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REFLEXIVITY OF BILATTICES

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Abstract. We study reflexivity of bilattices. Some examples of reflexive and non-reflexive bilattices are given. With a given subspace lattice $\mathscr{L}$ we may associate a bilattice $\Sigma_{\mathscr{L}}$. Similarly, having a bilattice Σ we may construct a subspace lattice $\mathscr{L}_{\Sigma}$. Connections between reflexivity of subspace lattices and associated bilattices are investigated. It is also shown that the direct sum of any two bilattices is never reflexive.

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1. Introduction

Let $\mathscr{H}$ be a separable complex Hilbert space. By $\mathscr{B}(\mathscr{H})$ we denote the algebra of all bounded linear operators on $\mathscr{H}$ and by $\mathscr{P}(\mathscr{H})$ the set of all orthogonal projections on $\mathscr{H}$.

Recall that for two closed subspaces $M, N \subset \mathscr{H}$ we can define *join* $M \vee N = \text{cl}\{f+g\colon f \in M, g \in N\}$ and *meet* $M \wedge N = M \cap N$. Now if we identify a closed linear subspace with the orthogonal projection onto it, then $\mathscr{P}(\mathscr{H})$ with the operations defined above forms a complete lattice. A SOT-closed sublattice of $\mathscr{P}(\mathscr{H})$ containing the trivial projections 0 and I is called a *subspace lattice*. Here, for a family of operators $\mathscr{S} \subseteq \mathscr{B}(\mathscr{H})$, we denote by $\operatorname{lat}\mathscr{S} = \{P \in \mathscr{P}(\mathscr{H});\ SP = PSP\ \forall\, S \in \mathscr{S}\}$ the collection of orthogonal projections onto the subspaces invariant for $\mathscr{S}$. For a subspace lattice $\mathscr{L}$, we denote by $\operatorname{alg}\mathscr{L}$ the algebra of all operators $A \in \mathscr{B}(\mathscr{H})$ satisfying $\mathscr{L} \subseteq \operatorname{lat}\{A\}$, i.e., operators that leave invariant the ranges of all projections in $\mathscr{L}$.

Reflexivity was first introduced for operator algebras ([5]). An algebra $\mathscr{A} \subset \mathscr{B}(\mathscr{H})$ containing the identity is called *reflexive* if $\mathscr{A} = \operatorname{alg}\operatorname{lat}\mathscr{A}$. Given an abstract lattice $\mathscr{L} \subset \mathscr{P}(\mathscr{H})$, one can also ask if there is an algebra $\mathscr{A}$ such that $\operatorname{lat}\mathscr{A} = \mathscr{L}$.

Such lattices are called reflexive. Namely, a subspace lattice $\mathscr{L}$ is called *reflexive*, if $\mathscr{L} = \operatorname{lat}\operatorname{alg}\mathscr{L}$ ([5]). Reflexivity for subspaces was defined in [4]: a subspace $\mathscr{S} \subset \mathscr{B}(\mathscr{H})$ is *reflexive* if $\mathscr{S} = \{T \in \mathscr{B}(\mathscr{H})\colon\ Th \in \overline{\mathscr{S}h}$ for all $h \in \mathscr{H}\}$.

We also define analogues of the above notions for bilattices. Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces. Recall that a bilattice is a set $\Sigma \subseteq \mathscr{P}(\mathscr{H}) \times \mathscr{P}(\mathscr{K})$ such that $(0, I), (I, 0), (0, 0) \in \Sigma$ and $(P_1 \wedge P_2, Q_1 \vee Q_2), (P_1 \vee P_2, Q_1 \wedge Q_2) \in \Sigma$ whenever $(P_1, Q_1), (P_2, Q_2) \in \Sigma$. Bilattices were introduced by Shulman in [6] and studied in [7] as subspace analogues of lattices. Here we consider only bilattices closed in the strong operator topology. For a bilattice $\Sigma \subseteq \mathscr{P}(\mathscr{H}) \times \mathscr{P}(\mathscr{K})$ we define

$$\operatorname{op}\Sigma = \{T \in \mathscr{B}(\mathscr{H}, \mathscr{K})\colon\ QTP = 0,\ \forall\,(P, Q) \in \Sigma\}.$$

Then $\operatorname{op}\Sigma$ is a reflexive subspace and all reflexive subspaces are of this form. The bilattice $\operatorname{bil}\mathscr{S}$ of a subspace $\mathscr{S} \subseteq \mathscr{B}(\mathscr{H}, \mathscr{K})$ is defined as the set

$$\operatorname{bil}\mathscr{S} = \{(P, Q)\colon\ Q\mathscr{S}P = \{0\}\}.$$

Definition 1.1. *Let $\Sigma \subseteq \mathscr{P}(\mathscr{H}) \times \mathscr{P}(\mathscr{K})$ be a bilattice. Then Σ is called reflexive if* $\operatorname{bil}\operatorname{op}\Sigma = \Sigma$.

2. Connections between the reflexivity of lattices and bilattices

For a bilattice $\Sigma \subset \mathscr{P}(\mathscr{H}) \times \mathscr{P}(\mathscr{K})$ we may consider the sets $\Sigma_l = \{P\colon\ (P, Q) \in \Sigma$ for some $Q\}$ and $\Sigma_r = \{Q\colon\ (P, Q) \in \Sigma$ for some $P\}$. Plainly both sets Σ_l and Σ_r are lattices. The natural question is: what is the relationship between the reflexivity of Σ_l and Σ_r and the reflexivity of Σ? The example below shows that even if both Σ_l and Σ_r are reflexive, Σ may be not reflexive.

Example 2.1. Let $\mathscr{L}$ be any subspace lattice in $\mathscr{P}(\mathscr{H})$. Then the set

$$\Sigma = \{(P, 0), (P, I)\colon\ P \in \mathscr{L}\},$$

is a non-reflexive bilattice. To see this it is enough to note that since $I \in \mathscr{L}$, $\operatorname{op}\Sigma = \{0\}$ and $\operatorname{bil}\operatorname{op}\Sigma = \mathscr{P}(\mathscr{H}) \times \mathscr{P}(\mathscr{K})$.

Remark 2.2. Note that for the lattice $\mathscr{L}$ in Example 2.2 we may take a nest (i.e. a linearly ordered lattice). Since the trivial lattice $\{0, I\}$ is also a nest, a bilattice given by two nests does not have to be reflexive. Note also that if (I, I) is in a bilattice $\Sigma \subsetneq \mathscr{P}(\mathscr{H}) \times \mathscr{P}(\mathscr{K})$, then $\operatorname{op}\Sigma = \{0\}$, so Σ is not reflexive. Moreover, if a bilattice Σ is reflexive, then the pairs $(P, 0)$, $(0, Q)$ must belong to Σ for all $P \in \mathscr{P}(\mathscr{H})$ and $Q \in \mathscr{P}(\mathscr{K})$.

Given a subspace lattice $\mathscr{L}$, one can form a billatice $\Sigma_{\mathscr{L}}$ by letting

$$\Sigma_{\mathscr{L}} = \{(P,Q)\colon \text{ there exists } L \in \mathscr{L} \text{ with } P \leqslant L \leqslant Q^{\perp}\}.$$

Note that for any $P, Q \in \mathscr{P}(\mathscr{H})$ the pairs $(P,0)$ and $(0,Q)$ belong to $\Sigma_{\mathscr{L}}$.

There is a dual construction as well: given a bilattice Σ we may consider a lattice defined by

$$\mathscr{L}_{\Sigma} = \{P \oplus Q^{\perp}\colon (P,Q) \in \Sigma\}.$$

Now we can ask what is the relationship between the reflexivity of $\mathscr{L}$ and the reflexivity of $\Sigma_{\mathscr{L}}$? Similarly, what is the relationship between the reflexivity of Σ and the reflexivity of $\mathscr{L}_{\Sigma}$?

Proposition 2.3. *If $\mathscr{L}$ is a subspace lattice, then* $\operatorname{op}\Sigma_{\mathscr{L}} = \operatorname{alg}\mathscr{L}$.

P r o o f. Let $T \in \operatorname{alg}\mathscr{L}$ and $E \in \mathscr{L}$. If $P, Q \in \mathscr{P}(\mathscr{H})$ are such that $P \leqslant E \leqslant Q^{\perp}$, then $QTP = 0$. Hence $T \in \operatorname{op}\Sigma_{\mathscr{L}}$.

On the other hand, if $T \in \operatorname{op}\Sigma_{\mathscr{L}}$ and $E \in \mathscr{L}$, then $(E, E^{\perp}) \in \Sigma_{\mathscr{L}}$. Hence $E^{\perp}TE = 0$, so $T \in \operatorname{alg}\mathscr{L}$. □

Proposition 2.4. *If $\mathscr{L}$ is a subspace lattice, then* $\operatorname{bil}\operatorname{op}\Sigma_{\mathscr{L}} = \Sigma_{\operatorname{lat}\operatorname{alg}\mathscr{L}}$.

P r o o f. Let $(P,Q) \in \Sigma_{\operatorname{lat}\operatorname{alg}\mathscr{L}}$. There is $E \in \operatorname{lat}\operatorname{alg}\mathscr{L}$ such that $P \leqslant E \leqslant Q^{\perp}$. Note that $QTP = 0$ for all $T \in \operatorname{alg}\mathscr{L} = \operatorname{op}\Sigma_{\mathscr{L}}$. Hence $(P,Q) \in \operatorname{bil}\operatorname{op}\Sigma_{\mathscr{L}}$.

Let now $(P,Q) \in \operatorname{bil}\operatorname{op}\Sigma_{\mathscr{L}}$. Since $Q\operatorname{alg}\mathscr{L}P = 0$ and $I \in \operatorname{alg}\mathscr{L}$, we have $QP = 0$ and $P \leqslant Q^{\perp}$. Denote by $L = [\operatorname{alg}\mathscr{L}P\mathscr{H}]$ (the projection on $\operatorname{alg}\mathscr{L}P\mathscr{H}$). Notice that $L^{\perp}\operatorname{alg}\mathscr{L}L = 0$, which implies that $L \in \operatorname{lat}\operatorname{alg}\mathscr{L}$. To prove that $(P,Q) \in \Sigma_{\operatorname{lat}\operatorname{alg}\mathscr{L}}$ it suffices to show that $P \leqslant L \leqslant Q^{\perp}$. Since $IPx \in \operatorname{alg}\mathscr{L}Px$ for any $x \in \mathscr{H}$, we have $L^{\perp}Px = 0$. Hence $P \leqslant L$. Similarly, for any $x \in \mathscr{H}$ we have $QLx = 0$, so $L \leqslant Q^{\perp}$. □

Corollary 2.5. *If $\mathscr{L}$ is a subspace lattice, then $\mathscr{L}$ is reflexive if and only if $\Sigma_{\mathscr{L}}$ is reflexive.*

This corollary allows us to construct easily examples of reflexive or non-reflexive bilattices.

Proposition 2.6. *Let Σ be a bilattice. If Σ is reflexive, then the lattice $\mathscr{L}_{\Sigma}$ is reflexive.*

P r o o f. Note that $(A_{ij})_{i,j=1,2} \in \operatorname{alg}\mathscr{L}_\Sigma$ if and only if $A_{11} \in \operatorname{alg}\Sigma_l$, $A_{12} = 0$, $A_{21} \in \operatorname{op}\Sigma$ and $A_{22} \in \operatorname{alg}(\Sigma_r)^\perp$. Take $P \in \operatorname{lat}\operatorname{alg}\mathscr{L}_\Sigma$. Note that P has a matrix form $\begin{pmatrix} P_1 & P_2 \\ P_2^* & P_3 \end{pmatrix}$, where P_1 and P_3 are projections. Let $A = \begin{pmatrix} \alpha I & 0 \\ B & \beta I \end{pmatrix} \in \operatorname{alg}\mathscr{L}_\Sigma$. Then $B \in \operatorname{op}\Sigma$. Since $P^\perp AP = 0$, putting $\alpha = 1$, $\beta = 0$ and $B = 0$ we obtain that $P_2 = 0$ and for $\alpha = \beta = 0$ we have that $P_3^\perp BP_1 = (I - P_3)BP_1 = 0$. Therefore $(P_1, P_3^\perp) \in \operatorname{bil}\operatorname{op}\Sigma = \Sigma$, which implies that $P \in \mathscr{L}_\Sigma$. □

The example below shows that the reflexivity of $\mathscr{L}_\Sigma$ does not imply the reflexivity of Σ.

Example 2.7. Let $\dim\mathscr{H} > 1$ and $\dim\mathscr{K} > 1$. Consider the bilattice $\Sigma = \{(0,0),(0,I),(I,0)\}$. Since $\operatorname{op}\Sigma = \mathscr{B}(\mathscr{H},\mathscr{K})$, for any non-trivial projection $P \in \mathscr{P}(\mathscr{H})$ the pair $(P,0) \in \operatorname{bil}\operatorname{op}\Sigma$. Hence Σ is not reflexive.

On the other hand, $\mathscr{L}_\Sigma = \{0\oplus I, I\oplus I, 0\oplus 0\}$ and $\operatorname{alg}\mathscr{L}_\Sigma = \begin{pmatrix} \mathscr{B}(\mathscr{H}) & 0 \\ \mathscr{B}(\mathscr{H},\mathscr{K}) & \mathscr{B}(\mathscr{K}) \end{pmatrix}$. It is easy to check that $\operatorname{lat}\operatorname{alg}\mathscr{L}_\Sigma = \mathscr{L}_\Sigma$, so $\mathscr{L}_\Sigma$ is reflexive.

3. The orthogonal sum of bilattices

By [2, Theorem 3.4], we know that the orthogonal sum preserves reflexivity of operator subspaces, i.e. the orthogonal sum of subspaces is reflexive if and only if each subspace is reflexive. Similar result was obtained in [1, Theorem 7.1] for subspace lattices. Hence one should expect that the same can be proved for bilattices. However, we will see that it is not true. First, we will need the following result.

Proposition 3.1. *Let $\Sigma_n \subset \mathscr{P}(\mathscr{H}_n) \times \mathscr{P}(\mathscr{K}_n)$ be bilattices, for $n \in \mathbb{N}$. Then $\operatorname{op}(\bigoplus \Sigma_n) = \bigoplus \operatorname{op}\Sigma_n$.*

P r o o f. Let $A_n \in \operatorname{op}\Sigma_n$. Define $A = \oplus A_n \in \bigoplus \operatorname{op}\Sigma_n$. Then for any $(P,Q) \in \bigoplus \Sigma_n$ we have $(P,Q) = \oplus(P_n,Q_n)$ and $QAP = \oplus Q_nA_nP_n = 0$. Hence $\bigoplus \operatorname{op}\Sigma_n \subset \operatorname{op}\bigoplus \Sigma_n$.

Let now $A \in \operatorname{op}\bigoplus\Sigma_n$ and $A = (A_{ij})$. Choose $i,j \in \mathbb{N}$ such that $i \neq j$. Set $P = 0\oplus\ldots\oplus I\oplus 0\oplus\ldots$, where I is on the j-th place, and $Q = 0\oplus\ldots\oplus I\oplus 0\oplus\ldots$, where I is on the i-th place. Note that the equation $QAP = 0$ implies that $A_{ij} = 0$. Hence A is decomposable to $\oplus A_{nn}$. Moreover, if $P = 0\oplus\ldots\oplus P_n\oplus 0\oplus\ldots$ and $Q = 0\oplus\ldots\oplus Q_n\oplus 0\oplus\ldots$, for $(P_n,Q_n) \in \Sigma_n$, then $QAP = 0$ implies that $Q_nA_{nn}P_n = 0$. Hence $A_{nn} \in \operatorname{op}\Sigma_n$, so $A \in \bigoplus \operatorname{op}\Sigma_n$. □

Theorem 3.2. *The orthogonal sum of any two bilattices is not reflexive.*

Before proving the theorem, let us consider the following example, which shows that the orthogonal sum of two reflexive bilattices does not have to be reflexive.

Example 3.3. Let

$$\Sigma = \{(P, 0), (0, P)\colon\ P \in \mathscr{P}(\mathscr{H})\}.$$

Note that $\operatorname{op}\Sigma = \mathscr{B}(\mathscr{H})$ and $\operatorname{bil}\operatorname{op}\Sigma = \Sigma$. Hence Σ is reflexive.

Denote by $\widetilde{\Sigma} = \Sigma \oplus \Sigma$. By Proposition 3.9

$$\operatorname{op}\widetilde{\Sigma} = \operatorname{op}\Sigma \oplus \operatorname{op}\Sigma = \mathscr{B}(\mathscr{H}) \oplus \mathscr{B}(\mathscr{H}).$$

If $\widetilde{\Sigma}$ is reflexive, then it must contain the pairs $(\widetilde{P}, 0)$ and $(0, \widetilde{P})$, for every orthogonal projection $\widetilde{P} \in \mathscr{P}(\mathscr{H} \oplus \mathscr{H})$. That would mean that each orthogonal projection on $\mathscr{H} \oplus \mathscr{H}$ is of the form $P_1 \oplus P_2$, but that is not true. Consider, for instance, two commuting, positive contractions $S, C \in \mathscr{B}(\mathscr{H})$ such that $S^2 + C^2 = I$, $S < C$ and $\ker S = \ker(C - S) = \{0\}$ and put $\widetilde{P} = \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix}$ (see [3]). Notice that $\widetilde{P}$ is an orthogonal projection on $\mathscr{H} \oplus \mathscr{H}$ and the pair $(\widetilde{P}, 0) \in \operatorname{bil}\operatorname{op}\widetilde{\Sigma}$ but $(\widetilde{P}, 0) \notin \widetilde{\Sigma}$. Hence $\widetilde{\Sigma}$ is not reflexive.

Now we can ask if it is possible that the orthogonal sum of any two bilattices is reflexive?

P r o o f of Theorem 3.10. Let us consider Hilbert spaces $\mathscr{H}_1$, $\mathscr{H}_2$, $\mathscr{K}_1$, $\mathscr{K}_2$ with orthogonal bases $\{e_i\colon\ i \in I\}$ for $\mathscr{H}_1$ and $\{f_j\colon\ j \in J\}$ for $\mathscr{H}_2$. Take $i_0 \in I$ and $j_0 \in J$. For $h = \sum h_i e_i \in \mathscr{H}_1$ and $g = \sum g_j f_j \in \mathscr{H}_2$ put $P(h \oplus g) = \frac{1}{2}(h_{i_0} + g_{j_0})e_{i_0} \oplus \frac{1}{2}(h_{i_0} + g_{j_0})f_{j_0}$. Notice that P is an orthogonal projection on $\mathscr{H}_1 \oplus \mathscr{H}_2$ but P cannot be decomposed to the sum $P = P_1 \oplus P_2$ for $P_k \in \mathscr{P}(\mathscr{H}_k)$, $k = 1, 2$. Hence the pair $(P, 0) \notin \Sigma_1 \oplus \Sigma_2$ for any bilattices $\Sigma_k \subset \mathscr{P}(\mathscr{H}_k) \times \mathscr{P}(\mathscr{K}_k)$ $(k = 1, 2)$. Therefore the sum $\Sigma_1 \oplus \Sigma_2$ cannot be reflexive, by Remark 2.3. □

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