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STABILITY FOR NON-AUTONOMOUS LINEAR EVOLUTION EQUATIONS WITH L^p-MAXIMAL REGULARITY

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Abstract. We study stability and integrability of linear non-autonomous evolutionary Cauchy-problem

$$(\mathrm{P})\begin{cases} \dot{u}(t)+A(t)u(t)=f(t) & t\text{-a.e. on } [0,\tau],\\ u(0)=0, \end{cases}$$

where $A\colon [0,\tau]\to\mathscr{L}(X,D)$ is a bounded and strongly measurable function and X, D are Banach spaces such that $D\underset{d}{\hookrightarrow}X$. Our main concern is to characterize L^p-maximal regularity and to give an explicit approximation of the problem (P).

Keywords: maximal regularity; on-autonomous evolution equation; stability for linear evolution equation; integrability for linear evolution equation

MSC 2010: 35K90, 47D06

1. Introduction

We study L^p-maximal regularity for non-autonomous evolutionary linear Cauchy-problems.

Let $(X,\|\cdot\|)$ and $(D,\|\cdot\|_D)$ be two Banach spaces such that D is continuously and densely embedded in X. Let $A\colon [0,\tau]\to\mathscr{L}(X,D)$ be a bounded and strongly measurable function. Let $p\in(1,\infty)$. We say that A has L^p-*maximal regularity* on the bounded real interval $[0,\tau]$ (and we write $A\in\mathscr{MR}_p(0,\tau)$) if for all subintervals $[a,b]$ of $[0,\tau]$ and for every $f\in L^p(a,b;X)$ there exists a unique $u\in MR_p(a,b):=$

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$L^p(a,b;D)\cap W^{1,p}(a,b;X)$ such that

$$\mathrm{CP}(a,b)\begin{cases}\dot u(t)+A(t)u(t)=f(t) & t\text{-a.e. on } [a,b],\\ u(a)=0.\end{cases}$$

In particular $\dot u$ and Au have the same regularity as the inhomogeneity f. This property is the reason for the name maximal regularity. Recall that $W^{1,p}(a,b;X)\subset C([a,b];X)$, so the condition $u(a)=0$ above makes sense.

For the autonomous case, that is if $A(\cdot)=A$ is independent of $t\in[0,\tau]$, L^p-maximal regularity is independent of the bounded interval $[a,b]$ and if $A\in\mathscr{MR}_p(0,\tau)$ for some $p\in(1,\infty)$ then $A\in\mathscr{MR}_p(0,\tau)$ for all $p\in(1,\infty)$ [22], [8]. Thus we denote by $\mathscr{MR}$ the set of all operators $A\in\mathscr{L}(D,X)$ having L^p-maximal regularity. It is also well known that if A has L^p-maximal regularity then A is closed as unbounded operator on X [6] and $-A$ generates a holomorphic C_0-semigroup on X [12] and [17]. De Simon [10] showed that the converse is true if X is a Hilbert space. However, the restriction to Hilbert spaces is essential by a result of Kalton and Lancien [16].

Maximal regularity has been studied by many authors in recent years. The reader may consults [1], [2], [4], [5], [6], [11], [14], [15], [19], [20] and the references therein for different sufficient conditions for L^p-maximal regularity in the non-autonomous case and for applications.

It is known [7, Lemma 1.2] that if $A\in\mathscr{MR}$ then there exists a constant $M(A)>0$ such that

$$\begin{aligned}\|(\varrho+\mathscr{A}+\mathscr{B})^{-1}\|_{\mathscr{L}(L^p(a,b;X),MR_p(a,b))}&\leqslant M(A)\quad\text{and}\\ \|(\varrho+\mathscr{A}+\mathscr{B})^{-1}\|_{\mathscr{L}(L^p(a,b;X))}&\leqslant\frac{M(A)}{1+\varrho}\end{aligned}\tag{1.1}$$

for all intervals $[a,b]\subset[0,\tau]$ and all $\varrho\geqslant0$, where $\mathscr{B}$ is the distributional derivative with domain $D(\mathscr{B})=\{u\in W^{1,p}(a,b;X),u(a)=0\}$ and $\mathscr{A}$ the multiplication operator with domain $L^p(a,b;D)$ defined by $(\mathscr{A}f)(s)=Af(s)$ a.e.

In the case where A is not constant, we obtain a comparable result. Indeed, if $A\in\mathscr{MR}_p(0,\tau)$, then we show in Proposition 2.2 below that $\varrho+A\in\mathscr{MR}_p(0,\tau)$ for all $\varrho\in\mathbb{C}$ and there exists $M(A)>0$ such that

$$\begin{aligned}\varrho^{1/p}\|(\varrho+\mathscr{A}+\mathscr{B})^{-1}\|_{\mathscr{L}(L^p(a,b;X))}&\leqslant M(A)\quad\text{and}\\ \|(\varrho+\mathscr{A}+\mathscr{B})^{-1}\|_{\mathscr{L}(L^p(a,b;X),L^p(a,b;D))}&\leqslant M(A)\end{aligned}$$

for all intervals $[a,b]\subset[0,\tau]$ and all $\varrho\geqslant0$.

In Lemma 4.1 we will see that the constant $M(A(t))$ in (1.1) corresponding to each $A(t) \in \mathscr{MR}$ does not depend on t provided that A is relatively continuous.

The notion of relative continuity was introduced recently in [7] by Arendt, Chill, Fornaro and Poupaud, who proved in [7, Theorem 2.7] L^p-maximal regularity assuming only that A is bounded, strongly measurable and relatively continuous, and that $A(t) \in \mathscr{MR}$ for every $t \in [0, \tau]$.

Theorem 2.7 in [7] establishes existence and uniqueness of a solution of the problem $\mathrm{CP}(0, \tau)$. But at least from a theoretical point of view, it is very important to exhibit an explicit approximation of this solution. Our goal is to characterize L^p-maximal regularity of $\mathrm{CP}(0, \tau)$. In particular, our approach gives an explicit approximation of the problem $\mathrm{CP}(0, \tau)$, which may have some interest.

Let $\Lambda = \lambda_0 < \lambda_1, \ldots, < \lambda_{n+1} = \tau$ be a subdivision of $[0, \tau]$ and $A_\Lambda \colon [0, \tau] \to \mathscr{L}(D, X)$ be given by

$$t \mapsto A_\Lambda(t) := \begin{cases} A_k & \text{for } \lambda_k \leqslant t < \lambda_{k+1}, \\ A_n & \text{for } t = \tau, \end{cases}$$

where

$$A_k x := \frac{1}{\lambda_{k+1} - \lambda_k} \int_{\lambda_k}^{\lambda_{k+1}} A(r)x \, \mathrm{d}r \quad (x \in D, \ k = 0, 1, \ldots, n).$$

The function A is said to be *relatively p-approximable* if for all $\varepsilon > 0$ there exist $\delta > 0, \eta \geqslant 0$ such that for all $f \in L^p(0, \tau; D)$ and all subdivisions Λ of $[0, \tau]$ of modulus $|\Lambda| := \max\limits_{j=0,1,\ldots,n} (\lambda_{j+1} - \lambda_j) \leqslant \delta$ we have

$$\|\mathscr{A}_\Lambda f - \mathscr{A} f\|_{L^p(0,\tau;D)} \leqslant \varepsilon \|f\|_{L^p(0,\tau;D)} + \eta \|f\|_{L^p(0,\tau;X)}.$$

Assume that A is relatively p-approximable. We show (see Proposition 3.4) that if $A \in \mathscr{MR}_p(0, \tau)$ then there exists $\delta_0 > 0$ such that $A_\Lambda \in \mathscr{MR}_p(0, \tau)$ for all subdivisions Λ of $[0, \tau]$ such that $|\Lambda| \leqslant \delta_0$. This implies in particular that the means A_k are in $\mathscr{MR}$, $k = 0, 1, \ldots, n$. Moreover, if for each $[a, b] \subset [0, \tau]$ the unique solution $u_\Lambda \in MR_p(a, b)$ (see Section 3) of

$$P_\Lambda(a, b) \begin{cases} \dot{u}_\Lambda(t) + A_\Lambda(t) u_\Lambda(t) = f(t) & t\text{-a.e. on } [a, b], \\ u_\Lambda(0) = 0 \end{cases}$$

converges in $MR_p(a, b)$ as $|\Lambda| \to 0$ then $A \in \mathscr{MR}_p(0, \tau)$. In this case $u := \lim\limits_{|\Lambda| \to 0} u_\Lambda$ is the unique solution of $\mathrm{CP}(a, b)$ belonging to $MR_p(a, b)$ (see Theorem 3.5). Our main result shows that this convergence holds if A is relatively continuous. This gives an alternative proof of Theorem 2.7 in [7]. We prove this result in Theorem 4.5

by a more general approach based on the stability of the problem CP$(0,\tau)$. An application to a non-autonomous diffusion equation is given in Section 5.

2. PRELIMINARIES

Throughout this paper $(D, \|\cdot\|_D)$ and $(X, \|\cdot\|)$ are two Banach spaces such that D is continuously and densely embedded into X. We write $D \underset{d}{\hookrightarrow} X$. Let $A\colon [0,\tau] \to \mathscr{L}(D,X)$ be a bounded, strongly Bochner measurable function. Let $p \in (1,\infty)$ be fixed throughout this section.

Definition 2.1. We say that A has *L^p-maximal regularity* on the bounded interval $[0,\tau]$, and we write $A \in \mathscr{MR}_p(0,\tau)$, if for all intervals $[a,b] \subset [0,\tau]$ and every $f \in L^p(a,b;X)$ there exists a unique function u belonging to the *maximal regularity space* $MR_p(a,b) := L^p(a,b;D) \cap W^{1,p}(a,b;X)$ such that

$$\mathrm{CP}(a,b)\begin{cases} \dot u(t) + A(t)u(t) = f(t) & t\text{-a.e. on } [a,b],\\ u(a) = 0. \end{cases}$$

The space $MR_p(a,b)$ is a Banach space for the norm

$$\|u\|_{MR} := \|u\|_{L^p(a,b;D)} + \|u\|_{W^{1,p}(a,b;X)}.$$

Let $MR_0(a,b)$ be the closed subspace of $MR_p(a,b)$ consisting of all u satisfying $u(a) = 0$.

It is useful to reformulate the property of L^p-maximal regularity in terms of sum methods, as initiated by Da Prato and Grisvard [9]. For this, consider for each interval $[a,b] \subset [0,\tau]$ the unbounded linear operators $\mathscr{A} = \mathscr{A}_{a,b}$ and $\mathscr{B} = \mathscr{B}_{a,b}$ with domains $D(\mathscr{A}) = L^p(a,b;D)$ and $D(\mathscr{B}) = \{u \in W^{1,p}(a,b;X), u(a) = 0\}$ defined by

$$(\mathscr{A}f)(t) = A(t)f(t) \quad \text{and} \quad (\mathscr{B}u)(t) = \dot u(t) \quad \text{for almost every } t \in [a,b].$$

In fact, if $C := \sup\limits_{t\in[0,\tau]} \|A(t)\|_{\mathscr{L}(D,X)}$, it is easy to see that $\mathscr{A}f$ is Bochner measurable and

$$\|(\mathscr{A}f)(t)\| \leqslant C\|f(t)\|_D \quad t\text{-a.e. on } [a,b],$$

for all $f \in L^p(a,b;D)$. It follows that $\|\mathscr{A}\|_{\mathscr{L}(L^p(a,b;D),L^p(a,b;X))} \leqslant C$.

Thus A has the property of L^p-maximal regularity if and only if for all $[a,b] \subset [0,\tau]$ the unbounded operator $\mathscr{A} + \mathscr{B}$ with domain $D(\mathscr{A} + \mathscr{B}) = MR_0(a,b)$ is invertible. It follows that if $A \in \mathscr{MR}_p(0,\tau)$ then for each subinterval $[a,b]$ of $[0,\tau]$ and $f \in L^p(a,b;X)$ the problem CP(a,b) has a unique solution $u \in MR_p(a,b)$ and

$$\|u\|_{MR} \leqslant c\|f\|_{L^p(a,b;X)} \tag{2.1}$$

for some constant $c > 0$ which is independent of f and of the interval $[a, b]$. We do not need to assume here that the operators $A(t)$ are closed. Observe that D is a Banach space and $A\colon [0, \tau] \to \mathscr{L}(D, X)$ is bounded and strongly measurable. By unique solvability of the problem $\mathrm{CP}(a, b)$, for every interval $[a, b] \subset [0, \tau]$ and every $f \in L^p(a, b; X)$, the operator $(A_{a,b} + B_{a,b})^{-1}$ can be seen as the restriction of the operator $(A_{0,\tau} + B_{0,\tau})^{-1}$ to the space of functions in $L^p(0, \tau; X)$ which vanish on $[0, a]$. This shows, in particular, that the constant c in (2.1) does not depend on the interval $[a, b] \subset [0, \tau]$.

The following proposition is used in the next sections.

Proposition 2.2. *Assume that $A \in \mathscr{MR}_p(0, \tau)$. Then the following holds.*

(i) *$A \in \mathscr{MR}_p(0, \tau)$ if and only if $\varrho + A \in \mathscr{MR}_p(0, \tau)$ for some (or all) $\varrho \in \mathbb{C}$.*

(ii) *There exists $M(A) > 0$ such that*

$$\begin{aligned}\varrho^{1/p}\|(\varrho + \mathscr{A} + \mathscr{B})^{-1}\|_{\mathscr{L}(L^p(a,b;X))} &\leqslant M(A) \quad \text{and}\\ \|(\varrho + \mathscr{A} + \mathscr{B})^{-1}\|_{\mathscr{L}(L^p(a,b;X),L^p(a,b;D))} &\leqslant M(A)\end{aligned}$$

for all intervals $[a, b] \subset [0, \tau]$ and all $\varrho \geqslant 0$.

Proof. (i) Let $f \in L^p(0, \tau; X)$, $\varrho \in \mathbb{C}$ and $g(t) := \mathrm{e}^{\varrho t} f(t)$. Let $p^* > 1$ be such that $1/p + 1/p^* = 1$. Then u satisfies

$$\dot{u} + A(t)u + \varrho u = f, \quad \text{a.e. on } [0, \tau],\ u(0) = 0 \tag{2.2}$$

if and only if $v(t) := \mathrm{e}^{\varrho t} u(t)$ satisfies

$$\dot{v} + A(t)v = g, \quad \text{a.e. on } [0, \tau],\ v(0) = 0 \tag{2.3}$$

which is assumed to have a unique solution in $MR_p(0, \tau)$. Thus (i) holds.

(ii) It suffices to prove the estimates in (ii) for $[a, b] = [0, \tau]$. From the proof of (i) we have that $u(t) = \mathrm{e}^{-\varrho t} v(t) = \mathrm{e}^{-\varrho t} \int_0^t \dot{v}(r)\,\mathrm{d}r$, where u and v are the solution of (2.2) and (2.3), respectively. Thus, for all $\varrho > 0$

$$\|u(t)\| \leqslant \mathrm{e}^{-\varrho t}\tau^{1/p^*}\|\dot{v}\|_{L^p(0,t,X)} \leqslant \mathrm{e}^{-\varrho t}\tau^{1/p^*}\|v\|_{MR(0,t)} \leqslant c\tau^{1/p^*}\mathrm{e}^{-\varrho t}\|g\|_{L^p(0,t,X)}.$$

Then

$$\begin{aligned}\|u\|^p_{L^p(0,\tau;X)} &\leqslant c^p\tau^{p/p^*}\int_0^\tau\int_0^t \mathrm{e}^{-p\varrho(t-r)}\|f(r)\|^p\,\mathrm{d}r\,\mathrm{d}t = c^p\tau^{p/p^*}\|\mathrm{e}^{-p\varrho}\|_1\|f\|^p_{L^p(0,\tau;X)}\\ &\leqslant c^p\frac{\tau^{p/p^*}}{p\varrho}\|f\|^p_{L^p(0,\tau;X)}.\end{aligned}$$

The first inequality follows. To prove the second inequality we integrate by parts. Since $\|u(t)\|_D = \mathrm{e}^{-\varrho t}\|v(t)\|_D$, then

$$\begin{aligned}\|u\|^p_{L^p(0,\tau;D)} &= \int_0^\tau \mathrm{e}^{-\varrho p t}\|v(t)\|^p_D \,\mathrm{d}t \\ &= \mathrm{e}^{-\varrho p\tau}\int_0^\tau \|v(t)\|^p_D \,\mathrm{d}t + \varrho p\int_0^\tau \mathrm{e}^{-\varrho p t}\int_0^t \|v(r)\|^p_D \,\mathrm{d}r\,\mathrm{d}t.\end{aligned}$$

It follows that

$$\begin{aligned}\|u\|^p_{L^p(0,\tau;D)} &\leqslant c^p(\mathrm{e}^{-\varrho p\tau}\|g\|^p_{L^p(0,\tau,X)} + \varrho p\int_0^\tau\int_0^t \mathrm{e}^{-\varrho p(t-r)}\|f(r)\|^p \,\mathrm{d}r\,\mathrm{d}t) \\ &\leqslant c^p(1+\varrho p/\varrho p)\|f\|^p_{L^p(0,\tau;X)} = 2c^p\|f\|^p_{L^p(0,\tau;X)}.\end{aligned}$$

Setting $M(A) := 2^{1/p}c(1+\tau^{1/p^*}/p)$, the proof is complete. ☐

We may also consider the initial value problems

$$\mathrm{CP}(a,b,x)\begin{cases}\dot{u}(t) + A(t)u(t) = f(t) & t\text{-a.e. on } [a,b],\\ u(a) = x \in X.\end{cases}$$

Assume that $A \in \mathscr{MR}_p(0,\tau)$. Then for all $0 \leqslant a \leqslant b \leqslant \tau$ the problem $\mathrm{CP}(a,b,x)$ has a unique solution in $MR_p(a,b)$ for all $f \in L^p(a,b;X)$ and for all x in the *trace space*

$$Tr = \{u(a),\ u \in MR_p(a,b)\}.$$

The trace space Tr is a Banach space with the norm $\|x\|_{Tr} := \inf\{\|u\|_{MR}\colon u(a) = x\}$. Note that the trace space does not depend on the interval $[a,b]$ and does not depend on the choice of the point where the functions $u \in MR_p(a,b)$ are evaluated. This means that for every $\tau' > 0$ and $t \in [0,\tau']$

$$Tr = \{u(t)\colon\ u \in MR(0,\tau')\}.$$

Note that Tr is isomorphic to the real interpolation space $(X,D)_{1/p*,p}$, where $1/p*+1/p = 1$ (see [18], Chapter 1 for more details). Moreover,

$$MR_p(0,\tau) \underset{d}{\hookrightarrow} C([0,\tau];Tr).$$

The two following lemmas will be used in the sequel.

Lemma 2.3. *Let* $A_n\colon [0,\tau]\to \mathscr{L}(D,X)$ *be a sequence of strongly measurable and bounded functions such that* $\|A_n(t)x\|\leqslant c\|x\|_D$ *for all* $n\in\mathbb{N}$, $x\in D$ *and* t*-a.e. for some constant* $c>0$. *Assume that for all* $x\in D$ *we have* $A_n(t)x\to A(t)x$ t*-a.e. on* $[0,\tau]$ *as* $n\to\infty$. *Then* $A_n(\cdot)w_n(\cdot)\to A(\cdot)w(\cdot)$ *in* $L^p(0,\tau;X)$ *as* $n\to\infty$ *if* $w_n\in L^p(0,\tau;D)$ *are such that* $w_n\to w$ *in* $L^p(0,\tau;D)$.

Proof. Let $A_n\colon [0,\tau]\to\mathscr{L}(D,X)$ $(n=0,1,2,\dots)$ be strongly measurable and bounded with $\|A_n(t)x\|\leqslant c\|x\|_D$ $(x\in D, n\in\mathbb{N})$. Let $x\in D$ and let Ω be a measurable subset of $[0,\tau]$. We set $w=x\otimes 1_\Omega$. Then $\|\mathscr{A}_n w-\mathscr{A}w\|_p^p=\int_\Omega\|A_n(t)x-A(t)x\|^p\,\mathrm{d}t\to 0$ as $n\to\infty$ by Lebesgue's Theorem. It follows that $\|\mathscr{A}_n w-\mathscr{A}w\|_p$ as $n\to\infty$ for all $\omega\in L^p(0,\tau;D)$. Let now $(\omega_n)_{n\in\mathbb{N}}\subset L^p(0,\tau;D)$ be such that $w_n\to w$ in $L^p(0,\tau;D)$. Then

$$\|\mathscr{A}_n\omega_n-\mathscr{A}\omega\|_p\leqslant c\|\omega_n-\omega\|_p+\|\mathscr{A}_n\omega-\mathscr{A}\omega\|_p$$

and the statement follows. □

Lemma 2.4. *Let* $A\colon[0,\tau]\to\mathscr{L}(D,X)$ *be a bounded and strongly Bochner measurable function. Assume that there exists a sequence* $A_n\colon[0,\tau]\to\mathscr{L}(D,X)$, $n\in\mathbb{N}$, *of strongly measurable functions such that*

(i) $A_n\in\mathscr{MR}_p(0,\tau)$ *for all* $n\in\mathbb{N}$,

(ii) *for each* $x\in D$ *one has* $\|A_n(t)x-A(t)x\|\to 0$ *as* $n\to\infty$ *a.e.,*

(iii) $\sup\limits_n\|A_n(t)x\|\leqslant c\|x\|_D$ *a.e. on* $[0,\tau]$ *for some constant* $c>0$ *and all* $x\in D$.

Assume that for each $[a,b]\subset[0,\tau]$ *and for each* $f\in L^p(a,b;X)$ *the unique solution* u_n *in* $MR_p(a,b)$ *of*

$$\dot u_n(t)+A_n(t)u_n(t)=f(t)\quad t\text{-a.e. on } [a,b],\quad u_n(a)=0$$

converges in $MR_p(a,b)$ *as* $n\to\infty$. *Then* $A\in\mathscr{MR}_p(0,\tau)$. *Moreover, for each* $[a,b]\subset[0,\tau]$ *the limit* $u:=\lim\limits_{n\to\infty}u_n$ *is the unique solution of the problem* $\mathrm{CP}(a,b)$.

Proof. Let $[a,b]\subset[0,\tau]$ and $f\in L^p(a,b;X)$.

Existence: Let u_n be the unique solution in $MR_p(a,b)$ of

$$\dot u_n(t)+A_n(t)u_n(t)=f(t)\quad t\text{-a.e. on } [a,b],\ u_n(a)=0.$$

Let $u\in MR_p(a,b)$ such that $u_n\to u$ in $MR_p(a,b)$ as $n\to\infty$. Hence $\dot u_n\to\dot u$ and by Lemma 2.3, $A_nu_n\to Au$ in $L^p(a,b;X)$ as $n\to\infty$. It follows that

$$\dot u(t)+A(t)u(t)=f(t)\quad t\text{-a.e. on } [a,b],\ u(a)=0. \tag{2.4}$$

Uniqueness: Since $(\mathscr{A}_n+\mathscr{B})^{-1}f=u_n$ converges in $MR_p(a,b)$ as $n\to\infty$ to some solution u of (2.4), it follows from the principle of uniform boundedness that

$$M:=\sup_{n\geqslant 0}\|(\mathscr{A}_n+\mathscr{B})^{-1}\|_{\mathscr{L}(L^p(a,b;X),MR_p(a,b))}<\infty.$$

Let $v\in MR_p(a,b)$ be such that

$$\dot v(t)+A(t)v(t)=0\quad t\text{-a.e. on }[a,b],\ v(a)=0.$$

Since $v=(\mathscr{A}_n+\mathscr{B})^{-1}(\mathscr{A}_n+\mathscr{B})v$, then $\|v\|_{MR}\leqslant M\|(\mathscr{A}_n+\mathscr{B})v\|_{L^p(a,b;X)}$. Letting $n\to\infty$ and using Lemma 2.3 we obtain $v=0$. $\square$

3. INTEGRABILITY

Let $A\colon[0,\tau]\to\mathscr{L}(D,X)$ be strongly Bochner measurable. We want to characterize L^p-maximal regularity under some additional regularity assumptions on A.

If $A\in\mathscr{MR}_p(0,\tau)$ is independent of t, the problem $\mathrm{CP}(a,b)$ being an autonomous Cauchy problem, then $-A$ seen as an unbounded operator on X with domain D generates an analytic C_0-semigroup $(T(s))_{s\geqslant 0}$ on X [6]. Hence $A\in\mathscr{MR}_p(0,\tau)$ if and only if for every $f\in L^p(0,\tau;X)$ the function

$$u(t):=\int_0^t T(t-r)f(r)\,\mathrm{d}r,\quad 0\leqslant t\leqslant\tau$$

belongs to $MR_p(0,\tau)$ and is the unique solution of the problem $\mathrm{CP}(0,\tau)$.

The case when A is a step function is also easy to understand. Let $\Lambda=\lambda_0<\lambda_1<\ldots<\lambda_{n+1}$ be a subdivision of $[0,\tau]$. Consider $A_k\in\mathscr{L}(D,X)$ for $k=0,1,\ldots,n$ and let A be given by $A(t)=A_\Lambda(t):=A_k$ for $\lambda_k\leqslant t<\lambda_{k+1}$ and $A(\tau)=A_\Lambda(\tau):=A_n$. Choosing f with support in $[\lambda_k,\lambda_{k+1})$, we obtain that L^p-maximal regularity of each A_k is a necessary condition on A to have L^p-maximal regularity. This condition is also sufficient. In fact, assume that each $A_k\in\mathscr{MR}$ and let $(T_k(s))_{s\geqslant 0}$ denote the C_0-semigroup on X generated by $-A_k$ (with domain D) for $k=0,1,\ldots,n$. For each interval $[a,b]\subset[0,\tau]$ such that $\lambda_{m-1}\leqslant a<\lambda_m<\ldots<\lambda_{l-1}\leqslant b<\lambda_l$ we define the operators $P_\Lambda(a,b)\in\mathscr{L}(X)$ by

$$P_\Lambda(a,b)=T_l(b-\lambda_{l-1})T_{l-1}(\lambda_{l-1}-\lambda_{l-2})\ldots T_{m+1}(\lambda_{m+1}-\lambda_m)T_m(\lambda_m-a),\tag{3.1}$$

and for $\lambda_{l-1}\leqslant a\leqslant b<\lambda_l$ by

$$P_\Lambda(a,b)=T_l(b-a).\tag{3.2}$$

It is easy to see that $(a,b) \mapsto P_\Lambda(a,b)$ is strongly continuous on X for $0 \leqslant a \leqslant b \leqslant \tau$. Moreover, for every $f \in L^p(a,b;X)$ the function

$$u_\Lambda(t) := \int_a^t P_\Lambda(r,t) f(r)\,\mathrm{d}r \tag{3.3}$$

belongs to $MR_p(a,b)$ and is the unique solution of problem

$$\mathrm{CP}_\Lambda(a,b)\begin{cases} \dot v(t) + A_\Lambda(t)v(t) = f(t) & t\text{-a.e. on } [a,b],\\ v(a) = 0. \end{cases}$$

Note also that, for all $x \in Tr$ and $f \in L^p(a,b,X)$ the function $v_\Lambda(t) = P_\Lambda(a,t)x + \int_a^t P_\Lambda(r,t)f(r)\,\mathrm{d}r$ belongs to $MR_p(a,b)$ and is the unique solution of the initial value problem

$$\mathrm{CP}_\Lambda(a,b,x)\begin{cases} \dot v(t) + A_\Lambda(t)v(t) = f(t) & t\text{-a.e. on } [a,b],\\ v(a) = x. \end{cases}$$

The product given by (3.1)–(3.2), and also the existence of a limit of this product as $|\Lambda|$ converges to 0 uniformly on $[a,b] \subset [0,T]$, was studied in our work [13]. This leads to a theory of integral product, comparable to that of the classical Riemann integral. The notion of product integral has been introduced by Vito Volterra at the end of the 19th century. We refer to Antonín Slavík [21] and the reference therein for a discussion of the work of V. Volterra and for more details on product integration theory.

Consider now the general case where $A\colon [0,\tau] \to \mathscr{L}(D,X)$ is bounded and strongly measurable. We want to approximate A by step functions as follows:

Let $\Lambda := \lambda_0 < \lambda_1 < \ldots < \lambda_{n+1}$ be a subdivision of $[0,\tau]$ and $A_\Lambda\colon [0,\tau] \to \mathscr{L}(D,X)$ be defined by $A_\Lambda(t) := A_k$ for $\lambda_k \leqslant t < \lambda_{k+1}$ and $A_\Lambda(\tau) := A_n$, where A_k is given by

$$A_k x := \frac{1}{\lambda_{k+1} - \lambda_k} \int_{\lambda_k}^{\lambda_{k+1}} A(r)x\,\mathrm{d}r \quad (x \in D,\ k = 0,1,\ldots,n). \tag{3.4}$$

The following lemma says that A_Λ converges strongly and almost everywhere to A as $|\Lambda| \to 0$.

Lemma 3.1. *Let $A\colon [0,\tau] \to \mathscr{L}(D,X)$ be bounded and strongly measurable. Then for all $x \in D$ we have $A_\Lambda(t)x \to A(t)x$ in X as $|\Lambda| \to 0$ t-a.e.*

Proof. Let $C \geqslant 0$ be such that $\|A(t)x\|_X \leqslant C\|x\|_D$ for all $x \in D$ and for almost every $t \in [0,\tau]$. Let Λ be any subdivision of $[0,\tau]$ and A_k be given by (3.4) for $k=0,1,\dots n$. We have $\|A_kx\|_X \leqslant C\|x\|_D$ for all $x\in D$. Let t be any Lebesgue point of $A(\cdot)x$. Let $k\in\{0,1,\dots,n\}$ be such that $t\in[\lambda_k,\lambda_{k+1})$. Then

$$\begin{aligned} A_\Lambda(t)x - A(t)x &= \frac{1}{\lambda_{k+1}-\lambda_k}\int_{\lambda_k}^{\lambda_{k+1}}(A(r)x-A(t)x)\,\mathrm{d}r \\ &= \frac{1}{\lambda_{k+1}-\lambda_k}\int_{\lambda_k}^{t}(A(r)x-A(t)x)\,\mathrm{d}r + \frac{1}{\lambda_{k+1}-\lambda_k}\int_{t}^{\lambda_{k+1}}(A(r)x-A(t)x)\,\mathrm{d}r \\ &= \frac{t-\lambda_k}{\lambda_{k+1}-\lambda_k}\frac{1}{t-\lambda_k}\int_{\lambda_k}^{t}(A(r)x-A(t)x)\,\mathrm{d}r \\ &\quad + \frac{\lambda_{k+1}-t}{\lambda_{k+1}-\lambda_k}\frac{1}{\lambda_{k+1}-t}\int_{t}^{\lambda_{k+1}}(A(r)x-A(t)x)\,\mathrm{d}r. \end{aligned}$$

Using [3, Proposition 1.2.2, page 16] we obtain that $A_\Lambda(t)x - A(t)x \to 0$ as $|\Lambda|\to 0$. The result follows since almost all points of $[0,\tau]$ are Lebesgue points of $A(\cdot)x$. $\square$

In order to prove results on the convergence of the solutions u_Λ of $\mathrm{CP}_\Lambda(0,\tau)$ we need more regularity on A.

Recall that the function A is relatively continuous (in the sense of [7, Definition 2.5]) if for each $t\in[0,T]$ and all $\varepsilon>0$ there exist $\delta>0$, $\eta\geqslant 0$ such that for all $s\in[0,T]$, $|t-s|\leqslant\delta$ implies that

$$\|A(t)x - A(s)x\| \leqslant \varepsilon\|x\|_D + \eta\|x\| \quad \text{for } x\in D.$$

The relative continuity on the compact interval $[0,\tau]$ is equivalent to uniform relative continuity, that is, for every $\varepsilon>0$ there exist $\delta>0$ and $\eta\geqslant 0$ such that for all $x\in D$ and for all $t,s\in[0,T]$ one has

$$\|A(t)x - A(s)x\| \leqslant \varepsilon\|x\|_D + \eta\|x\|$$

whenever $|t-s|\leqslant\delta$. If A is relatively continuous then A is bounded (see [7, Remark 2.6]).

Next we give some sufficient and necessary conditions for L^p-maximal regularity. This is based on the following definition.

Definition 3.2. A function $A\colon [0,\tau]\mapsto \mathscr{L}(D,X)$ is called *relatively p-approximable* if for all $\varepsilon>0$ there exist $\delta>0$, $\eta\geqslant 0$ such that for all $f\in L^p(0,\tau;D)$ and for all subdivisions Λ of $[0,\tau]$, $|\Lambda|\leqslant\delta$ implies that

$$\|\mathscr{A}_\Lambda f - \mathscr{A} f\|_{L^p(0,\tau;D)} \leqslant \varepsilon\|f\|_{L^p(0,\tau;D)} + \eta\|f\|_{L^p(0,\tau;X)}. \tag{3.5}$$

The relative p-approximability is weaker than relative continuity. Indeed each relatively continuous function A is relatively p-approximable. The converse is not true, a counterexample is given by step functions.

Proposition 3.3. *Assume that* $A\colon [0,\tau] \to \mathscr{L}(D,X)$ *is relatively continuous. Then* A *is relatively* p*-approximable.*

Proof. Let $\varepsilon > 0$. By the relative continuity there exist $\delta \geqslant 0$ and $\eta \geqslant 0$ such that $|t-t'| \leqslant \delta$ implies $\|A(t)x - A(t')x\| \leqslant \varepsilon\|x\|_D + \eta\|x\|_X$ $(x \in D)$. Let Λ be a subdivision of $[0,\tau]$ with $|\Lambda| < \delta$ and let $t \in [\lambda_k, \lambda_{k+1})$. Since

$$\begin{aligned}\|A_\Lambda(t)f(t) - A(t)f(t)\|_X &= \int_{\lambda_k}^{\lambda_{k+1}} \|A(r)f(t) - A(t)f(t)\|_X \,\frac{\mathrm{d}r}{\lambda_{k+1}-\lambda_k}\\ &\leqslant \varepsilon\|f(t)\|_D + \eta\|f(t)\|_X,\end{aligned}$$

it follows that $\|\mathscr{A}_\Lambda f - \mathscr{A} f\|_{L^p(0,\tau;D)} \leqslant \varepsilon\|f\|_{L^p(0,\tau;D)} + \eta\|f\|_{L^p(0,\tau;X)}$. □

Proposition 3.4. *Let* $A\colon [0,\tau] \to \mathscr{L}(D,X)$ *be strongly measurable and relatively* p*-approximable. Assume that* $A \in \mathscr{MR}_p(0,\tau)$. *Then there exists* $\delta_0 > 0$ *such that for each subdivision* Λ *of* $[0,\tau]$ *with* $|\Lambda| \leqslant \delta_0$ *we have* $A_\Lambda \in \mathscr{MR}_p(0,\tau)$.

Proof. According to point (1) of Proposition 2.2, it suffices to prove the proposition for $\varrho + A_\Lambda$ for some $\varrho \geqslant 0$. Let $M(A)$ be the constant from Proposition 2.2. For $\varepsilon_0 = 1/[2M(A)]$, since $A(\cdot)$ is relatively p-approximable, there exist $\delta_0 > 0$ and $\eta_0 \geqslant 0$ such that $|\Lambda| \leqslant \delta_0$ implies that

$$\|\mathscr{A}_\Lambda f - \mathscr{A} f\|_{L^p(0,\tau;D)} \leqslant \varepsilon_0\|f\|_{L^p(0,\tau;D)} + \eta_0\|f\|_{L^p(0,\tau;X)}$$

for all $f \in L^p(0,\tau;D)$. Let Λ be a subdivision of $[0,\tau]$ such that $|\Lambda| \leqslant \delta_0$ and let $f \in L^p(0,\tau;X)$. Then

$$\begin{aligned}&\|(\mathscr{A}_\Lambda - \mathscr{A})(\varrho + \mathscr{A} + \mathscr{B})^{-1} f\|_{L^p(0,\tau;X)}\\ &\qquad\leqslant \frac{1}{2M(A)}\|(\varrho + \mathscr{A} + \mathscr{B})^{-1} f\|_{L^p(0,\tau;D)} + \eta_0\|(\varrho + \mathscr{A} + \mathscr{B})^{-1} f\|_{L^p(0,\tau;X)}\\ &\qquad\leqslant \frac{1}{2}\|f\|_{L^p(0,\tau;X)} + \frac{\eta_0 M(A)}{\varrho^{1/p}}\|f\|_{L^p(0,\tau;X)}.\end{aligned}$$

Thus, for $\varrho \geqslant \varrho_0 := (4M(A)\eta_0)^p$ we have $\|(\mathscr{A}_\Lambda - \mathscr{A})(\varrho + \mathscr{A} + \mathscr{B})^{-1}\|_{\mathscr{L}(L^p(0,\tau;X))} \leqslant 3/4$. Therefore, $(\varrho + \mathscr{A}_\Lambda + \mathscr{B})$ is invertible whenever $|\Lambda| \leqslant \delta_0$ and $\varrho \geqslant \varrho_0$. □

The main result of this section is the following.

Theorem 3.5. *Let $A\colon [0,\tau] \to \mathscr{L}(D,X)$ be strongly measurable and relatively p-approximable. Then the following assertions are equivalent.*

(i) $A \in \mathscr{MR}_p(0,\tau)$.

(ii) *There exists $\delta_0 > 0$ such that $A_\Lambda \in \mathscr{MR}_p(0,\tau)$ for all subdivisions Λ of $[0,\tau]$ such that $|\Lambda| \leqslant \delta_0$ and for each $[a,b] \subset [0,\tau]$ the solution u_Λ of $\mathrm{CP}_\Lambda(a,b)$, given by (3.3), converges in $MR_p(a,b)$ as $|\Lambda| \to 0$.*

Proof. (i)$\Longrightarrow$(ii) Let $\delta_0 > 0$ and $\varrho = \varrho_0$ be as in the proof of Proposition 3.4. Let Λ and Γ be two subdivisions of $[0,\tau]$ such that $|\Lambda|$, $|\Gamma| \leqslant \delta_0$. Let $[a,b] \subset [0,\tau]$. We have $\|(\mathscr{A}_\Lambda - \mathscr{A})(\varrho + \mathscr{A} + \mathscr{B})^{-1}\|_{\mathscr{L}(L^p(a,b;X))} \leqslant 3/4$. Hence

$$(\varrho + \mathscr{A}_\Lambda + \mathscr{B})^{-1} = (\varrho + \mathscr{A} + \mathscr{B})^{-1} \sum_{k=0}^{\infty} ((\mathscr{A}_\Lambda - \mathscr{A})(\varrho + \mathscr{A} + \mathscr{B})^{-1})^k.$$

The same is also true if we replace Λ by Γ. Let now $\varepsilon > 0$ and $f \in L^p(a,b;X)$. Let $n_0 \in \mathbb{N}$ be such that

$$\Bigl\| \sum_{k=n_0+1}^{\infty} ((\mathscr{A}_\Lambda - \mathscr{A})(\varrho + \mathscr{A} + \mathscr{B})^{-1})^k - \sum_{k=n_0+1}^{\infty} ((\mathscr{A}_\Gamma - \mathscr{A})(\varrho + \mathscr{A} + \mathscr{B})^{-1})^k \Bigr\|_{\mathscr{L}(L^p(a,b;X))} \leqslant \frac{\varepsilon}{2M(A)}.$$

We set $I_{k,\Lambda} := ((\mathscr{A}_\Lambda - \mathscr{A})(\varrho + \mathscr{A} + \mathscr{B})^{-1})^k$ and $I_{k,\Gamma} := ((\mathscr{A}_\Gamma - \mathscr{A})(\varrho + \mathscr{A} + \mathscr{B})^{-1})^k$ for $k = 0, 1, \ldots, n_0$.

By Lemma 3.1 we conclude that $I_{1,\Lambda} f - I_{1,\Gamma} f = (\mathscr{A}_\Lambda - \mathscr{A}_\Gamma)(\varrho + \mathscr{A} + \mathscr{B})^{-1} f$ converges to 0 on $L^p(a,b;X)$ as $|\Lambda|, |\Gamma| \to 0$. It is not difficult to deduce that also all $I_{k,\Lambda} f - I_{k,\Gamma} f$ converge to 0 as $|\Lambda|, |\Gamma| \to 0$.

Then let $\delta' > 0$ be such that

$$|\Lambda|, |\Gamma| \leqslant \delta' \Longrightarrow \|I_{k,\Lambda} f - I_{k,\Gamma} f\|_{L^p(a,b;X)} \leqslant \varepsilon((n_0+1)M(A))^{-1} \quad \text{for every } 0 \leqslant k \leqslant n_0.$$

We deduce that $\|u_\Lambda - u_\Gamma\|_{MR} \leqslant \varepsilon/2\|f\| + \varepsilon/2$ whenever $|\Lambda|, |\Gamma| \leqslant \min\{\delta_0, \delta'\}$.

The implication (ii)$\Longrightarrow$(i) is given by Lemma 2.4. □

4. STABILITY AND MAXIMAL REGULARITY

In this section we give a stability result for the L_p-maximal regularity.

Throughout this section we assume that $A\colon\ [0,\tau] \mapsto \mathscr{L}(D,X)$ is strongly measurable and relatively continuous and $A(t) \in \mathscr{MR}$ for all $t \in [0,\tau]$. We assume also that there exists an approximation $A_n\colon\ [0,\tau] \mapsto \mathscr{L}(D,X)$ (strongly measurable) of A with the following properties.

(H$_1$) There exists $C > 0$ such that $\|A_n(t)\|_{\mathscr{L}(D,X)} \leqslant C$ for all $t \in [0,\tau]$ and $n \in \mathbb{N}$.

(H$_2$) For each $x \in D$ one has $A_n(t)x \to A(t)x$ as $n \to \infty$ in X t-a.e. on $[0,\tau]$.

(H$_3$) For every $\varepsilon > 0$ there exist $\eta \geqslant 0$, $n_0 \in \mathbb{N}$ such that for all $x \in D, n \geqslant n_0, t \in [0,\tau]$ one has

$$\|A_n(t)x - A(t)x\| \leqslant \varepsilon\|x\|_D + \eta\|x\|.$$

(H$_4$) $A_n \in \mathscr{MR}(0,\tau)$ for all $n \in \mathbb{N}$.

We have seen in Lemma 2.4 that if there exists a sequence A_n satisfying the assumptions (H$_1$)–(H$_4$) then $A \in \mathscr{MR}_p(0,\tau)$ provided that for each $[a,b] \subset [0,\tau]$ and for every $f \in L^p(a,b;X)$ the unique solution u_n in $MR_p(a,b)$ of

$$\dot{u}_n(t) + A_n(t)u_n(t) = f(t) \quad t\text{-a.e. on } [a,b],\ u_n(a) = 0$$

converges in $MR_p(a,b)$ as $n \to \infty$. The main result is Theorem 4.5 which says, in particular, that this convergence holds if A is relatively continuous. We also show that $A_\Lambda\colon\ [0,\tau] \mapsto \mathscr{L}(D,X)$ defined in the previous section satisfies (H$_1$)–(H$_4$) as $|\Lambda| \to 0$ provided that A is relatively continuous. This gives an alternative proof of Theorem 2.7 in [7].

We begin with the following useful auxiliary result. For each $t_0 \in [a,b] \subset [0,\tau]$ we denote by $\mathscr{A}(t_0) = \mathscr{A}(t_0)_{a,b}$ the unbounded operator on $L^p(a,b;X)$ with domain $L^p(a,b;D)$ defined by $(\mathscr{A}(t_0)f)(s) = A(t_0)f(s)$ s-a.e.

Lemma 4.1. *Let $A\colon\ [0,\tau] \mapsto \mathscr{L}(D,X)$ be strongly measurable and relatively continuous. Assume that $A(t) \in \mathscr{MR}$ for all $t \in [0,\tau]$. Then there exist $M \geqslant 0$, $\varrho_0 \geqslant 0$ independent of $t \in [0,\tau]$ such that*

$$\|(\varrho + \mathscr{A}(t) + \mathscr{B})^{-1}\|_{\mathscr{L}(L^p(a,b;X),MR_p(a,b))} \leqslant M$$
$$\text{and}\quad \|(\varrho + \mathscr{A}(t) + \mathscr{B})^{-1}\|_{\mathscr{L}(L^p(a,b;X))} \leqslant \frac{M}{1+\varrho},$$

for all intervals $[a,b] \subset [0,\tau]$ and all $\varrho \geqslant \varrho_0$.

Proof. a) Let $[a,b] \subset [0,\tau]$. For each $t \in [a,b]$ there exist $\eta_t \geqslant 0$, $\delta_t > 0$ such that for all $s \in [t-\delta_t, t+\delta_t]$ we have

$$\|A(t)x - A(s)x\| \leqslant \frac{1}{2M(A(t))}\|x\|_D + \delta_t\|x\| \quad (x \in D),$$

where $M(A(t))$ is the constant in (1.1) (see Section 1). By compactness we find $t_i \in [a,b]$ such that $[a,b] \subset \bigcup\limits_{i=0}^{n} [t_i - \delta_{t_i}, t_i + \delta_{t_i}]$. We may assume that this covering is minimal. Thus $t_i \neq t_j$ for $i \neq j$. We arrange then t_i in such way that $a \leqslant t_0 < t_1 < t_2 < \ldots < t_n \leqslant b$. Thus

$$t_i - \delta_{t_i} \leqslant t_{i+1} - \delta_{t_{i+1}} \leqslant t_i + \delta_{t_i} \leqslant t_{i+1} + \delta_{t_{i+1}}.$$

Setting $\tau_0 = a$, $\tau_i = \max\{t_{i-1}, t_i - \delta_i\}, i = 1, \ldots, n-1$ and $\tau_n = b$ we obtain that $\tau_0 < \tau_1 < \ldots < \tau_n$ form a subdivision of $[0,\tau]$ with $t_i \in [\tau_i, \tau_{i+1}] \subset [t_i - \delta_{t_i}, t_i + \delta_{t_i}]$ and for all $t \in [\tau_i, \tau_{i+1}]$.

$$\|A(t)x - A(t_i)x\| \leqslant \frac{1}{2M(A(t_i))}\|x\|_D + \delta_{t_i}\|x\| \quad (x \in D,\ i = 0, 1, \ldots, n).$$

b) Let $[a,b] \in [0,\tau]$ and let $f \in L^p(a,b;X)$. Let $t \in [a,b]$ and let i be such that $t \in [\tau_i, \tau_{i+1}]$. It follows from step a) that

$$\begin{aligned}
&\|(\mathscr{A}(t) - \mathscr{A}(t_i))(\varrho + \mathscr{A}(t_i) + \mathscr{B})^{-1} f\|_{L^p(a,b;X)} \\
&\quad \leqslant \frac{1}{2M(A(t_i))}\|(\varrho + \mathscr{A}(t_i) + \mathscr{B})^{-1} f\|_{L^p(a,b;D)} + \eta_{t_i}\|(\varrho + \mathscr{A}(t_i) + \mathscr{B})^{-1} f\|_{L^p(a,b;X)} \\
&\quad \leqslant \frac{1}{2M(A(t_i))}\|(\varrho + \mathscr{A}(t_i) + \mathscr{B})^{-1} f\|_{MR_p(a,b)} + \eta_{t_i}\|(\varrho + \mathscr{A}(t_i) + \mathscr{B})^{-1} f\|_{L^p(a,b;X)} \\
&\quad \leqslant \frac{1}{2}\|f\|_{L^p(a,b;X)} + \frac{\eta_{t_i} M(A(t_i))}{1+\varrho}\|f\|_{L^p(a,b;X)}
\end{aligned}$$

for all $\varrho \geqslant 0$. Hence we find $\varrho_0 \geqslant 0$ such that for all $\varrho \geqslant \varrho_0$ we have $\|(\mathscr{A}(t) - \mathscr{A}(t_i))(\varrho + \mathscr{A}(t_i) + \mathscr{B})^{-1}\|_{\mathscr{L}(L^p(a,b;X))} \leqslant 3/4$. Thus,

$$\begin{aligned}
(\varrho + \mathscr{A}(t) + \mathscr{B})^{-1} &= (\varrho + \mathscr{A}(t_i) + \mathscr{B} + \mathscr{A}(t) - \mathscr{A}(t_i))^{-1} \\
&= (\varrho + \mathscr{A}(t_i) + \mathscr{B})^{-1}(I + (\mathscr{A}(t) - \mathscr{A}(t_i))(\varrho + \mathscr{A}(t_i) + \mathscr{B})^{-1})^{-1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|(\varrho+\mathscr{A}(t)+\mathscr{B})^{-1}\|_{\mathscr{L}(L^p(a,b;X),MR_p(a,b))}\\
&\qquad\leqslant \|(\varrho+\mathscr{A}(t)+\mathscr{B})^{-1}\|_{\mathscr{L}(L^p(a,b;X),MR_p(a,b))}\\
&\qquad\quad\times\|(I+(\mathscr{A}(t)-\mathscr{A}(t_i))(\varrho+\mathscr{A}(t_i)+\mathscr{B})^{-1})^{-1}\|_{\mathscr{L}(L^p(a,b;X))}\\
&\qquad\leqslant \sup_{j=1,\dots,n} M(A(t_j))\sum_{k=0}^{\infty}\|(\mathscr{A}(t)-\mathscr{A}(t_i))(\varrho+\mathscr{A}(t_i)+\mathscr{B})^{-1}\|^k_{\mathscr{L}(L^p(a,b;X))}\\
&\qquad\leqslant M:=4\sup_{j=1,\dots,n} M(A(t_j)).
\end{aligned}$$

This completes the proof. □

We now show that the problems CP(a,b) are well posed in $L^p(a,b;X)$ for all subintervals $[a,b]$ which are small enough, provided (H$_1$)–(H$_4$) hold and A is relatively continuous. For the proof we need the following Lemma.

Lemma 4.2. *Assume that the family A_n, $n\in\mathbb{N}$ satisfies the condition* (H$_3$). *Then there exist $\delta>0$, $\varrho_1\geqslant 0$ and $n_0\in\mathbb{N}$ such that for each $[a,b]\subset[0,\tau]$, $|b-a|\leqslant\delta$ implies that*

$$\|(\mathscr{A}_n-\mathscr{A}(t))(\varrho+\mathscr{A}(t)+\mathscr{B})^{-1}\|_{\mathscr{L}(L^p(a,b;X))}\leqslant 3/4,$$

for all $t\in[a,b]$, $n\geqslant n_0$ and all $\varrho\geqslant\varrho_1$.

Proof. Let $\varepsilon:=1/(4M)$, where M is the constant from Lemma 4.1. By the assumption on A, there exist $\delta>0$ and $\eta_1\geqslant 0$ such that for all $s_1,s_2\in[0,\tau]$, $|s_2-s_1|\leqslant\delta$ implies

$$\|A(s_1)x-A(s_2)x\|\leqslant\frac{1}{4M}\|x\|_D+\eta_1\|x\|\quad(x\in D).$$

By the assumption (H$_3$) there exist $\eta_2\geqslant 0$ and $n_0\in\mathbb{N}$ such that for all $x\in D$, $n\geqslant n_0$ and $t\in[0,\tau]$ one has

$$\|A_n(t)x-A(t)x\|\leqslant\frac{1}{4M}\|x\|_D+\eta_2\|x\|\quad(x\in D).$$

Let now $[a,b]$ be a subinterval of $[0,\tau]$ such that $|b-a|\leqslant\delta$. Let $f\in L^p(a,b;X)$ and $\varrho\geqslant\varrho_0$ (with ϱ_0 from Lemma 4.1). Using Lemma 4.1 we obtain that for each

$t \in [a,b]$ and $n \geqslant n_0$

$$\begin{aligned}
&\|(\mathscr{A}_n - \mathscr{A}(t))(\varrho + \mathscr{A}(t) + \mathscr{B})^{-1} f\|_{L^p(a,b;X)} \\
&\quad = \Bigl(\int_a^b \|(A_n(s) - A(t))((\varrho + \mathscr{A}(t) + \mathscr{B})^{-1} f)(s)\|_X^p \,\mathrm{d}s\Bigr)^{1/p} \\
&\quad \leqslant \Bigl(\int_a^b \Bigl[\frac{1}{2M}\|((\varrho + \mathscr{A}(t) + \mathscr{B})^{-1} f)(s)\|_D \\
&\qquad + (\eta_2 + \eta_1)\|((\varrho + \mathscr{A}(t) + \mathscr{B})^{-1} f)(s)\|_X\Bigr]^p \,\mathrm{d}s\Bigr)^{1/p} \\
&\quad \leqslant \frac{1}{2M}\|(\varrho + \mathscr{A}(t) + \mathscr{B})^{-1} f\|_{L^p(a,b;D)} + (\eta_2 + \eta_1)\|(\varrho + \mathscr{A}(t) + \mathscr{B})^{-1} f\|_{L^p(a,b;X)} \\
&\quad \leqslant \frac{1}{2}\|f\|_{L^p(a,b;X)} + \frac{(\eta_1 + \eta_2)M}{\varrho + 1}\|f\|_{L^p(a,b;X)}.
\end{aligned}$$

Hence for all $\varrho \geqslant \varrho_1 := \max\{\varrho_0, 4(\eta_2 + \eta_1)M\}$ we have $\|(\mathscr{A}_n(\cdot) - \mathscr{A}(t))(\varrho + \mathscr{A}(t) + \mathscr{B})^{-1}\|_{\mathscr{L}(L^p(a,b;X))} \leqslant 3/4$. □

Theorem 4.3. *Assume that the family $A_n, n \in \mathbb{N}$ satisfies the conditions* (H$_1$)–(H$_4$). *Then there exists $\eta > 0$ such that for all $[a,b] \subset [0,\tau]$ with $|b-a| < \eta$ and all $f \in L^p(a,b;X)$ the unique solution u_n in $MR_p(a,b)$ of*

$$\dot u_n(t) + A_n(t)u_n(t) = f(t) \quad t\text{-a.e. on } [a,b], \quad u_n(a) = 0 \tag{4.1}$$

converges in $MR_p(a,b)$ as $n \to \infty$ and $u := \lim\limits_{n\to\infty} u_n$ is the unique solution of CP(a,b).

Proof. We use the same idea as in the proof of Theorem 3.5. Let δ, ϱ_1 and n_0 be the constants given by Lemma 4.2. According to Proposition 2.2 we can assume that $\varrho_1 = 0$. Let $[a,b] \subset [0,\tau]$ be such that $|b-a| \leqslant \delta$. Let $t_0 \in [a,b]$ and $f \in L^p(a,b;X)$ be fixed. Let $\varepsilon > 0$ and $k_0 \in \mathbb{N}$ be such that

$$\Bigl\|\sum_{k=k_0+1}^{\infty} (\mathscr{A}_n - \mathscr{A}(t_0))(\mathscr{A}(t_0) + \mathscr{B})^{-1}\Bigr\|^k_{\mathscr{L}(L^p(a,b;X))} \leqslant \frac{\varepsilon}{2M} \tag{4.2}$$

where M is the constant in Lemma 4.1. We have the following equality

$$u_n = (\mathscr{A}_n + \mathscr{B})^{-1} f = (\mathscr{A}(t_0) + \mathscr{B})^{-1}(I + (\mathscr{A}_n - \mathscr{A}(t_0))(\mathscr{A}(t_0) + \mathscr{B})^{-1})^{-1}. \tag{4.3}$$

For each $k \in \{0,1,\ldots,k_0\}$ and $n \in \mathbb{N}$ we set

$$I_{k,n} := ((\mathscr{A}_n - \mathscr{A}(t_0))(\mathscr{A}(t_0) + \mathscr{B})^{-1})^k.$$

By the hypothesis (H_2) we have $I_{1,n}f - I_{1,m}f = (\mathscr{A}_n - \mathscr{A}_m)(\mathscr{A}(t_0) + \mathscr{B})^{-1}f$, and thus all $I_{n,k}f - I_{m,k}f$ converge to 0 on $L^p(a,b;X)$ as $n, m \to \infty$. Let $N > 0$ be such that

$$(4.4)\quad n, m \geqslant N \Longrightarrow \|I_{n,k}f - I_{m,k}f\|_{L^p(a,b;X)} \leqslant \frac{\varepsilon}{(k_0+1)M} \quad \text{for every } 0 \leqslant k \leqslant n_0.$$

From (4.2), (4.3) and (4.4) we deduce that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence on the Banach space $MR_p(a,b)$. The last claim follows from Lemma 2.4. ☐

We are now ready for the proof of our main results. Let δ be the constant given by Lemma 4.2 and $[a,b]$ be a subinterval of $[0,\tau]$ such that $|a-b| \leqslant \delta$. Then we have the following stability result.

Theorem 4.4. *Assume that A is relatively continuous and $A(t) \in \mathscr{MR}$ for all $t \in [0,\tau]$. We also assume that the A_n satisfy the hypothese* (H_1)–(H_4). *Let $x_n \in Tr$ and $f_n \in L^p(a,b;X)$ be such that $x_n \to x$ in Tr and $f_n \to f$ in $L^p(a,b;X)$. Then the solution u_n of*

$$(4.5)\qquad \dot{u}_n(t) + A_n(t)u_n(t) = f_n(t) \quad \text{a.e. on } [a,b], \quad u_n(a) = x_n$$

converges in $MR_p(a,b)$ and $u := \lim\limits_{n\to\infty} u_n$ is the unique solution of

$$(4.6)\qquad \dot{u}(t) + A(t)u(t) = f(t) \quad \text{a.e. on } [a,b], \quad u(a) = x.$$

Proof. (a) Let $f_n \in L^p(a,b;X)$ be such that $f_n \to f$ in $L^p(a,b;X)$. We have

$$u_n = (\mathscr{A}_n + \mathscr{B})^{-1}f_n = (\mathscr{A}_n + \mathscr{B})^{-1}(f_n - f) + (\mathscr{A}_n + \mathscr{B})^{-1}f.$$

Theorem 4.3 implies that the second term on the right-hand side of the above equality converges in $MR_p(a,b)$ to $(\mathscr{A} + \mathscr{B})^{-1}f$. Using the Banach-Steinhaus Theorem we obtain

$$\lim_{n\to\infty} \|(\mathscr{A}_n + \mathscr{B})^{-1}f_n - (\mathscr{A} + \mathscr{B})^{-1}f\|_{MR} = 0.$$

(b) Now let $x_n \to x$ and $f_n \to f$, respectively, in Tr and in $L^p(a,b;X)$. There exist $w_n, w \in MR_p(a,b)$ such that $w_n(a) = x_n$, $w(a) = x$ and $\lim\limits_{n\to\infty} \|w_n - w\|_{MR} = 0$. Let $u_n \in MR_p(a,b)$ be such that

$$\dot{u}_n(t) + A_n(t)u_n(t) = f_n(t) \quad \text{a.e. on } [a,b], \quad u_n(a) = x_n.$$

There exists a unique $v_n \in MR_p(a,b)$ such that

$$\dot{v}_n(t) + A_n(t)v_n(t) = -\dot{w}_n(t) - A_n(t)w_n(t) + f_n(t) \quad \text{a.e. on } [a,b], \quad v_n(a) = 0.$$

By unique solvability we have $u_n = v_n + w_n$. The assumption (H$_2$) implies that $\dot{w}_n + A_n w_n + f_n \to \dot{w} + Aw + f$ in $L^p(a,b;X)$. Thus from (a) it follows that $v_n \to v$ in $MR_p(a,b)$ and v is the unique solution in $MR_p(a,b)$ of

$$\dot{v}(t) + A(t)v(t) = -\dot{w}(t) - A(t)w(t) + f(t) \quad \text{a.e. on } [a,b], \quad v(a) = 0.$$

Thus $u_n \to u := v + w$ in $MR_p(a,b)$ and

$$\dot{u}(t) + A(t)u(t) = f(t) \quad \text{a.e. on } [a,b], \quad u(a) = x.$$

The uniqueness follows from (a). $\square$

From Theorem 4.4 we deduce the following global stability result.

Theorem 4.5. *Let* $A\colon [0,\tau] \to \mathscr{L}(D,X)$ *be strongly measurable and relatively continuous. Assume that* $A(t) \in \mathscr{MR}$ *for all* $t \in [0,\tau]$ *and* A_n *satisfy the hypothese* (H$_1$)–(H$_4$). *Let* $x_n \in Tr$ *and* $f_n \in L^p(0,\tau;X)$ *be such that* $x_n \to x$ *in* Tr *and* $f_n \to f$ *in* $L^p(0,\tau;X)$. *Then the unique solution* u_n *of*

$$\dot{u}_n(t) + A_n(t)u_n(t) = f_n(t) \quad \text{a.e. on } [0,\tau], \quad u_n(0) = x_n \tag{4.7}$$

converges in $MR_p(0,\tau)$ *and* $u := \lim\limits_{n\to\infty} u_n$ *is the unique solution of*

$$\dot{u}(t) + A(t)u(t) = f(t) \quad \text{a.e. on } [0,\tau], \quad u(0) = x. \tag{4.8}$$

Proof. Let u_n be the solution of (4.7). From Theorem 4.4, u_n converges in $MR_p(a,b)$ for all $[a,b] \subset [0,\tau]$ such that $|b-a| \leqslant \delta$. We put

$$\tau_1 := \max\{0 \leqslant \tau' \leqslant \tau, \text{ such that } u_n \to u \text{ in } MR(0,\tau')\}.$$

Thus $\tau_1 \geqslant \delta$. We show that $\tau_1 = \tau$. Indeed, we assume by contradiction that $\tau_1 < \tau$ and choose $\tau_1' < \tau_1$ such that $\tau_1 - \tau_1' \leqslant \delta/2$. Then $u_n \to u$ in $MR(0,\tau_1')$. On the other hand, u_n coincides on the interval $[\tau_1', (\tau_1'+\delta)\wedge\tau]$ with the solution of

$$\dot{u}(t) + A_n(t)u(t) = f_n(t) \quad \text{a.e. on } [\tau_1', (\tau_1'+\delta)\wedge\tau], \quad u(\tau_1') = u_n(\tau_1') \in Tr$$

which converges by Theorem 4.4 on $MR(\tau_1', (\tau_1'+\delta)\wedge\tau)$. Then $u_n \to u$ on $MR(0, (\tau_1'+\delta)\wedge\tau)$. Thus $(\tau_1'+\delta)\wedge\tau \leqslant \tau_1$, which is a contradiction to the definition of τ_1. $\square$

We now consider the approximation $A_\Lambda\colon [0,\tau] \mapsto \mathscr{L}(D,X)$ introduced in Section 2. We have proved in Lemma 3.1 and Proposition 3.3 that A_Λ satisfies (H_2) and (H_3). Moreover, since $A\colon [0,\tau] \to \mathscr{L}(D,X)$ is relatively continuous, there exists $\delta > 0$ such that the coefficients A_k which are defined for all $k = 0,1,\dots,n$ by

$$A_k x := \frac{1}{\lambda_{k+1}-\lambda_k}\int_{\lambda_k}^{\lambda_{k+1}} A(r)x\,\mathrm{d}r \quad (x \in D),$$

belong to $\mathscr{MR}$ provided $|\Lambda| \leqslant \delta$. Indeed, for $t \in [\lambda_k,\lambda_{k+1}]$

$$\varrho + \mathscr{A}_k + \mathscr{B} = (Id + (\mathscr{A}_k - \mathscr{A}(t))(\varrho + \mathscr{A}(t) + \mathscr{B})^{-1})(\varrho + \mathscr{A}(t) + \mathscr{B}).$$

By an analogous argument as in the proof of Lemma 4.2, we obtain that

$$\|(\mathscr{A}_k - \mathscr{A}(t))(\varrho + \mathscr{A}(t) + \mathscr{B})^{-1}\|_{\mathscr{L}(L^p(0,\tau;X))} \leqslant 3/4$$

for all $\varrho \geqslant \varrho_0$ and $|\Lambda| \leqslant \delta$ for some $\varrho_0 \geqslant 0$ and $\delta > 0$. Thus $A_k \in \mathscr{MR}$, $k = 0,1,\dots,n$. This is equivalent as proved in Section 3 to the fact that $A_\Lambda \in \mathscr{MR}_p(0,\tau)$. Thus $A_\Lambda\colon [0,\tau] \mapsto \mathscr{L}(D,X)$ as defined above satisfies the hypothese (H_1)–(H_4) for all subdivisions Λ of $[0,\tau]$ such that $|\Lambda| < \delta$. We have thus proved the following.

Corollary 4.6. *Assume that $A\colon [0,\tau] \to \mathscr{L}(D,X)$ is strongly measurable and relatively continuous and $A(t) \in \mathscr{MR}$ for all $t \in [0,\tau]$. Then $A \in \mathscr{MR}(0,\tau)$, and for each $[a,b] \subset [0,\tau]$ the unique solution u in $MR_p(a,b)$ of* CP(a,b) *satisfies*

$$\lim_{|\Lambda|\to 0} \|u - u_\Lambda\|_{MR} = 0,$$

where u_Λ is given by (3.3).

5. An example

Let $\Omega \subset \mathbb{R}^n$ be an open set such that $\partial\Omega$ is bounded and of class C^2. As example we consider the non-autonomous diffusion equation which is described in [7]

$$\left\{\begin{array}{ll} \partial_t u(t,x) - \mathscr{A}(t,x,D)u(t,x) = f(t,x) & \text{a.e. on } (0,\tau)\times\Omega, \\ u(t)(x) = 0 & \text{on } (0,\tau)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{a.e. on } \Omega, \end{array}\right. \tag{5.1}$$

where $\mathscr{A}(t,x,D)$ is the partial differential operator defined by

$$\mathscr{A}(t,x,D)u(x) := \sum_{i,j=1}^{n} a_{ij}(t,x)\partial_i\partial_j u(x) + \sum_{i,j=1}^{n} b_j(t,x)\partial_j u(x) + b_0(t,x)u(x), \tag{5.2}$$

such that $a_{ij} \in C([0,\tau] \times \overline{\Omega})$ for $i, j = 1, \dots, n$ is uniformly continuous, bounded and uniformly elliptic, i.e.

$$\sum_{i,j=1}^{n} a_{ij}(t,x)\xi_i\xi_j \geqslant \beta|\xi|^2$$

for some $\beta > 0$ and all $\xi \in \mathbb{R}^n, x \in \overline{\Omega}, t \in [0,\tau]$, and $b_j \in L^\infty((0,\tau) \times \Omega)$ for $j = 0, 1, \dots, n$.

Recall that if X and D are two Banach spaces and Y an intermediate space such that $D \hookrightarrow Y \hookrightarrow X$ then we say that Y is close to X compared with D if for each $\varepsilon > 0$ there exists $\eta \geqslant 0$ such that $\|x\|_Y \leqslant \varepsilon\|x\|_D + \eta\|x\|_X$, $x \in D$ (see [7] for more details and several examples).

Let $p, q \in (1, \infty)$. Let $D := W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. The space $W^{1,p}(\Omega)$ is close to $L^p(\Omega)$ compared with $W^{2,p}(\Omega)$. We deduce from [7, Theorem 2.10, Theorem 1.3] that $A\colon [0,\tau] \to (D, L^p(\Omega))$ given by

$$A(t)u := -\sum_{i,j=1}^{n} a_{i,j}(t,x)\partial_i\partial_j u - \sum_{i,j=1}^{n} b_j(t,\cdot)\partial_j u - b_0(t,\cdot)u \quad (u \in D)$$

is relatively continuous and $A(t) \in \mathscr{MR}$ for all $t \in [0,\tau]$. Thus the problem (5.1) is stable in the sense of Theorem 4.5 for all initial data $u_0 \in B_{pq}^{2/q^*} \cap \mathring{B}_{pq}^{1/q^*}(\Omega)$ (see [24] for the Besov spaces B_{pq}^{2/q^*}). Moreover, the unique solution

$$u \in C([0,\tau]; B_{pq}^{2/q^*} \cap B_{pq}^{1/q^*}(\Omega)) \cap W^{1,q}(0,\tau; L^p(\Omega)) \cap L^q(0,\tau; W^{2,p} \cap W_0^{1,p}(\Omega))$$

of (5.1) can be explicitly approximated as follows:

Let $\Lambda := (\lambda_0, \lambda_1, \dots, \lambda_m)$ be a subdivision of $[0,\tau]$ and define

$$\begin{aligned} a_{ij}^k(\cdot) &:= \frac{1}{\lambda_{k+1} - \lambda_k} \int_{\lambda_k}^{\lambda_{k+1}} a_{ij}(r,\cdot)\,\mathrm{d}r, \\ b_j^k(\cdot) &:= \frac{1}{\lambda_{k+1} - \lambda_k} \int_{\lambda_k}^{\lambda_{k+1}} b_j(r,\cdot)\,\mathrm{d}r, \quad \text{and} \\ b_0^k(\cdot) &:= \frac{1}{\lambda_{k+1} - \lambda_k} \int_{\lambda_k}^{\lambda_{k+1}} b_0(r,\cdot)\,\mathrm{d}r \end{aligned}$$

for $i, j = 0, 1, \dots, n$ and $k = 0, 1, \dots, m$. The coefficients A_k, introduced in Section 3, are then given in the situation of (5.2) as follows

$$A_k u := -\sum_{i,j=1}^{n} a_{ij}^k(\cdot)\partial_i\partial_j u\,\mathrm{d}r - \sum_{j=1}^{n} b_j^k(\cdot)\partial_j u - b_0^k(\cdot)u \quad (u \in D).$$

Let u_Λ be the unique solution in $W^{1,q}(0,\tau;L^p(\Omega)) \cap L^q(0,\tau;W^{2,p} \cap W_0^{1,p}(\Omega))$ of the approximated problem

$$\begin{cases} \partial_t u(t,x) - \mathscr{A}_\Lambda(t,x,D)u(t,x) = f(t,x) & \text{a.e. on } (0,\tau)\times\Omega, \\ u(t)(x) = 0 \text{ on } (0,\tau)\times\partial\Omega, & \\ u(0,x) = u_0(x) & \text{a.e. on } \Omega. \end{cases}$$

Then u_Λ is given explicitly by (3.3) where

$$A_\Lambda(t) := \begin{cases} A_k & \text{for } \lambda_k \leqslant t < \lambda_{k+1},\ k = 0,1,\dots,m, \\ A_m & \text{for } t = \tau, \end{cases}$$

and by Corollary 4.6

$$\lim_{|\Lambda|\to 0} \|u - u_\Lambda\|_{MR} = 0.$$

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