying the exact functor $D := \operatorname{Hom}_R(-, E_R(R/\mathfrak{m}))$ to the exact sequence (2.4.3) we get the exact sequence

$$0 \to C \to H^1_{\mathfrak{m}}(R/\mathfrak{p}) \to \bigoplus_{i=1}^{n} E_{R/\mathfrak{p}}(R/\mathfrak{m}) \to 0, \tag{2.4.5}$$

where $C := \operatorname{Hom}_R(B, E_R(R/\mathfrak{m}))$ is an $R$-module of finite length and $E_{R/\mathfrak{p}}(R/\mathfrak{m}) \simeq \operatorname{Hom}_R(R/\mathfrak{p}, E_R(R/\mathfrak{m}))$. Consider the exact sequence

$$\operatorname{Hom}_R(M, H^1_{\mathfrak{m}}(R/\mathfrak{p})) \to \operatorname{Hom}_R\left(M, \bigoplus_{i=1}^{n} E_{R/\mathfrak{p}}(R/\mathfrak{m})\right) \to \operatorname{Ext}^1_R(M, C). \tag{2.4.6}$$

By hypothesis the $R$-module $\operatorname{Hom}_R(M, H^1_{\mathfrak{m}}(R/\mathfrak{p}))$ is of finite length. Also by Corollary 2.3 the $R$-module $\operatorname{Ext}^1_R(M, C)$ has finite length. Therefore the exact sequence (2.4.6) implies that the $R$-module $\operatorname{Hom}_R(M, \bigoplus_{i=1}^{n} E_{R/\mathfrak{p}}(R/\mathfrak{m}))$ is of finite length. On the other hand,

$$\operatorname{Hom}_R\left(M, \bigoplus_{i=1}^{n} E_{R/\mathfrak{p}}(R/\mathfrak{m})\right) \simeq \bigoplus_{i=1}^{n} \operatorname{Hom}_R(M, E_R(R/\mathfrak{m})),$$

since $\mathfrak{p}M = 0$. Therefore the $R$-module $\operatorname{Hom}_R(M, E_R(R/\mathfrak{m}))$ is of finite length and so the $R$-module $M$ is of finite length, and so the $R$-module $0 :_M \mathfrak{p}$ is of finite length which is a contradiction. Now suppose, inductively, that $\lambda(\mathfrak{p}, M) = t \geqslant 1$, and the result has been proved for all values smaller than $t$. By an argument similar to that in the first step it is enough to prove that the $R$-module $\mathrm{Hom}_R(M, H^1_{\mathfrak{m}}(R/\mathfrak{p}))$ is not of finite length. We suppose that the $R$-module $\mathrm{Hom}_R(M, H^1_{\mathfrak{m}}(R/\mathfrak{p}))$ is of finite length and again look for a contradiction. Since $\mathfrak{p}H^1_{\mathfrak{m}}(R/\mathfrak{p}) = 0$ it follows that $\mathrm{Hom}_R(R/\mathfrak{p}, H^1_{\mathfrak{m}}(R/\mathfrak{p})) \simeq H^1_{\mathfrak{m}}(R/\mathfrak{p})$ and so we have

$$\begin{aligned}\mathrm{Hom}_R(M, H^1_{\mathfrak{m}}(R/\mathfrak{p})) &\simeq \mathrm{Hom}_R(M, \mathrm{Hom}_R(R/\mathfrak{p}, H^1_{\mathfrak{m}}(R/\mathfrak{p})) \\ &\simeq \mathrm{Hom}_R(M \otimes_R R/\mathfrak{p}, H^1_{\mathfrak{m}}(R/\mathfrak{p})) \simeq \mathrm{Hom}_R(M/\mathfrak{p}M, H^1_{\mathfrak{m}}(R/\mathfrak{p})).\end{aligned}$$

The argument now proceeds like that used in the first step with the $R$-module $M_1 = M/\mathfrak{p}M$. Since $\mathfrak{p}M_1 = 0$, so by a similar argument for $M$, the $R$-module

$$\mathrm{Hom}_R(M_1, H^1_{\mathfrak{m}}(R/\mathfrak{p}))$$

is of finite length, and consequently the $R$-module $M_1$ is of finite length. By [4, Corollary 7.2.12] and [4, Exercise 7.2.6], we deduce that $\mathrm{Att}_R(M) \cap V(\mathfrak{p}) = \mathrm{Att}_R(M/\mathfrak{p}M) \subseteq \{\mathfrak{m}\}$. In particular, $\mathfrak{p} \nsubseteq \bigcup_{q \in \mathrm{Att}_R(M) \setminus \{\mathfrak{m}\}} q$. Now by assumption $\lambda(\mathfrak{p}, M) = t \geqslant 1$, so there exist $x_1, \ldots, x_t$ in $\mathfrak{p}$ such that

$$\mathfrak{p} + \mathrm{Ann}_R M/\mathrm{Ann}_R M = (x_1, \ldots, x_t) + \mathrm{Ann}_R M/\mathrm{Ann}_R M.$$

Since $\mathfrak{p} \nsubseteq \bigcup_{q \in \mathrm{Att}_R(M) \setminus \{\mathfrak{m}\}} q$,

$$\mathfrak{p} + \mathrm{Ann}_R M = (x_1, \ldots, x_t) + \mathrm{Ann}_R M \nsubseteq \bigcup_{q \in \mathrm{Att}_R(M) \setminus \{\mathfrak{m}\}} q,$$

and therefore using the fact that $\mathrm{Ann}_R M \subseteq \bigcap_{q \in \mathrm{Att}_R(M) \setminus \{\mathfrak{m}\}} q$, it follows that

$$(x_1, \ldots, x_t) \nsubseteq \bigcup_{q \in \mathrm{Att}_R(M) \setminus \{\mathfrak{m}\}} q.$$

Consequently, in view of [13, Exercise 16.8], there exists $y_1 \in (x_2, \ldots, x_t)$ such that $z_1 \notin \bigcup_{q \in \mathrm{Att}_R(M) \setminus \{\mathfrak{m}\}} q$, where $z_1 = x_1 + y_1$. Clearly $z_1 \in \mathfrak{p}$ and $(x_1, \ldots, x_t) = (z_1, x_2, \ldots, x_t)$, hence

$$\mathfrak{p} + \mathrm{Ann}_R M/\mathrm{Ann}_R M = (z_1, x_2, \ldots, x_t) + \mathrm{Ann}_R M/\mathrm{Ann}_R M.$$

Now we have

$$\mathrm{Att}_R(M/z_1M) \subseteq \mathrm{Att}_R(M) \cap V(Rz_1) \subseteq \{\mathfrak{m}\}.$$

By [4, Corollary 7.2.12] $M/z_1M$ is of finite length. The exact sequence

$$0 \to z_1M \to M \to M/z_1M \to 0 \tag{2.4.7}$$

induces an exact sequence

$$\mathrm{Ext}^i_R(M, R/\mathfrak{p}) \to \mathrm{Ext}^i_R(z_1M, R/\mathfrak{p}) \to \mathrm{Ext}^{i+1}_R(M/z_1M, R/\mathfrak{p})$$

for each $i \geqslant 2$. By assumption, $\mathrm{Ext}^i_R(M, R/\mathfrak{p})$ for all $i \geqslant 2$ is of finite length. Also $\mathrm{Ext}^{i+1}_R(M/z_1M, R/\mathfrak{p})$ is of finite length and so for all $i \geqslant 2$ the $R$-module $\mathrm{Ext}^i_R(z_1M, R/\mathfrak{p})$ is of finite length. The exact sequence

$$0 \to 0 :_M z_1 \to M \to z_1M \to 0 \tag{2.4.8}$$

induces an exact sequence

$$\mathrm{Ext}^i_R(M, R/\mathfrak{p}) \to \mathrm{Ext}^i_R(0 :_M z_1, R/\mathfrak{p}) \to \mathrm{Ext}^{i+1}_R(z_1M, R/\mathfrak{p})$$

for each $i \geqslant 2$. This shows that for each $i \geqslant 2$, the $R$-module $\mathrm{Ext}^i_R(0 :_M z_1, R/\mathfrak{p})$ is of finite length. Since $z_1 \in \mathfrak{p}$, it follows that $0 :_{(0:_M z_1)} \mathfrak{p} = 0 :_M \mathfrak{p}$ is not finitely generated. Now $Rz_1 + (0 :_R M) \subseteq 0 :_R (0 :_M z_1)$, so $\lambda(\mathfrak{p}, 0 :_M z_1) \leqslant \lambda(\mathfrak{p}, M) - 1 = t - 1$. Hence, by induction hypothesis the $R$-module $\mathrm{Ext}^1_R(0 :_M z_1, R/\mathfrak{p})$ is not of finite length. The exact sequence (2.4.8) induces an exact sequence

$$\mathrm{Ext}^1_R(M, R/\mathfrak{p}) \to \mathrm{Ext}^1_R(0 :_M z_1, R/\mathfrak{p}) \to \mathrm{Ext}^2_R(z_1M, R/\mathfrak{p}).$$

Consequently, as the $R$-module $\mathrm{Ext}^2_R(z_1M, R/\mathfrak{p})$ is of finite length it follows that the $R$-module $\mathrm{Ext}^1_R(M, R/\mathfrak{p})$ is not of finite length, hence the $R$-module $\mathrm{Hom}_R(M, H^1_{\mathfrak{m}}(R/\mathfrak{p}))$ is not of finite length, which is a contradiction. □

An immediate consequence of Theorem 4.2 is the following theorem.

**Theorem 2.5.** *Let $(R, \mathfrak{m})$ be a complete Noetherian local ring and $M$ an Artinian $R$-module. Let $\mathfrak{p}$ be a prime ideal of $R$ such that $\dim R/\mathfrak{p} = 1$. Then the following are equivalent:*

(i) *The $R$-module $M$ is $\mathfrak{p}$-cofinite.*

(ii) *For all $i \geqslant 0$, the $R$-module $\mathrm{Ext}^i_R(M, R/\mathfrak{p})$ has finite length.*

Proof. (i) → (ii) Follows from Lemma 2.2.

(ii) → (i) Suppose that $M$ is not $\mathfrak{p}$-cofinite. Then by [14, Proposition 4.1] the $R$-module $0 :_M \mathfrak{p}$ is not finitely generated. Now by Theorem 2.4 there exists $i \geqslant 1$ such that $\mathrm{Ext}^i_R(M, R/\mathfrak{p})$ is not of finite length, which is a contradiction. □

The following theorem is needed in the proof of the main result of this paper.

**Theorem 2.6.** *Let $(R, \mathfrak{m})$ be a complete Noetherian local ring and $I$ an ideal of $R$. Let $M$ be an Artinian $R$-module. Then the following are equivalent:*
(i) *The $R$-module $M$ is $I$-cofinite.*
(ii) *For every finitely generated $R$-module $N$ with $\mathrm{Supp}(N) \subseteq V(I)$ and for all $i \geqslant 0$, the $R$-module $\mathrm{Ext}^i_R(M, N)$ is of finite length.*

Proof. (i) → (ii) Follows from Lemma 2.2.

(ii) → (i) Suppose that $M$ is not $I$-cofinite. Then by [15, Theorem 1.6] there exists $q \in \mathrm{Att}_R(M)$ such that $\dim R/I + q \geqslant 1$. Therefore there exists $\mathfrak{p} \in V(I + q)$ such that $\dim R/\mathfrak{p} = 1$. By [15, Theorem 1.6], $M$ is not $\mathfrak{p}$-cofinite. So by Theorem 2.5 there exists $i \geqslant 0$ such that $\mathrm{Ext}^i_R(M, R/\mathfrak{p})$ is not of finite length. But $R/\mathfrak{p}$ is finitely generated and $\mathrm{Supp}(R/\mathfrak{p}) \subseteq V(I)$, which is a contradiction. □

Now we are ready to state and to prove the main result of this paper.

**Theorem 2.7.** *Let $(R, \mathfrak{m})$ be a complete Noetherian local ring and $I$ an ideal of $R$. Let $M$ be an Artinian $R$-module. Then the following are equivalent:*
(i) *For all finitely generated $R$-modules $N$ with $\mathrm{Supp}(N) \subseteq V(I)$ and all integers $i \geqslant 0$, the $R$-modules $\mathrm{Ext}^i_R(N, M)$ are of finite length.*
(ii) *For all finitely generated $R$-modules $N$ with $\mathrm{Supp}(N) \subseteq V(I)$ and all integers $i \geqslant 0$, the $R$-modules $\mathrm{Ext}^i_R(M, N)$ are of finite length.*

Proof. The assertion follows from Theorem 2.6 and [11, Lemma 1]. □

**Acknowledgment.** The authors are deeply grateful to the referee for his/her careful reading the paper and valuable suggestions. Also, we would like to thank Marand branch, Islamic Azad University for the Financial support of this research, which is based on a research project contract.

*References*

[1] *R. Abazari, K. Bahmanpour*: Cofiniteness of extension functors of cofinite modules. J. Algebra *330* (2011), 507–516.
[2] *K. Bahmanpour, R. Naghipour*: Cofiniteness of local cohomology modules for ideals of small dimension. J. Algebra. *321* (2009), 1997–2011.
[3] *K. Bahmanpour, R. Naghipour, M. Sedghi*: On the category of cofinite modules which is Abelian. To appear in Proc. Am. Math. Soc.
[4] *M. P. Brodmann, R. Y. Sharp*: Local Cohomology. An Algebraic Introduction with Geometric Applications. Cambridge Studies in Advanced Mathematics 60, Cambridge University Press, Cambridge, 1998.
[5] *D. Delfino*: On the cofiniteness of local cohomology modules. Math. Proc. Camb. Philos. Soc. *115* (1994), 79–84.

[6] *D. Delfino, T. Marley*: Cofinite modules and local cohomology. J. Pure Appl. Algebra *121* (1997), 45–52.
[7] *A. Grothendieck*: Local Cohomology. A seminar given by A. Grothendieck, Harvard University, Fall 1961. Notes by R. Hartshorne. Lecture Notes in Mathematics 41, Springer, Berlin, 1967.
[8] *R. Hartshorne*: Affine duality and cofiniteness. Invent. Math. *9* (1970), 145–164.
[9] *C. Huneke, J. Koh*: Cofiniteness and vanishing of local cohomology modules. Math. Proc. Camb. Philos. Soc. *110* (1991), 421–429.
[10] *Y. Irani, K. Bahmanpour*: Finiteness properties of extension functors of cofinite modules. Bull. Korean Math. Soc. *50* (2013), 649–657.
[11] *K.-I. Kawasaki*: On the finiteness of Bass numbers of local cohomology modules. Proc. Am. Math. Soc. *124* (1996), 3275–3279.
[12] *K.-I. Kawasaki*: On a category of cofinite modules which is abelian. Math. Z. *269* (2011), 587–608.
[13] *H. Matsumura*: Commutative Ring Theory. Transl. from the Japanese by M. Reid. Cambridge Studies in Advanced Mathematics 8, Cambridge University Press, Cambridge, 1986.
[14] *L. Melkersson*: Modules cofinite with respect to an ideal. J. Algebra *285* (2005), 649–668.
[15] *L. Melkersson*: Properties of cofinite modules and applications to local cohomology. Math. Proc. Camb. Philos. Soc. *125* (1999), 417–423.
[16] *K. I. Yoshida*: Cofiniteness of local cohomology modules for ideals of dimension one. Nagoya Math. J. *147* (1997), 179–191.

*Authors' addresses*: Behrouz Sadeghi, Department of Mathematics, Marand branch, Islamic Azad University, Marand, Iran, e-mail: `b-sadeqi@marandiau.ac.ir`; Kamal Bahmanpour (corresponding author), Faculty of Mathematical Sciences, Department of Mathematics, University of Mohaghegh Ardabili, 56199-11367, Ardabil, Iran, e-mail: `bahmanpour.k@gmail.com`; Jafar A'zami, Faculty of Mathematical Science, Department of Mathematics, University of Mohaghegh Ardabili, 56199-11367, Ardabil, Iran, e-mail: `jafar.azami@gmail.com`, `azami@uma.ac.ir`.