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HEXAVALENT (G, s)-TRANSITIVE GRAPHS

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Abstract. Let X be a finite simple undirected graph with a subgroup G of the full automorphism group $\mathrm{Aut}(X)$. Then X is said to be (G, s)-transitive for a positive integer s, if G is transitive on s-arcs but not on $(s+1)$-arcs, and s-transitive if it is $(\mathrm{Aut}(X), s)$-transitive. Let G_v be a stabilizer of a vertex $v \in V(X)$ in G. Up to now, the structures of vertex stabilizers G_v of cubic, tetravalent or pentavalent (G, s)-transitive graphs are known. Thus, in this paper, we give the structure of the vertex stabilizers G_v of connected hexavalent (G, s)-transitive graphs.

Keywords: symmetric graph; s-transitive graph; (G, s)-transitive graph

MSC 2010: 05C25, 20B25

1. Introduction

Throughout this paper, we consider undirected finite graphs without loops or multiple edges. For a graph X, we use $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$ to denote its vertex set, edge set, and its full automorphism group, respectively. An *s-arc* in a graph X is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_{s-1}, v_s)$ of vertices of X such that v_{i-1} is adjacent to v_i for $1 \leqslant i \leqslant s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leqslant i \leqslant s-1$. A 1-arc is called an *arc* for short and a *0-arc* is a vertex. For a subgroup $G \leqslant \mathrm{Aut}(X)$, X is said to be *(G, s)-arc-transitive* and *(G, s)-regular* if G is transitive and regular on the set of s-arcs in X, respectively. (G, s)-arc-transitive is simply called *G-symmetric*. A (G, s)-arc-transitive graph is said to be *(G, s)-transitive* if the graph is not $(G, s+1)$-arc-transitive. A graph X is called *s-arc-transitive*, *s-regular* and

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s-transitive if it is $(\mathrm{Aut}(X), s)$-arc-transitive, $(\mathrm{Aut}(X), s)$-regular and $(\mathrm{Aut}(X), s)$-transitive, respectively. In particular, X is said to be *vertex-transitive* and *symmetric* if it is $(\mathrm{Aut}(X), 0)$-arc-transitive and $(\mathrm{Aut}(X), 1)$-arc-transitive, respectively.

Let X be a connected (G, s)-transitive graph for some $s \geqslant 1$ and let G_v be the stabilizer of $v \in V(X)$ in G. It is well known that $s \leqslant 7$ and $s \neq 6$, which is due to several authors. In 1947 Tutte [13] showed that if X is cubic then (G, s)-transitive means (G, s)-regular and $1 \leqslant s \leqslant 5$. Gardiner in [5], [6], [7] obtained that $s \leqslant 7$ and $s \neq 6$ for valency $p + 1$ with p an odd prime. Until 1981, Weiss [17] extended this result to general valency, and showed that if $s \geqslant 4$ then X has valency $p^n + 1$ with p a prime and n a positive integer.

As we all know a graph is G-symmetric if and only if G is vertex-transitive and G_v is transitive on the neighborhood of v. Thus, to investigate G-symmetric graphs, we need the information about the vertex stabilizers of such graphs. Gardiner [5], [6], [7] characterized the structure of G_v for valency $p+1$ with p an odd prime. For valency 5, Weiss [14], [15] obtained an upper bound of the order $|G_v|$, which is $2^{17} \cdot 3^2 \cdot 5$. After that, Weiss [18] conjectured that, for a finite vertex-transitive locally-primitive graph X, the order of the vertex stabilizer is bounded above by some function of the valency of X. Although many results about the vertex stabilizers of arc-transitive graphs have been achieved, this conjecture is still unsettled. For example, Weiss [18] described the structure of G_v for $s \geqslant 4$. Weiss [16] showed that if X has prime valency $p \geqslant 5$ and G_v is solvable, then the order $|G_v| \bigm| p(p-1)^2$. Up to now, we have already known the exact structure of G_v with valency 3, 4 or 5: see [4] for valency 3; [10, Theorem 4] and [19, Theorem 1.1] for valency 4 and $s \geqslant 2$; and [8, Theorem 1.1] for valency 5. For the case of valency 4 and $s = 1$, this is particularly difficult because the action of the vertex stabilizer on the neighborhood may not be primitive. In this case G_v is a 2-group and has no upper bound. Potočnik, Spiga and Verret [11] constructed two families of tetravalent 1-transitive graphs with arbitrarily large vertex stabilizers. In this paper, we determine the structure of G_v when X is of valency 6.

2. Preliminaries

In this section we collect some notation and preliminary results which will be used later in the paper. In view of [20, Proposition 4.4], we have the following proposition.

Proposition 2.1. *Let G be an abelian transitive group on Ω. Then G is self-centralizing in the symmetric group S_Ω.*

For a graph X, let $G \leqslant \mathrm{Aut}(X)$ and let S be a subset of $V(X)$. Denote by $G_{(S)}$ the subgroup of G fixing S pointwise. In particular, for $u, v, w \in V(X)$, write

$G_u = G_{(\{u\})}$, $G_{uv} = G_{(\{u,v\})}$ and $G_{uvw} = G_{(\{u,v,w\})}$. For $u, v \in V(X)$, $\{u, v\}$ is the edge incident to u and v in X, and $N(v)$ is the *neighborhood* of v in X. The next proposition is from [21, Lemma 2.7].

Proposition 2.2. *Let X be a connected symmetric graph and let $e = \{u, v\} \in E(X)$. Suppose that $H \leqslant \mathrm{Aut}(X)$ is transitive on $N(v)$ and $K \leqslant \mathrm{Aut}(X)$ is transitive on $N(u)$. Then the group $\langle H, K\rangle \leqslant \mathrm{Aut}(X)$ is transitive on $E(X)$.*

Let G be a transitive permutation group on a set Ω and let $\alpha \in \Omega$. If the stabilizer G_α is transitive on $\Omega \setminus \{\alpha\}$ then G is called 2-transitive on Ω. The following proposition is about sufficient and necessary conditions for symmetric graphs. Its proof is straightforward and left to the reader.

Proposition 2.3. *Let X be a graph and $G \leqslant \mathrm{Aut}(X)$. Then we have:*
(1) *X is G-arc-transitive if and only if G is vertex-transitive and the vertex stabilizer G_v is transitive on $N(v)$ for each $v \in V(X)$.*
(2) *X is $(G, 2)$-arc-transitive if and only if G is vertex-transitive and G_v is 2-transitive on $N(v)$ for each $v \in V(X)$.*

For two groups M and N, $N \rtimes M$ stands for a semidirect product of N by M. Let X be a graph with $G \leqslant \mathrm{Aut}(X)$ and $\{u, v\} \in E(X)$. Write $G_v^* = G_{(\{v\}\cup N(v))}$ and $G_{uv}^* = G_{(\{u,v\}\cup N(v)\cup N(u))}$.

Lemma 2.4. *Let X be a connected hexavalent (G, s)-transitive graph with $G \leqslant \mathrm{Aut}(X)$ and $v \in V(X)$. Then $s \leqslant 4$ and*
(1) *if $s = 3$ then $G_{uv}^* = 1$;*
(2) *if $s = 4$ then $G_v \cong \mathrm{AGL}(2, 5)$.*

Proof. Since X is hexavalent, by [17, Theorem] we have $s \leqslant 4$, and by [5, Lemma 3.3] and [6, Section 1: Theorem] we can easily deduce that if $s = 3$ then $G_{uv}^* = 1$.

Finally, let $s = 4$. Then by [5, Lemma 3.7] and [7, Lemma 4.3 (i)], G_v^* has a normal Sylow 5-subgroup $\mathbb{Z}_5^2$, and by [7, Corollary 3.6], $\mathrm{SL}(2, 5) \leqslant G_v/\mathbb{Z}_5^2 \leqslant \mathrm{GL}(2, 5)$. Since $G_v = \mathbb{Z}_5^2 \cdot H$ is a split extension by [7, Lemma 4.7], we have $\mathbb{Z}_5^2 \rtimes \mathrm{SL}(2, 5) \leqslant G_v \leqslant \mathbb{Z}_5^2 \rtimes \mathrm{GL}(2, 5)$, and since H acts irreducibly on $\mathbb{Z}_5^2$ by [7, Lemma 4.11], we have $\mathrm{ASL}(2, 5) \leqslant G_v \leqslant \mathrm{AGL}(2, 5)$. Finally, by [7, Lemma 4.8], $G_v/G_v^* = \mathrm{PGL}(2, 5)$, which forces that $G_v = \mathrm{AGL}(2, 5)$. $\square$

3. Main result

In this section, we give the main result of the paper. Let p be a prime and n a positive integer. We denote by $\mathbb{Z}_n$ the cyclic group of order n, by $\mathbb{Z}_p^n$ the elementary abelian group of order p^n, by D_{2n} the dihedral group of order $2n$, by F_n the Frobenius group of order n, and by A_n and S_n the alternating group and the symmetric group of degree n.

Theorem 3.1. *Let X be a connected hexavalent (G, s)-transitive graph for some $G \leqslant \mathrm{Aut}(X)$ and $s \geqslant 1$. Let $v \in V(X)$. Then $s \leqslant 4$ and one of the following statements holds:*

(1) *For $s = 1$, G_v is a $\{2, 3\}$-group.*

(2) *For $s = 2$, $G_v \cong \mathrm{PSL}(2, 5)$, $\mathrm{PGL}(2, 5)$, A_6 or S_6.*

(3) *For $s = 3$, $G_v \cong D_{10} \times \mathrm{PSL}(2, 5)$, $F_{20} \times \mathrm{PGL}(2, 5)$, $\mathrm{A}_5 \times \mathrm{A}_6$, $\mathrm{S}_5 \times \mathrm{S}_6$, $(D_{10} \times \mathrm{PSL}(2, 5)) \cdot \mathbb{Z}_2$ with $D_{10} \cdot \mathbb{Z}_2 = F_{20}$ and $\mathrm{PSL}(2, 5) \cdot \mathbb{Z}_2 = \mathrm{PGL}(2, 5)$, or $(\mathrm{A}_5 \times \mathrm{A}_6) \rtimes \mathbb{Z}_2$ with $\mathrm{A}_5 \rtimes \mathbb{Z}_2 = \mathrm{S}_5$ and $\mathrm{A}_6 \rtimes \mathbb{Z}_2 = \mathrm{S}_6$.*

(4) *For $s = 4$, $G_v \cong \mathbb{Z}_5^2 \rtimes \mathrm{GL}(2, 5) = \mathrm{AGL}(2, 5)$.*

Proof. Clearly, $s \leqslant 4$ and (4) holds by Lemma 2.4. Thus, we only need to prove (1), (2) and (3). In what follows we may assume that $s \leqslant 3$. Denote by $G_v^{N(v)}$ the constituent of G_v on $N(v)$, that is, the permutation group induced by G_v on $N(v)$. Since X is hexavalent, we have $G_v^{N(v)} = G_v/G_v^* \leqslant \mathrm{S}_6$.

Let $s = 1$. Then by Proposition 2.3, $G_v^{N(v)}$ is a transitive, but not a 2-transitive permutation group of degree 6, which implies that $6 \mid |G_v^{N(v)}|$ and G_v is not a $\{2\}$-group. Let p be a prime factor of order $|G_v|$. Then there exists an element g of order p in G_v. Suppose that $p > 5$. Then g fixes each vertex in $N(v)$ and $g \in G_v^*$, that is, for any vertex $u \in N(v)$ we have $g \in G_u$. Again g fixes each vertex in $N(u)$ because $p > 5$. By the connectivity of X, g fixes each vertex in $V(X)$ and hence $g = 1$, a contradiction. Suppose that $p = 5$. Then if g fixes each vertex in $N(u)$ for any $u \in V(X)$ then $g = 1$, a contradiction. Thus, there exists a vertex $w \in V(X)$ such that g has an orbit of length 5 because g has order $o(g) = 5$. It follows that G_w is 2-transitive on $X_1(w)$, and hence G_v is 2-transitive on $N(v)$, contrary to our assumption. Thus, $p \leqslant 3$. This implies that G_v is a $\{2, 3\}$-group and (1) holds.

Let $s \geqslant 2$. Then by Proposition 2.3, $G_v^{N(v)}$ is a 2-transitive permutation group of degree 6 and hence $5 \cdot 6 \mid |G_v^{N(v)}|$. Since X has valency 6, we have $G_v^{N(v)} = G_v/G_v^* \leqslant \mathrm{S}_6$. By Atlas [2], $G_v^{N(v)} = \mathrm{PSL}(2, 5)$, $\mathrm{PGL}(2, 5)$, A_6 or S_6. It follows that $G_{uv}^{N(v)\setminus\{u\}} = D_{10}$, F_{20}, A_5 or S_5. Note that each non-trivial normal subgroup of D_{10}, F_{20}, A_5 or S_5 is transitive on $N(v) \setminus \{u\}$ and $G_u^* \trianglelefteq G_{uv}$. Thus, G_u^* acts trivially or transitively on $N(v) \setminus \{u\}$.

Suppose that G_u^* acts trivially on $N(v) \setminus \{u\}$. Then $G_u^* \leqslant G_v^*$ and hence $G_u^* = G_v^*$ because the transitivity of G on $V(X)$ implies $|G_u^*| = |G_v^*|$. Since $G_v^* \trianglelefteq G_v$ and $G_u^* \trianglelefteq G_u$, we have $G_v^* \trianglelefteq \langle G_u, G_v \rangle$. By Proposition 2.2, $\langle G_u, G_v \rangle$ is transitive on $E(X)$. Since G_v^* fixes the edge $\{u, v\}$, it is easy to see that G_v^* fixes each edge in X, forcing that $G_v^* = 1$. Thus, $G_v = \mathrm{PSL}(2, 5)$, $\mathrm{PGL}(2, 5)$, A_6 or S_6. Let u and w be two distinct vertices in $N(v)$. Then $G_{uvw} = \mathbb{Z}_2$, $\mathbb{Z}_4$, A_4 or S_4, and hence G_{uvw} cannot act transitively on $N_u \setminus \{v\}$ because $|N_u \setminus \{v\}| = 5$. It follows that G is not 3-arc-transitive. Therefore, $s = 2$ and (2) holds.

Suppose that G_u^* acts transitively on $N(v) \setminus \{u\}$. Then by the symmetry of X, G_v^* acts transitively on $N(u) \setminus \{v\}$, and by [6, Section 1: Theorem], $|G_{uv}^*| = 1$. In particular, $G_u^* \neq G_v^*$. This implies that $(N(u) \setminus \{v\}) \cap (N(v) \setminus \{u\}) = \emptyset$, that is, X has no 3-cycles. Let (v_0, v_1, v_2, v_3) and (u_0, u_1, u_2, u_3) be two 3-arcs in X. Then (v_0, v_1, v_2, v_3) and (u_0, u_1, u_2, u_3) are not 3-cycles. Since $s \geqslant 2$, there exists an element $g \in G$ such that $(v_0, v_1, v_2)^g = (u_0, u_1, u_2)$. Clearly, $(v_0, v_1, v_2, v_3)^g = (u_0, u_1, u_2, v_3^g)$ is a 3-arc. Note that $G_{u_1}^*$ fixes u_0, u_1 and u_2, and acts on $N(u_2) \setminus \{u_1\}$ transitively. Thus, there exists an element $h \in G_{u_1}^*$ such that $v_3^{gh} = u_3$, that is, $(v_0, v_1, v_2, v_3)^{gh} = (u_0, u_1, u_2, u_3)$. It follows that X is $(G, 3)$-arc-transitive. Recall that we assume $s \leqslant 3$. Thus, in this case $s = 3$.

Note that the kernel of the action of G_u^* acting on $N(v) \setminus \{u\}$ equals $G_u^* \cap G_v^* = G_{uv}^* = 1$. Thus, G_u^* acts faithfully and transitively on $N(v) \setminus \{u\}$. Set $H = \langle G_z^*;\ z \in N(v) \rangle$. Since all of the G_z^* $(z \in N(v))$ are conjugate to each other in G_v, $H \trianglelefteq G_v$, and since for each $z \in N(v)$, G_z^* is transitive on $N(v) \setminus \{z\}$, we have H is transitive on $N(v)$. Recall that $G_v^{N(v)} = \mathrm{PSL}(2, 5)$, $\mathrm{PGL}(2, 5)$, A_6 or S_6. Thus, $H^{N(v)} = \mathrm{PSL}(2, 5)$, $\mathrm{PGL}(2, 5)$, A_6 or S_6. Let $\alpha \in G_v^*$ and $\beta \in G_z^*$. Then for each $x \in N(v)$, we have $x^{\alpha^{-1}\beta^{-1}\alpha\beta} = x^{\beta^{-1}\alpha\beta} = (x^{\beta^{-1}})^{\alpha\beta} = (x^{\beta^{-1}})^{\beta} = x$ and also this is true for any $x \in X_1(z)$ because $x^{\alpha^{-1}} \in N(z)$. Thus, $\alpha^{-1}\beta^{-1}\alpha\beta \in [G_v^*, G_z^*] \leqslant G_{vz}^* = 1$ and hence $[G_v^*, H] = 1$. It follows that $H \cap G_v^* \leqslant Z(G_v^*)$, the center of G_v^*. Since $G_v^* \trianglelefteq G_{uv}$, we have $1 \neq G_v^* \cong G_v^*/G_{uv}^* \cong G_v^*G_u^*/G_u^* \trianglelefteq G_{uv}^{N(u)}$. Note that $G_{uv}^{N(u)} = D_{10}$, F_{20}, A_5 or S_5. Thus, $G_v^* = \mathbb{Z}_5$, D_{10}, F_{20}, A_5 or S_5. Take $h \in H$ such that h fixes u with two 2-cycles on $N(v)$ and has order 2-power. Then h fixes some vertex $y \in N(u)$ with $y \neq v$. Note that G_v^* is transitive on $N(u) \setminus \{v\}$ with a regular subgroup $\mathbb{Z}_5$ and h commutes with every element in G_v^*. If h acts on $N(u) \setminus \{v\}$ non-trivially, then h induces a 5-cycle that lies in $\mathbb{Z}_5$ by Proposition 2.1. This is impossible because h fixes $y \in N(u) \setminus \{v\}$ and the order of h has 2-power. Thus, $h \in G_u^*$ and $2 \mid |G_u^*|$. It follows that $G_v^* = D_{10}$, F_{20}, A_5 or S_5 and $H \cap G_v^* \leqslant Z(G_v^*) = 1$. This implies that $G_v^*H = G_v^* \times H$ and H acts faithfully on $N(v)$, that is, $H = H^{N(v)} = \mathrm{PSL}(2, 5)$, $\mathrm{PGL}(2, 5)$, A_6 or S_6. By the definition of H, we have $G_u^* \leqslant H_u$ for $u \in N(v)$. If $H_u \neq G_u^*$ then $1 \neq H_u/G_u^*$ is a permutation group on $N(u) \setminus \{v\}$. It follows that there exists an element $g \in H_u$ such that g

acts on $N(u) \setminus \{v\}$ non-trivially. Since $G_v^* H = G_v^* \times H$, we have that g commutes with G_v^*. Recall that G_v^* acting on $N(u) \setminus \{v\}$ has a regular subgroup $\mathbb{Z}_5$. Thus, g centralizes $\mathbb{Z}_5$, which is impossible by Proposition 2.1. Thus, $H_u = G_u^*$ and hence $|H| = 6 \cdot |H_u| = 6 \cdot |G_u^*| = 6 \cdot |G_v^*|$. It forces that if $G_v^* = D_{10}$ then $H = \mathrm{PSL}(2,5)$; if $G_v^* = F_{20}$ then $H = \mathrm{PGL}(2,5)$; if $G_v^* = \mathrm{A}_5$ then $H = \mathrm{A}_6$; if $G_v^* = \mathrm{S}_5$ then $H = \mathrm{S}_6$.

Assume that $G_v^* = D_{10}$. Then $H = \mathrm{PSL}(2,5)$. Recall that $G_v/G_v^* = \mathrm{PSL}(2,5)$, $\mathrm{PGL}(2,5)$, A_6 or S_6. Since $H \cong G_v^* H/G_v^* \trianglelefteq G_v/G_v^*$, we have $G_v/G_v^* = \mathrm{PSL}(2,5)$ or $\mathrm{PGL}(2,5)$. If $G_v/G_v^* = \mathrm{PSL}(2,5)$ then $G_v^* H/G_v^* = G_v/G_v^*$. It follows that $G_v^* H = G_v$ and hence $G_v = D_{10} \times \mathrm{PSL}(2,5)$.

If $G_v/G_v^* = \mathrm{PGL}(2,5)$ then $|G_v : G_v^* H| = 2$ and $G_{uv}^{N(v)\setminus\{u\}} = F_{20}$. Thus, there exists a 2-element $g \in G_{uv}$ such that g induces a 4-cycle on $N(v)\setminus\{u\}$. Since g is a 2-element, g induces the identity, a transposition, two 2-cycles or a 4-cycle on $N(u) \setminus \{v\}$. If g induces the identity on $N(u)\setminus\{v\}$ then $g \in G_u^*$, which is impossible because $G_u^* = D_{10}$ acts faithfully on $N(v)$. If g induces a transposition on $N(u) \setminus \{v\}$ then $gG_u^* \in G_u/G_u^* \cong \mathrm{PGL}(2,5)$, which is impossible because a primitive permutation group of degree 6 containing a transposition must be S_6 (see [3, Theorem 3.3A]). If g induces two 2-cycles on $N(u) \setminus \{v\}$ then g induces an even permutation on $N(u)\setminus\{v\}$. Note that $G_{uv}^{N(v)\setminus\{u\}} = F_{20}$ and $G_v^* = D_{10}$ acts faithfully and transitively on $N(u)\setminus\{v\}$. It forces that $g \in G_v^*$, which is impossible because g induces a 4-cycle on $N(v)$. Thus, g induces a 4-cycle on $N(u)\setminus\{v\}$. It follows that $g^4 \in G_{uv}^* = 1$, that is, g has order 4. Since g^2 induces two 2-cycles on both $N(u) \setminus \{v\}$ and $N(v) \setminus \{u\}$, we have $g^2 \in G_v^* \times H$, and since $G_v^* < G_v^* \cdot \langle g \rangle = G_v^* \cdot \mathbb{Z}_2 \leqslant G_{uv}^{N(u)\setminus\{v\}} = F_{20}$, we have $G_v^* \cdot \langle g \rangle = F_{20}$. Thus, we have $G_v = (D_{10} \times \mathrm{PSL}(2,5)) \cdot \langle g \rangle = (D_{10} \times \mathrm{PSL}(2,5)) \cdot \mathbb{Z}_2$ with $D_{10} \cdot \mathbb{Z}_2 = F_{20}$ and $\mathrm{PSL}(2,5) \cdot \mathbb{Z}_2 = \mathrm{PGL}(2,5)$.

Assume that $G_v^* = F_{20}$. Then $H = \mathrm{PGL}(2,5)$. Since $HG^*/G_v^* \trianglelefteq G_v/G_v^*$, we have $G_v/G_v^* = \mathrm{PGL}(2,5)$ and hence $G_v = G_v^* H = F_{20} \times \mathrm{PGL}(2,5)$.

Assume that $G_v^* = \mathrm{A}_5$. Then $H = \mathrm{A}_6$. Since $H \cong G_v^* H/G_v^* \trianglelefteq G_v/G_v^* \leqslant \mathrm{S}_6$, we have $G_v/G_v^* = \mathrm{A}_6$ or S_6. If $G_v/G_v^* = \mathrm{A}_6$ then $G_v = G_v^* H = \mathrm{A}_5 \times \mathrm{A}_6$.

If $G_v/G_v^* = \mathrm{S}_6$ then $|G_v : G_v^* H| = 2$. Clearly, $G_v^* H = \mathrm{A}_5 \times \mathrm{A}_6 \lhd G_v$, $G_v^* \lhd G_v$ and $H \lhd G_v$. Since $G_v^{N(v)} = G_v/G_v^* = \mathrm{S}_6$, there exists an element $g_1 \in G_{uv}$ such that g_1 induces a transposition on $N(v)$. Since $G_v^* = \mathrm{A}_5$ and G_v^* acts faithfully on $N(u)$, there exists an element $g_2 \in G_v^*$ such that $g_1 g_2$ induces the identity or a transposition on $N(u)\setminus\{v\}$. For the former, $g_1 g_2 \in G_u^*$ and $g_1 g_2$ induces the same transposition as g_1 on $N(v)$, contrary to the fact that $G_u^* = \mathrm{A}_5$, acting faithfully on $N(v)$. Set $g = g_1 g_2$. Thus, $g \in G_{uv}$ induces a transposition on both $N(u)$ and $N(v)$. Furthermore, $g^2 \in G_{uv}^* = 1$ and hence g is an involution. It follows that $H\langle g \rangle = \mathrm{S}_6$ and $G_v^* \langle g \rangle = \mathrm{S}_5$ because $G_v^* \lhd G_v$ and $H \lhd G_v$. Thus, $(\mathrm{A}_5 \times \mathrm{A}_6) \rtimes \mathbb{Z}_2$ with $G_v = \mathrm{A}_5 \rtimes \mathbb{Z}_2 = \mathrm{S}_5$ and $\mathrm{A}_6 \rtimes \mathbb{Z}_2 = \mathrm{S}_6$.

Assume that $G_v^* = \mathrm{S}_5$. Then $H = \mathrm{S}_6$. Since $\mathrm{S}_6 \leqslant G_v^*H/G_v^* \leqslant G_v/G_v^* \leqslant \mathrm{S}_6$, we have $G_v = G_v^*H$ and hence $G_v = G_v^*H = \mathrm{S}_5 \times \mathrm{S}_6$. Thus, (3) holds. $\square$

4. Examples

Let X be a connected hexavalent (G, s)-transitive graph and let $v \in V(X)$. In this section, we show that each type of G_v in Theorem 3.1 can be realized. Let n be a positive integer. Denote by C_n, K_n and $K_{n,n}$ the cycle of order n, the complete graph of order n and the complete bipartite graph of order $2n$, respectively. The first example is a connected hexavalent $(G, 1)$-transitive graph with G_v a $\{2, 3\}$-group and the order $|G_v|$ having no upper bound.

Example 4.1. The lexicographic product $C_n[3K_1]$ is defined as the graph with vertex set $V(C_n) \times V(3K_1)$ such that for any two vertices $u = (x_1, y_1)$ and $v = (x_2, y_2)$ in $V(C_n[3K_1])$, u is adjacent to v in $C_n[3K_1]$ if and only if $\{x_1, x_2\} \in E(C_n)$. Then $C_n[3K_1]$ is a connected hexavalent 1-transitive graph with $\mathrm{Aut}(C_n[3K_1]) = \mathrm{S}_3^n \rtimes D_{2n}$ and a vertex stabilizer $\mathrm{Aut}(C_n[3K_1])_v$ of $v \in V(C_n[3K_1])$ in $\mathrm{Aut}(C_n[3K_1])$ isomorphic to $(\mathrm{S}_3^{n-1} \cdot \mathbb{Z}_2) \rtimes \mathbb{Z}_2$.

Next we give a connected hexavalent $(G, 2)$-transitive graph with G_v isomorphic to A_6 or S_6.

Example 4.2. Let $X = K_7$. Then $A = \mathrm{Aut}(X) = \mathrm{S}_7$. Clearly, A has an arc-transitive subgroup B isomorphic to A_7. Thus, the vertex stabilizers A_v and B_v of $v \in V(K_7)$ in A and B are isomorphic to S_6 and A_6, respectively.

The following example is a connected hexavalent G-arc-transitive graph with G_v isomorphic to $\mathrm{PSL}(2, 5)$, $\mathrm{PGL}(2, 5)$, $D_{10} \times \mathrm{PSL}(2, 5)$, $F_{20} \times \mathrm{PGL}(2, 5)$, $\mathrm{A}_5 \times \mathrm{A}_6$, $\mathrm{S}_5 \times \mathrm{S}_6$, $(D_{10} \times \mathrm{PSL}(2, 5)) \cdot \mathbb{Z}_2$ with $D_{10} \cdot \mathbb{Z}_2 = F_{20}$ and $\mathrm{PSL}(2, 5) \cdot \mathbb{Z}_2 = \mathrm{PGL}(2, 5)$, or $(\mathrm{A}_5 \times \mathrm{A}_6) \rtimes \mathbb{Z}_2$ with $\mathrm{A}_5 \rtimes \mathbb{Z}_2 = \mathrm{S}_5$ and $\mathrm{A}_6 \rtimes \mathbb{Z}_2 = \mathrm{S}_6$.

Example 4.3. Let $X = K_{6,6}$ with bipartite sets $\{1, 3, 5, 7, 9, 11\}$ and $\{2, 4, 6, 8, 10, 12\}$. Then $A = \mathrm{Aut}(X) \cong \mathrm{S}_6 \wr \mathrm{S}_2$ and $A_1 = \mathrm{S}_5 \times \mathrm{S}_6$. Clearly, A has a 3-transitive subgroup $B \cong \mathrm{A}_6 \wr \mathrm{S}_2$ and $B_1 = \mathrm{A}_5 \times \mathrm{A}_6$. Let $C = \langle B, (1,3)(2,12) \rangle$. Then C is 3-transitive and $C_1 = (\mathrm{A}_5 \times \mathrm{A}_6) \rtimes \mathbb{Z}_2$ with $\mathrm{A}_5 \rtimes \mathbb{Z}_2 = \mathrm{S}_5$ and $\mathrm{A}_6 \rtimes \mathbb{Z}_2 = \mathrm{S}_6$.

Take the following elements in A:

$$
\begin{aligned}
&a = (2, 4, 6, 8, 12), && b = (1, 12)(2, 3)(4, 5)(6, 7)(8, 9)(10, 11), \\
&c = (2, 8)(10, 12), && d = (2, 4)(6, 8)(10, 12), \\
&e = (1, 11)(2, 4)(3, 5)(6, 8)(7, 9)(10, 12). &&
\end{aligned}
$$

Then by Magma [1], $G = \langle a, b, c\rangle = \mathrm{PSL}(2,5) \wr \mathrm{S}_2$, $H = \langle B, d\rangle = \mathrm{PGL}(2,5) \wr \mathrm{S}_2$ and $K = \langle a, b, e\rangle$ are 3-transitive. Furthermore, $G_1 = D_{10} \times \mathrm{PSL}(2,5)$, $H_1 = F_{20} \times \mathrm{PGL}(2,5)$ and $K_1 = (D_{10} \times \mathrm{PSL}(2,5)) \cdot \mathbb{Z}_2$ with $D_{10} \cdot \mathbb{Z}_2 = F_{20}$ and $\mathrm{PSL}(2,5) \cdot \mathbb{Z}_2 = \mathrm{PGL}(2,5)$.

Take the following elements in A:

$$
\begin{aligned}
w &= (1,9,7,3,5)(2,6,4,8,12), & x &= (1,2)(3,4)(5,6)(7,8)(9,12)(10,11),\\
y &= (1,3)(4,12)(6,10)(9,11), & z &= (1,5,11)(3,7,9)(6,10,12),\\
g &= (1,9)(2,12)(3,7)(5,11).
\end{aligned}
$$

Let $M = \langle w, x, y, z\rangle$ and $N = \langle M, g\rangle$. Then by Magma [1], M and N are 2-transitive with $M_1 = \mathrm{PSL}(2,5)$ and $N_1 = \mathrm{PGL}(2,5)$.

Let G be a finite group, H a subgroup of G and $D = D^{-1}$ a union of several double-cosets of the form HgH with $g \notin H$. The *coset graph* $X = \mathrm{Cos}(G, H, D)$ of G with respect to H and D is defined to have vertex set $V(X) = [G : H]$, the set of the right cosets of H in G, and edge set $E(X) = \{\{Hg, Hdg\};\ g \in G, d \in D\}$. Then X is well defined and has valency $|D|/|H|$. Furthermore, X is connected if and only if D generates G. Note that G acts on $V(X)$ by right multiplication and so we can view G/H_G as a subgroup of $\mathrm{Aut}(X)$, where H_G is the largest normal subgroup of G contained in H. It is easy to see that G is transitive on the arcs of X if and only if $D = HgH$ for some $g \in G \setminus H$. Denote by 5^{1+2}_{+} the unique non-abelian group of order 125 with exponent 5. The following example is extracted from [9, Section 2] (also see [12]).

Example 4.4. Let $G = \mathrm{Ru}$. Then G has a maximal subgroup $H = \mathrm{AGL}(2,5)$. Let p be a Sylow 5-subgroup of H. Then by Atlas [2], $L = N_H(P) = 5^{1+2}_{+}.(\mathbb{Z}_4 \cdot \mathbb{Z}_4)$ and $N_G(P) = 5^{1+2}_{+}.(\mathbb{Z}_4 \cdot \mathrm{S}_4)$. Let M be a $\{2,5\}$-subgroup of $N_G(P)$ such that $L \leqslant M$. Then $|M : L| = 2$ and there exists a 2-element $g \in M \setminus L$ such $g^2 \in L$ and $L^g = L$. It follows that $L = H \cap H^g$, $HgH = Hg^{-1}H$ and $|H : H \cap H^g| = 6$. Since H is maximal in G, we have $\langle H, g\rangle = G$. Thus, the coset graph $\mathrm{Cos}(G, H, HgH)$ is connected, hexavalent and $(G, 4)$-transitive with $H = \mathrm{AGL}(2,5)$ as a vertex stabilizer in G.

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