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Czechoslovak Mathematical Journal, Vol. 63 (2013), No. 4, 1057–1085

Persistent URL: http://dml.cz/dmlcz/143617

Czechoslovak Mathematical Journal, 63 (138) (2013), 1057–1085

CURVES IN BANACH SPACES WHICH ALLOW A $C^{1,\mathrm{BV}}$ PARAMETRIZATION OR A PARAMETRIZATION WITH FINITE CONVEXITY

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(Received October 3, 2012)

Abstract. We give a complete characterization of those $f\colon [0,1] \to X$ (where X is a Banach space) which allow an equivalent $C^{1,\mathrm{BV}}$ parametrization (i.e., a C^1 parametrization whose derivative has bounded variation) or a parametrization with bounded convexity. Our results are new also for $X = \mathbb{R}^n$. We present examples which show applicability of our characterizations. For example, we show that the $C^{1,\mathrm{BV}}$ and C^2 parametrization problems are equivalent for $X = \mathbb{R}$ but are not equivalent for $X = \mathbb{R}^2$.

Keywords: curve in Banach spaces; $C^{1,\mathrm{BV}}$ parametrization; parametrization with bounded convexity

MSC 2010: 26E20, 26A51, 53A04

1. Introduction

Let X be a (real) Banach space, and let a continuous curve $f\colon [a,b] \to X$ be given. More than sixty years ago several authors (i.e. Ward, Zahorski, Choquet, Tolstov) investigated (in the case $X = \mathbb{R}^n$) conditions under which f allows an equivalent parametrization which is "smooth of the first order" (i.e., it is differentiable, or C^1). For more information and generalizations of these results to the case of an arbitrary X see [7]; the characterization in the C^1 case is recalled below (Theorem 6.3).

The problem of C^n ($n \in \mathbb{N}$) parametrizations (and other types of "higher order smooth" parametrizations) in the case $X = \mathbb{R}$ was settled in [14], [15]; see Section 4

The research of the first author was supported in part by ISF. The research of the second author was supported in part by grant MSM 0021620839 and the grants GAČR 201/06/0198 and GAČR P201/12/0436.

below. (The problem of n-times differentiable parametrizations in the case $X = \mathbb{R}$ was solved in [6].)

The problem of "higher order smooth" parametrizations in the vector case (even for $X = \mathbb{R}^2$) is essentially more difficult than in the case $X = \mathbb{R}$ and the problem of vector C^n parametrization for $n \geqslant 3$ is still open.

However, the case of vector C^2 parametrizations (for X having a Fréchet smooth norm) was settled in [10], the case of a $C^{1,\alpha}$ $(0 < \alpha \leqslant 1)$ parametrizations in [9], and the case of twice differentiable parametrizations in [5].

In the present article, we will characterize those $f\colon [a,b] \to X$ which allow parametrizations with bounded convexity (see Theorem 5.4) as well as functions that allow $C^{1,\mathrm{BV}}$ parametrizations (see Theorem 6.5). (Our exposition is essentially an improved version of a part of the unpublished preprint [8], where the bounded convexity parametrization problem was considered together with the C^2 problem.)

Here a function $f\colon [a,b] \to X$ is called a $C^{1,\mathrm{BV}}$ function if f is C^1 and f' has bounded variation on $[a,b]$. The notion of real functions with bounded convexity goes back to de la Vallée Poussin (1908) and Riesz (1911) (see [18, p. 28]) and its natural generalization to Banach-valued case was studied in detail in [21]. Note that a continuous $f\colon [a,b] \to X$ has bounded convexity if and only if f'_+ has bounded variation on $[a,b)$ (see Lemma 2.5 below), and thus f is $C^{1,\mathrm{BV}}$ if and only if f is C^1 and has bounded convexity.

If $X = \mathbb{R}$, we show (see Theorem 4.1 below) that f allows a $C^{1,\mathrm{BV}}$ parametrization if and only if f allows a parametrization with bounded convexity, if and only if f allows a C^2 parametrization. However, if $X = \mathbb{R}^2$, these parametrization problems are pairwise non-equivalent (see Example 6.1, 6.7 and 7.4).

Our characterizations are based on the notion of a turn of a curve and on a special type of 1/2-variation with a constraint (see Definition 3.5). The notion of a turn is a generalization of the classical notion of integral curvature (see Remark 2.8 for further information).

The structure of the present article is as follows. In Section 2, we introduce basic definitions and recall or prove the needed (essentially well-known) properties of curves with bounded convexity and curves with finite turn. In Section 3, we prove special lemmas needed in our main arguments. In short Section 4, we solve our parametrization problems in the easy case of real-valued functions. Section 5 contains the characterization of curves that allow parametrizations with finite convexity and Section 6 deals with curves allowing a parametrization in $C^{1,\mathrm{BV}}$. Finally, Section 7 contains examples that show applicability of our results.

2. Preliminaries

By λ we will denote the Lebesgue measure on $\mathbb{R}$. Throughout the whole article, X will always be a (real) Banach space. By $\mathcal{H}^1$ we will denote the 1-dimensional Hausdorff measure.

A mapping is *L-Lipschitz* provided it is Lipschitz with some constant L (not necessarily the minimal one). If $M \subset A \subset \mathbb{R}$ and $f\colon A \to X$ are given, then we define the *variation of f on M* as

$$V(f, M) := \sup \left\{ \sum_{i=1}^{n} \|f(x_i) - f(x_{i-1})\| \right\},$$

where the supremum is taken over all $(x_i)_{i=0}^n \subset M$ such that $x_0 < x_1 < \ldots < x_n$. (We set $V(f, M) := 0$, if M is empty or a singleton.) We say that $f\colon [a, b] \to X$ is *BV* (or *has bounded variation*), provided $V(f, [a, b]) < \infty$.

For basic well-known properties of variation, see, i.e., [11] and [3]. In particular, we will need the additivity of variation (see [3, (P3) on p. 263]):

$$V(f, M) = V(f, M \cap (-\infty, t]) + V(f, M \cap [t, \infty)), \quad \text{whenever } t \in M. \tag{2.1}$$

If $f\colon [a, b] \to X$ is BV, then we define $v_f(x) := V(f, [a, x])$, $x \in [a, b]$. If f is also continuous, then v_f is continuous as well ([11], [3]). Moreover, clearly v_f is (strictly) increasing, if and only if f is not constant on any subinterval of $[a, b]$. We say that $f\colon [a, b] \to X$ *is parametrized by the arc-length*, if $V(f, [u, v]) = v - u$ for every $a \leqslant u < v \leqslant b$. Obviously, each such f is 1-Lipschitz ([3, p. 267]).

Definition 2.1.

(a) Let $f\colon [a, b] \to X$ be a continuous mapping. We say that $f^*\colon [c, d] \to X$ is a parametrization of f if there exists an increasing homeomorphism $h\colon [c, d] \to [a, b]$ such that $f^* = f \circ h$. If f^* is moreover parametrized by the arc-length, we say that f^* is an *arc-length parametrization of f*.

(b) If $f\colon [a, b] \to X$ is nonconstant, continuous and BV, then there exists (see [11, §2.5.16] or [3, Theorem 3.1]) a unique $F\colon [0, l] \to X$ (where $l := v_f(b)$) such that $f = F \circ v_f$. We will denote this associated mapping F by $\mathcal{A}_f$.

Several times, we will apply the following easy lemma. For a proof, see [9, Lemma 2.2].

Lemma 2.2. *Let $f\colon [a,b] \to X$ be continuous. Then the following assertions hold.*

(i) *The function f has an arc-length parametrization if and only if f is BV and f is not constant on any $[u,v] \subset [a,b]$. In this case, $\mathcal{A}_f$ is an arc-length parametrization of f, $\mathcal{A}_f = f \circ (v_f)^{-1}$, and a general arc-length parametrization of f is of the form $F^s(x) = \mathcal{A}_f(x-s)$, $x \in [s, s+l]$, where $l := v_f(b)$ and $s \in \mathbb{R}$.*

(ii) *If f is BV on $[a,b]$, and it is not constant on any subinterval of an interval $[\alpha,\beta] \subset [a,b]$, then $\mathcal{A}_f|_{[v_f(\alpha),v_f(\beta)]} = f \circ (v_f|_{[\alpha,\beta]})^{-1}$ is an arc-length parametrization of $f|_{[\alpha,\beta]}$.*

Let $f\colon [a,b] \to X$. The derivative f' and the one-sided derivatives $f'_\pm$ are defined in the usual way; at the endpoints we take $f'(a) := f'_+(a)$, and $f'(b) := f'_-(b)$. We say that $f\colon [a,b] \to X$ is C^1 provided $f'(x)$ exists for all $x \in [a,b]$ and f' is continuous on $[a,b]$. We say that $f\colon [a,b] \to X$ is C^2 provided f' is C^1. We say that f is $C^{1,\mathrm{BV}}$ provided f is C^1, and f' is BV on $[a,b]$. Clearly, if f is C^2, then f is $C^{1,\mathrm{BV}}$.

It is well known (see i.e. [21, p. 2], or use [12, Theorem 7] together with [11, Theorem 2.10.13]) that if $f\colon [a,b] \to X$ is Lipschitz and $f'(x)$ exists for almost all $x \in [a,b]$, then

$$V(f,[a,b]) = \int_a^b \|f'(x)\| \,\mathrm{d}x. \tag{2.2}$$

For a proof of the following well-known version of Sard's Theorem, see e.g. [12, Theorem 7].

Lemma 2.3. *Let $f\colon [0,1] \to X$ be arbitrary. Let $C := \{x \in [0,1]\colon f'(x) = 0\}$. Then $\mathcal{H}^1(f(C)) = 0$.*

Let $I = [a,b]$ and let a continuous $f\colon I \to X$ be given. The *right and left unit tangent vector of f at $x \in I$* and the *unit tangent vector of f at $x \in I$* are defined, respectively, as the limits

$$\tau_+(f,x) = \lim_{t \to 0+} \frac{f(x+t)-f(x)}{\|f(x+t)-f(x)\|}, \qquad \tau_-(f,x) = \lim_{t \to 0-} -\frac{f(x+t)-f(x)}{\|f(x+t)-f(x)\|},$$
$$\tau(f,x) = \lim_{\substack{t \to 0 \\ x+t \in I}} \operatorname{sgn}(t) \cdot \frac{f(x+t)-f(x)}{\|f(x+t)-f(x)\|},$$

and f is said to be *tangentially smooth* if the function $\tau(f,x)$ is defined and continuous on I.

Clearly $\tau(f,x)$ exists if and only if $\tau_+(f,x) = \tau_-(f,x)$.

If $f'(x)$ or $f'_+(x)$ exists and is not equal to 0, then clearly $\tau(f,x) = f'(x)/\|f'(x)\|$ or $\tau_+(f,x) = f'_+(x)/\|f'_+(x)\|$, respectively. So, if $f\colon I \to X$ is C^1 and $f'(x) \neq 0,\ x \in I$, then f is tangentially smooth.

The notion of "convexity" goes back to de la Vallée Poussin (1908) and F. Riesz (1911); see [18, p. 28].

Definition 2.4. Let X be a Banach space and let $f\colon [a,b] \to X$ be given. For every partition $D = \{a = x_0 < x_1 < \ldots < x_n = b\}$ of $[a,b]$ we put

$$K(f,D) = \sum_{i=1}^{n-1} \Big\| \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} - \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \Big\|.$$

The *convexity of* f *on* $[a,b]$ we define as $K(f,[a,b]) = \sup K(f,D)$, where the supremum is taken over all partitions D of $[a,b]$ with $\#D \geqslant 3$. We say that f *has a bounded (or finite) convexity*, if $K(f,[a,b]) < \infty$.

Banach space-valued functions of bounded convexity were considered in [19], [20] and [4]; their properties are studied in detail in [21].

The following basic fact immediately follows from [21, Theorem 3.1, Proposition 3.3, and Proposition 3.4 (iv)].

Lemma 2.5. *Let X be a Banach space and let $f\colon [a,b] \to X$ be continuous. Then the following conditions are equivalent.*
(i) $K_a^b f < \infty$.
(ii) *$f'_+(x)$ exists for each $x \in [a,b)$ and $V(f'_+,[a,b)) < \infty$.*
(iii) *$f'_+(x)$ exists for each $x \in (a,b)$ and $V(f'_+,(a,b)) < \infty$.*
(iv) *The mapping $g(x) := f'_+(x),\ x \in [a,b),\ g(b) := f'_-(b)$ is well defined and $V_a^b g < \infty$.*

Moreover,

$$K_a^b f = V(f'_+,[a,b)) = V(f'_+,(a,b)) = V(g,[a,b]), \tag{2.3}$$

if one of the numbers is finite.

By Lemma 2.5,

(2.4) a function f is $C^{1,\mathrm{BV}}$ iff f is C^1 and has bounded convexity.

If f is C^2 on $[a,b]$, then (see [21, Theorem 3.8])

$$K(f,[a,b]) = \int_a^b \|f''(x)\|\,\mathrm{d}x. \tag{2.5}$$

Remark 2.6. Note also (see [4, Lemma 5.5] or [21, Remark 3.2]) that f is of bounded convexity if and only if f is a restriction of a mapping $g\colon (a-1, b+1) \to X$ that is delta-convex in the sense of [20], which holds if and only if f is d.c. with a Lipschitz control function (see [21, Theorem 3.1]). However, we will not use these facts.

We will need the following facts (see [21, Proposition A and Proposition 3.4]).

Lemma 2.7. *Let $f\colon [a,b] \to X$ have bounded convexity. Then f is Lipschitz and*

(i) *$f'_+(x)$ and $f'_-(x)$ exist at each point of $[a,b)$ and $(a,b]$, respectively;*
(ii) *$\lim\limits_{t\to x+} f'_\pm(t) = f'_+(x)$ for $x \in [a,b)$, and $\lim\limits_{t\to x-} f'_\pm(t) = f'_-(x)$ for $x \in (a,b]$;*
(iii) *the set $A := \{x \in (a,b)\colon f'_+(x) \neq f'_-(x)\}$ is countable and $\sum\limits_{x\in A} \|f'_+(x) - f'_-(x)\| < \infty$;*
(iv) *if $[c,d] \subset [a,b)$, then $V(f'_+, [c,d]) = K(f, [c,d]) + \|f'_+(d) - f'_-(d)\|$.*

Our characterization of curves which allow a $C^{1,\mathrm{BV}}$ parametrization or a parametrization with bounded convexity is based on the notion of the turn of a curve, which generalizes the classical notion of total (integral) curvature (see (2.10)) to non-smooth curves.

Remark 2.8. The "angular" turn of general curves was investigated and used, i.e., in [17] and [1] for curves in $\mathbb{R}^n$. We use the definition of turn which does not use the notion of an angle between vectors and so is defined in a general Banach space. Our ("nonangular") turn T does not equal the "angular" turn even in $\mathbb{R}^n$. However, the results from [4] imply that in case X is a Hilbert space, the "angular" turn T_a and the turn T which we use are equivalent in the sense that $T(f,I) \leqslant T_a(f,I) \leqslant (\pi/2)T(f,I)$ (see [4, Remark 1.1]). Thus, it is easy to see that, if X is a Hilbert space, we could also work with the "angular" turn. Indeed, our main results clearly hold also if the variation $W^\delta(f,G)$ and the notion of an (f,δ,K)-partition are defined using the "angular" turn.

Definition 2.9. Let $f\colon [a,b] \to X$ be continuous, and suppose that $g(x) := \tau_+(f,x)$ exists for all $x \in [a,b)$, and that $g(b) := \tau_-(f,b)$ also exists. Then we define the (*tangential*) *turn* $T(f,[a,b])$ *of* f *on* $[a,b]$ as $V(g,[a,b])$. We say that f *has finite turn on* $[a,b]$ provided $T(f,[a,b]) < \infty$. If $f\colon G \to X$, where $G \subset \mathbb{R}$ is open, then we say that f *has locally finite turn on* G provided $T(f,[c,d]) < \infty$ for each interval $[c,d] \subset G$.

We will need the following fact.

Lemma 2.10. *Let $f\colon [a,b] \to X$ be continuous, let $\tau_+(f,x)$ exist for each $x \in (a,b)$, and $V(\tau_+(f,\cdot),(a,b)) < \infty$. Then f has finite turn on $[a,b]$ and*

$$T(f,[a,b]) = V(\tau_+(f,\cdot),(a,b)) = V(\tau_+(f,\cdot),[a,b)). \tag{2.6}$$

Proof. The definition of variation and completeness of X easily imply (see [21, Lemma 2.6 and Remark 2.7]) that $\lim\limits_{x\to a+} \tau_+(f,x) =: w$ and $\lim\limits_{x\to b_-} \tau_+(f,x) =: z$ exist. By [4, Theorem 3.5], we obtain $\tau_+(f,a) = w$ and $\tau_-(b) = z$. Let g be as in Definition 2.9, and choose $c \in (a,b)$. Using [21, Lemma 2.6 and Remark 2.7]) and additivity of variation (2.1), we obtain

$$T(f,[a,b]) = V(g,[a,c]) + V(g,[c,b]) = V(g,(a,c]) + V(g,[c,b)) = V(\tau_+(f,\cdot),(a,b)).$$

By an obvious modification of the argument, also the second equality of (2.6) follows. □

Using Lemma 2.10, we easily obtain that if $T(f,[a,b]) < \infty$ for an $f\colon [a,b] \to X$, then

$$T(f,[c,d]) \leqslant T(f,[a,b]) \quad \text{whenever } [c,d] \subset [a,b]. \tag{2.7}$$

Suppose that $f\colon [a,b] \to X$ is continuous and h is an increasing homeomorphism of $[c,d]$ onto $[a,b]$. Since clearly $\tau_+(f \circ h, t) = \tau_+(f, h(t))$ (whenever $t \in [a,b)$ and one of the vectors is defined), (2.6) implies

$$T(f \circ h, [c,d]) = T(f,[a,b]), \tag{2.8}$$

if one side of this equality is defined. So, the notion of turn is a "geometrical one" unlike the notion of convexity. However, these notions are very closely connected:

Lemma 2.11 ([4, Proposition 5.7]). *Let X be a Banach space and let $F\colon [a,b] \to X$ be parametrized by the arc-length. Then F has finite turn if and only if F has bounded convexity. Moreover,*

$$T(F,[a,b]) = K(F,[a,b]) \quad \textit{if one of the numbers is finite.} \tag{2.9}$$

By (2.8), Lemma 2.11 and (2.5) we immediately obtain the following fact which shows that the turn is a generalization of the classical total (integral) curvature.

Lemma 2.12. *Let $f\colon [a,b] \to X$ be continuous and let $F\colon [c,d] \to X$ be an arc-length parametrization of f. If F is C^2 smooth on $[c,d]$, then*

$$T(f,[a,b]) = \int_c^d \|F''(x)\|\,\mathrm{d}x. \tag{2.10}$$

We will need also the following two easy lemmas on curves with finite turn.

Lemma 2.13. *Let $f\colon [a,b] \to X$ be a continuous function with finite turn on $[a,b]$. Then f is BV and is not constant on any interval. Let $f_1\colon [c,d] \to X$ be an arc-length parametrization of f, and let $\eta\colon [c,d] \to [a,b]$ be the increasing homeomorphism for which $f_1 = f \circ \eta$. Then:*

(i) *$(f_1)'_+(x)$ and $(f_1)'_-(x)$ exist for all $x \in [c,d)$ and $x \in (c,d]$, respectively, and $(f_1)'_+(x) = \tau_+(f,\eta(x))$ and $(f_1)'_-(x) = \tau_-(f,\eta(x))$ (and so $\|(f_1)'_+(x)\| = 1$ and $\|(f_1)'_-(x)\| = 1$) at all such points.*

(ii) *$K(f_1,[c,d]) = T(f,[a,b]) < \infty$.*

(iii) *$\tau_+(f,\cdot)$ is continuous from the right at all $x \in [a,b)$.*

(iv) *If $a < \xi < b$, then $T(f,[a,b]) = T(f,[a,\xi]) + T(f,[\xi,b]) + \|\tau_+(f,\xi) - \tau_-(f,\xi)\|$.*

(v) *If $\tau(f,x)$ exists for all $x \in [a,b]$, then f is tangentially smooth on $[a,b]$.*

Proof. The function f is BV by [4, Corollary 3.4 and Lemma 4.4 (ii)]. Clearly (by the definition of the turn and the definition of $\tau_+(f,x)$), the function f is not constant on any subinterval of $[a,b]$. Part (i) follows from [4, Lemma 4.5 (i)]. Since $T(f,[a,b]) = T(f_1,[c,d])$ by (2.8), part (ii) follows from (2.9). Part (iii) follows from (i), (ii), and Lemma 2.7 (ii). For part (iv), by (ii), (2.3), and (2.1), we have

$$\begin{aligned} T(f,[a,b]) &= K(f_1,[c,d]) = V((f_1)'_+,[c,d)) \\ &= V((f_1)'_+,[c,u]) + V((f_1)'_+,[u,d)), \end{aligned} \tag{2.11}$$

where $u := \eta^{-1}(\xi)$. By Lemma 2.7(iv), we obtain

$$V((f_1)'_+,[c,u]) = K(f_1,[c,u]) + \|(f_1)'_+(u) - (f_1)'_-(u)\|. \tag{2.12}$$

Using (ii) (on suitable intervals), we have that $K(f_1,[c,u]) = T(f,[a,\xi])$ and $V((f_1)'_+,[u,d)) = K(f_1,[u,d]) = T(f,[\xi,b])$. Since (i) implies $\tau_\pm(f,\xi) = (f_1)'_\pm(u)$, the conclusion follows by (2.11) and (2.12). Part (v) easily follows by (i), (ii) and Lemma 2.7(ii). □

Lemma 2.14. *Suppose that $f\colon [a,b]\to X$ is a continuous function with locally finite turn in (a,b). Let $\mu\in(a,b)$, $g(x):=T(f,[\mu,x])$ for $x\in(\mu,b)$ and $h(x):=T(f,[x,\mu])$ for $x\in(a,\mu)$. Then*

(i) *g is continuous from the left at each $x\in(\mu,b)$ and $g(\mu+)=0$,*

(ii) *$|g(x+)-g(x)|\leqslant 2$ for each $x\in(\mu,b)$,*

(iii) *h is continuous from the right at each $x\in(a,\mu)$, $h(\mu-)=0$, and*

(iv) *$|h(x-)-h(x)|\leqslant 2$ for each $x\in(a,\mu)$.*

P r o o f. All the statements easily follow from the corresponding assertions of [21, Proposition 3.7] on indefinite convexity via Lemma 2.13 (i), (ii). For example, to prove (ii), fix $x\in(\mu,b)$ and choose $\nu\in(x,b)$. Then $f^*:=f|_{[\mu,\nu]}$ has finite turn. Let $f_1\colon [c,d]\to X$ be an arc-length parametrization of f^*, and let $\eta\colon [c,d]\to[\mu,\nu]$ be the increasing homeomorphism for which $f_1=f^*\circ\eta$. Denote $y:=\eta^{-1}(x)$ and $p(t):=K_c^t f_1$, $t\in(c,d)$. We have $p(t)=g(\eta(t))$ by Lemma 2.13 (ii). So, since $|p(y+)-p(y)|=\|(f_1)'_+(y)-(f_1)'_-(y)\|$ by [21, Proposition 3.7 (ii)], we obtain by Lemma 2.13 (i)

$$|g(x+)-g(x)|=|p(y+)-p(y)|=\|(f_1)'_+(y)-(f_1)'_-(y)\|\leqslant 2.$$

□

Let now $f\colon [0,1]\to X$ with bounded convexity and $[a,b]\subset[0,1]$ be given. Then, only for the use in the present paper, we will define

$$K^*(f,[a,b]):=\|f'_-(a)-f'_+(a)\|+K(f,[a,b])+\|f'_-(b)-f'_+(b)\|,\tag{2.13}$$

where we put $f'_-(0):=f'_+(0)$ and $f'_+(1):=f'_-(1)$. Using (2.3), (2.1) and Lemma 2.7 (iii), it is easy to see that

$$K^*(f):=\sup\Big\{\sum_{I\in\mathcal{S}}K^*(f,I)\Big\}<\infty,\tag{2.14}$$

where the supremum is taken over all finite systems $\mathcal{S}$ of closed pairwise non-overlapping subintervals of $[0,1]$.

3. Lemmas

Lemma 3.1. *Let $f\colon [0,1]\to X$ have bounded convexity. Let $C := \{x\in[0,1]\colon f'_+(x)=0$ or $f'_-(x)=0\}$. Then $\mathcal{H}^1(f(C))=0$.*

Proof. Denote $C^* := \{x\in[0,1]\colon f'(x)=0\}$. Lemma 2.3 implies $\mathcal{H}^1(f(C^*)) = 0$. Since $C\setminus C^*$ is countable by Lemma 2.7 (i), (iii), the assertion follows. $\square$

The following lemma is proved in [10, Lemma 2.5].

Lemma 3.2. *Let $f\colon [a,b]\to X$ be continuous. Let $\emptyset\neq G\subset(a,b)$ be an open set, $H:=[a,b]\setminus G$ and let (a_t,b_t), $t\in T$, be all (pairwise different) components of G. Then:*

(i) *If $\mathcal{H}^1(f(H))=0$, then $V(f,[a,b])=\sum\limits_{t\in T}V(f,[a_t,b_t])$.*
(ii) *If $V(f,[a,b])=\sum\limits_{t\in T}V(f,[a_t,b_t])<\infty$, then $\mathcal{H}^1(f(H))=0$.*
(iii) *If $\mathcal{H}^1(f(H))=0$ and f is L-Lipschitz on each $[a_t,b_t]$, then f is L-Lipschitz on $[a,b]$.*
(iv) *If f is BV and $\mathcal{H}^1(f(H))=0$, then $\lambda(v_f(H))=0$.*

The following easy inequality is well known (see i.e. [16, Lemma 5.1]):

$$\text{if } u,v\in X\setminus\{0\},\ \text{then } \left\|\frac{u}{\|u\|}-\frac{v}{\|v\|}\right\|\leqslant\frac{2}{\|u\|}\,\|u-v\|. \tag{3.1}$$

Lemma 3.3. *Let $I=[a,b]\subset[0,1]$ and let $f\colon I\to X$ have bounded convexity on I. Define $i:=\inf\limits_{x\in[a,b)}\|f'_+(x)\|$, and $S:=\sup\limits_{x\in[a,b)}\|f'_+(x)\|$. Then*

(i) *$S-i\leqslant K(f,I)$,*
(ii) *$V(f,I)\leqslant S\cdot\lambda(I)$, and*
(iii) *if $i>0$, then $i\cdot T(f,I)\leqslant 2\cdot K(f,I)$.*

Proof. Recall that f is Lipschitz by Lemma 2.7.

To prove (i), observe that for each $\varepsilon>0$ we can choose $p,q\in I$ such that $S-i<\|f'_+(q)-f'_+(p)\|+\varepsilon$. Thus the conclusion follows by equality (2.3).

By (2.2) and Lemma 2.7 (iii) we obtain $V(f,I)=\int_a^b\|f'_+(t)\|\,\mathrm{d}t\leqslant S\cdot\lambda(I)$.

Finally, for part (iii), consider arbitrary points $x_0<\ldots<x_m$ in $[a,b)$. Applying (3.1) to $u=f'_+(x_{j+1})$ and $v=f'_+(x_j)$, we obtain $\|\tau_+(f,x_{j+1})-\tau_+(f,x_j)\|\leqslant 2i^{-1}\|f'_+(x_{j+1})-f'_+(x_j)\|$, so (2.6) and (2.3) imply (iii). $\square$

Our first basic lemma is the following ($K^*(f,I)$ is defined in (2.13)):

Lemma 3.4. *Let $f\colon [0,1]\to X$ have bounded convexity and let $I=[a,b]\subset[0,1]$.*

(i) *If* $\inf\{\|f'_+(x)\|\colon x \in I \setminus \{1\}\} = 0$ *or* $\inf\{\|f'_-(x)\|\colon x \in I \setminus \{0\}\} = 0$, *then we have* $\sqrt{V(f,I)} \leqslant K^*(f,I)/2 + \lambda(I)/2$.
(ii) *If* $0 < \delta \leqslant T(f,I) < \infty$, *then* $\sqrt{V(f,I)} \leqslant (2+\delta)/(2\delta)K^*(f,I) + \lambda(I)/2$.

Proof. Denote $K^* := K^*(f,I)$ and put $f'_-(0) := f'_+(0)$ and $f'_+(1) := f'_-(1)$. For part (i), let $\varepsilon > 0$ and $w \in I$ be such that $\|f'_+(w)\| < \varepsilon$ or $\|f'_-(w)\| < \varepsilon$. Distinguishing the cases $w = a$, $w = b$ and $w \in (a,b)$, by Lemma 2.7 (ii) we can clearly find a point $w^* \in [a,b)$ with $\|f'_+(w^*)\| < \varepsilon + \|f'_+(a) - f'_-(a)\| + \|f'_+(b) - f'_-(b)\|$. Thus, also using (2.3), we obtain $\|f'_+(x)\| \leqslant \|f'_+(x) - f'_+(w^*)\| + \|f'_+(w^*)\| \leqslant K^* + \varepsilon$ for each $x \in [a,b)$. Thus by (2.2) we have

$$\sqrt{V(f,I)} = \sqrt{\int_I \|f'_+(x)\|\,\mathrm{d}x} \leqslant \sqrt{\lambda(I)(K^*+\varepsilon)} \leqslant \frac{1}{2}(K^*(f,I) + \varepsilon + \lambda(I)).$$

Now we complete the proof by sending $\varepsilon \to 0$.

To prove part (ii), we can assume that $\inf\{\|f'_+(x)\|\colon x \in [a,b)\} =: i > 0$ (otherwise the conclusion follows by part (i)). Let $S := \sup\{\|f'_+(x)\|\colon x \in [a,b)\}$. By Lemma 3.3 (iii), it follows that $i \cdot \delta \leqslant i \cdot T(f,I) \leqslant 2 \cdot K(f,I)$, and by part (i) of the same lemma that $S - i \leqslant K(f,I)$. It follows that $S \leqslant S - i + i \leqslant (1 + 2/\delta)K(f,I)$. By Lemma 3.3 (ii) we have $V(f,I) \leqslant S \cdot \lambda(I)$. So, we obtain

$$\sqrt{V(f,I)} \leqslant \frac{1}{2}\Big(S + \frac{V(f,I)}{S}\Big) \leqslant \frac{1}{2}\Big(\Big(1 + \frac{2}{\delta}\Big)K(f,I) + \lambda(I)\Big)$$

and the assertion of (ii) follows. □

Now we define a special type of a "1/2-variation", which is crucial in our solution of the parametrization problems considered.

Definition 3.5. Let $f\colon [0,1] \to X$ be continuous and BV. Let $\emptyset \neq G \subset (0,1)$ be an open set and $0 < \delta < \infty$. If f has locally finite turn in G, then we define

$$W^\delta(f,G) = \sup\Big\{\sum_{k=1}^{n} \sqrt{V(f,I_k)}\Big\},$$

where the supremum is taken over all non-overlapping systems $I_1, \ldots, I_n$ of compact intervals with $\mathrm{int}(I_k) \subset G$ such that $T(f,I_k) \geqslant \delta$ whenever $I_k \subset G$.

Remark 3.6. Let f, G and $\delta > 0$ be as in Definition 3.5, and let $\omega\colon [0,1] \to [0,1]$ be an increasing homeomorphism. Then

$$W^\delta(f,G) = W^\delta(f \circ \omega, \omega^{-1}(G)).$$

This equality easily follows if we use (2.8) and observe that $v_{f\circ\omega} = v_f \circ \omega$.

Remark 3.7. Let f, G and $\delta > 0$ be as in Definition 3.5, and let $\mathcal{I}$ be the family of all components of G. Then, using only Definition 3.5, we clearly obtain

$$\sum_{I\in\mathcal{I}} \sqrt{V(f,I)} \leqslant W^{\delta}(f,G).$$

Lemma 3.8. *Let $f\colon [0,1] \to X$ have bounded convexity, let $\emptyset \neq G \subset (0,1)$ be an open set, and let f have locally finite turn on G. Let $f'_{+}(e) = 0$ or $f'_{-}(e) = 0$ whenever $e \in (0,1)$ is an endpoint of any component of G. Then $W^{\delta}(f,G) < \infty$ for each $\delta > 0$.*

Proof. Let $I_1,\ldots,I_n$ be a system of pairwise non-overlapping compact intervals with $\operatorname{int}(I_k) \subset G$ such that $T(f,I_k) \geqslant \delta$ whenever $I_k \subset G$. Consider an I_k with $I_k \cap \{0,1\} = \emptyset$. Observing that if $I_k \setminus G \neq \emptyset$, then f'_{+} or f'_{-} vanishes at an endpoint of I_k, by Lemma 3.4 we obtain that $\sqrt{V(f,I_k)} \leqslant (2+\delta)/(2\delta)K^*(f,I_k) + \lambda(I_k)/2$. So (2.14) implies $\sum\limits_{k=1}^{n} \sqrt{V(f,I_k)} \leqslant (2+\delta)/(2\delta)K^*(f) + 1/2 + 2\sqrt{V(f,[0,1])} < \infty$, and thus $W^{\delta}(f,G) < \infty$. □

Lemma 3.9. *Let a_i $(i \in I)$, b_j, c_j $(j \in J)$ be non-negative numbers, I countable, and J finite. Then*

$$\sqrt{\sum_{i\in I} a_i} \leqslant \sum_{i\in I} \sqrt{a_i} \quad \text{and} \quad \sum_{j\in J} \sqrt{b_j c_j} \leqslant \sqrt{\sum_{j\in J} b_j \cdot \sum_{j\in J} c_j}. \tag{3.2}$$

Proof. The first inequality is clear. The other is an immediate consequence of the Cauchy-Schwartz inequality. □

Lemma 3.10. *Let a continuous $f\colon [0,1] \to X$ have bounded variation and let it also have locally bounded turn in an open set $\emptyset \neq G \subset (0,1)$. Suppose that $\mathcal{S}$ is a family of pairwise non-overlapping compact intervals such that $\operatorname{int}(J) \subset G$ for each $J \in \mathcal{S}$ and*

$$\sum_{J\in\mathcal{S}} V(f,J) = V(f,[0,1]), \quad \sum_{J\in\mathcal{S}} \sqrt{V(f,J)} < \infty, \quad \sum_{J\in\mathcal{S}} \sqrt{V(f,J)\cdot T(f,J)} < \infty. \tag{3.3}$$

Then $W^{\delta}(f,G) < \infty$ for each $\delta > 0$.

(Note that (3.3) implies that f has finite turn on each $J \in \mathcal{S}$.)

Proof. Let $\delta > 0$ and consider a finite system $\mathcal{K}$ of non-overlapping compact intervals with $\operatorname{int}(I) \subset G$ $(I \in \mathcal{K})$ such that $T(f,I) \geqslant \delta$ whenever $I \in \mathcal{K}$ and $I \subset G$. For each $J \in \mathcal{S}$, let $\mathcal{K}_J := \{I \in \mathcal{K}\colon\ I \subset \operatorname{int}(J)\}$. Set $\mathcal{K}_1 := \bigcup\{\mathcal{K}_J\colon\ J \in \mathcal{S}\}$ and

$\mathcal{K}_2 := \mathcal{K} \setminus \mathcal{K}_1$. For each $J \in \mathcal{S}$, we obtain by the second inequality of (3.2) and Lemma 2.13 (iv) that

$$\sqrt{\delta} \sum \{\sqrt{V(f,I)} \colon I \in \mathcal{K}_J\} \leqslant \sum \{\sqrt{T(f,I) \cdot V(f,I)} \colon I \in \mathcal{K}_J\} \leqslant \sqrt{T(f,J) \cdot V(f,J)}.$$

Therefore,

$$\sum \{\sqrt{V(f,I)} \colon I \in \mathcal{K}_1\} \leqslant (1/\sqrt{\delta}) \sum \{\sqrt{T(f,J) \cdot V(f,J)} \colon J \in \mathcal{S}\}. \tag{3.4}$$

For each $I \in \mathcal{K}_2$, denote by $\mathcal{S}_I$ the set of all $J \in \mathcal{S}$ such that $J \cap \operatorname{int}(I) \neq \emptyset$. If $I = [a,b] \in \mathcal{K}_2$, put $a^* := \min J_a$, if there exists $J_a \in \mathcal{S}$ with $a \in \operatorname{int}(J_a)$, and $a^* := a$, if such J_a does not exist. Similarly, put $b^* := \max J_b$, if there exists $J_b \in \mathcal{S}$ with $b \in \operatorname{int}(J_b)$, and $b^* := b$, if such J_b does not exist. The equality of (3.3) easily implies that $V(f,[a^*,b^*]) = \sum \{V(f,J) \colon J \in \mathcal{S}_I\}$. (This can be proved either directly, or using first Lemma 3.2 (ii) and then Lemma 3.2 (i).) Thus the first inequality of (3.2) implies $\sqrt{V(f,[a^*,b^*])} \leqslant \sum \{\sqrt{V(f,J)} \colon J \in \mathcal{S}_I\}$. Observing that, for each $J \in \mathcal{S}$, the set $\{I \in \mathcal{K}_2 \colon J \in \mathcal{S}_I\}$ contains at most two intervals, we obtain

$$\sum \{\sqrt{V(f,I)} \colon I \in \mathcal{K}_2\} \leqslant 2 \sum \{\sqrt{V(f,J)} \colon J \in \mathcal{S}\}. \tag{3.5}$$

Now (3.3), (3.4) and (3.5) imply $W^\delta(f,G) < \infty$. □

Another important technical notion is the following.

Definition 3.11.

(i) We say that $\mathcal{I} \subset \mathbb{Z}$ is a $\mathbb{Z}$-interval, if $\mathcal{I} = (l,m) \cap \mathbb{Z}$, where $l,m \in \mathbb{Z} \cup \{-\infty,\infty\}$.

(ii) We will say that a family $\mathcal{P}$ of compact intervals is a *generalized partition of a bounded interval* (a,b), if there exists a system $(x_i)_{i \in \mathcal{I}}$ such that $\mathcal{I}$ is an $\mathbb{Z}$-interval, the function $i \mapsto x_i$, $i \in \mathcal{I}$, is strictly increasing, $\inf_{i \in \mathcal{I}} x_i = a$, $\sup_{i \in \mathcal{I}} x_i = b$ and $\mathcal{P} = \{[x_k, x_{k+1}] \colon k, k+1 \in \mathcal{I}\}$.

(iii) We will say that a family $\mathcal{P}$ of compact intervals is a *generalized partition of a bounded open set* $\emptyset \neq G \subset \mathbb{R}$, if $\bigcup \{\operatorname{int}(I) \colon I \in \mathcal{P}\} \subset G$, and for each component (a,b) of G, the family $\{I \in \mathcal{P} \colon \operatorname{int}(I) \subset (a,b)\}$ is a generalized partition of (a,b).

(iv) Suppose that $f \colon [0,1] \to X$ has locally finite turn in an open set $\emptyset \neq G \subset (0,1)$, $\mathcal{P}$ is a generalized partition of G, $0 \leqslant \delta \leqslant K \leqslant \infty$, and $\delta \in \mathbb{R}$. Then we say that $\mathcal{P}$ is an (f,δ,K)*-partition of* G, if $T(f,I) < \infty$, $T(f,I) \leqslant K$ for each $I \in \mathcal{P}$, and $T(f,I) \geqslant \delta$ for each $I \in \mathcal{P}$ with $I \subset G$.

Remark 3.12. Notice that in part (iv) of Definition 3.11, if $I = [c, d] \in \mathcal{P}$ but $c \notin G$ or $d \notin G$, then we *do not require* that $T(f, I) \geqslant \delta$.

The following lemma immediately follows from definitions.

Lemma 3.13. *Let $f\colon [0,1] \to X$ be continuous and BV, let $\emptyset \neq G \subset (0,1)$ be open, and let f have locally finite turn in G. Let $0 < \delta < \infty$ and let $\mathcal{P}$ be an (f, δ, ∞)-partition of G. Then $\sum\limits_{I \in \mathcal{P}} \sqrt{V(f, I)} \leqslant W^{\delta}(f, G)$.*

Lemma 3.14. *Let $\emptyset \neq G \subset (0,1)$ be open, and let $f\colon [0,1] \to X$ be a continuous BV function. Let f have locally finite turn on G and $0 < \delta < \infty$.*

Then there exists a generalized partition $\mathcal{P}$ of G which is an $(f, \delta, \delta + 3)$-partition of G.

P r o o f. Without any loss of generality, we can assume that $G = (a, b) \subset (0, 1)$. Let $x_0 := (a + b)/2$. We will construct points x_i $(i \in \mathbb{Z})$ with

$$a \leqslant \ldots \leqslant x_{-2} \leqslant x_{-1} < x_0 < x_1 \leqslant x_2 \leqslant x_3 \leqslant \ldots \leqslant b \tag{3.6}$$

such that $T(f, [x_i, x_{i+1}]) \leqslant \delta + 3$ if $x_i < x_{i+1}$, and, moreover, with

$$T(f, [x_i, x_{i+1}]) \geqslant \delta \quad \text{if } x_i < x_{i+1},\ x_i \neq a \text{ and } x_{i+1} \neq b. \tag{3.7}$$

First, we will construct points x_n $(n \in \mathbb{N})$ by induction. So suppose that $n \in \mathbb{N}$ and x_{n-1} was constructed. If $x_{n-1} = b$, then put $x_n := b$. If $x_{n-1} < b$ and $\sup\{T(f, [x_{n-1}, x])\colon x \in (x_{n-1}, b)\} \leqslant \delta + 3$, then also put $x_n := b$. Then clearly, $V(\tau_+(f, \cdot), (x_{n-1}, x_n)) \leqslant \delta + 3$ and thus $T(f, [x_{n-1}, x]) \leqslant \delta + 3$ by Lemma 2.10. If $x_{n-1} < b$ and $\sup\{T(f, [x_{n-1}, x])\colon x \in (x_{n-1}, b)\} > \delta + 3$, then set $x_n := \sup\{x \in (x_{n-1}, b)\colon T(f, [x_{n-1}, x]) \leqslant \delta + 3\}$. Using Lemma 2.14 (i) (applied to $\mu = x_{n-1}$), we obtain $x_n > x_{n-1}$. Further, using (2.7), we easily obtain $x_n < b$.

Lemma 2.14 (i) implies $T(f, [x_{n-1}, x_n]) \leqslant \delta + 3$. Moreover, $T(f, [x_{n-1}, x_n]) \geqslant \delta$. Indeed, otherwise Lemma 2.14 (ii) (with $\mu = x_{n-1}$, $x = x_n$) yields a contradiction with the definition of x_n.

We define the points x_{-n} $(n \in \mathbb{N})$ in a quite symmetrical way (using now Lemma 2.14 (iii), (iv)). Since f has locally finite turn in G, Lemma 2.13 (iv) and (3.7) easily imply that $\sup\limits_{n \in \mathbb{N}} x_n = b$ and $\inf\limits_{n \in \mathbb{N}} x_n = a$. So it is easy to check that $\mathcal{P} := \{[x_i, x_{i+1}]\colon i \in \mathbb{Z},\ x_i < x_{i+1}\}$ is a generalized partition of (a, b) with the desired properties. □

Lemma 3.15. *Let $f\colon [0,1] \to X$ be continuous and BV, and let $\emptyset \neq G \subset (0,1)$ be open. Suppose that f is nonconstant on each interval contained in G and $F := \mathcal{A}_f$ (see Definition 2.1) is C^2 on $v_f(G)$.*

If $W^\delta(f,G) < \infty$ for some $0 < \delta < \infty$, then $\int_{v_f(G)} \sqrt{\|F''\|} < \infty$.

Proof. By Lemma 2.12 we obtain, for each closed interval $I \subset G$,

$$T(f,I) = \int_{v_f(I)} \|F''\| < \infty. \tag{3.8}$$

Suppose that $W^\delta(f,G) < \infty$. Choose (by Lemma 3.14 (i)) an $(f,\delta,\delta+3)$-partition $\mathcal{P}$ of G. Lemma 3.13 implies $\sum\limits_{I\in\mathcal{P}} \sqrt{V(f,I)} < \infty$. By the Cauchy-Schwartz inequality and (3.8) we obtain, for each closed interval $I \subset G$,

$$\begin{aligned}\int_{v_f(I)} 1\cdot\sqrt{\|F''\|} &\leqslant \sqrt{\lambda(v_f(I))}\cdot\sqrt{\int_{v_f(I)}\|F''\|}\\ &= \sqrt{V(f,I)}\cdot\sqrt{T(f,I)} \leqslant \sqrt{V(f,I)}\cdot\sqrt{\delta+3}.\end{aligned}$$

Now it is easy to see that that the same inequalities hold for each $I \in \mathcal{P}$ (also for those with $I \setminus G \neq \emptyset$).

Consequently, $\int_{v_f(G)} \sqrt{\|F''\|} \leqslant \sqrt{\delta+3}\cdot \sum\limits_{I\in\mathcal{P}} \sqrt{V(f,I)} < \infty$. $\square$

In the basic constructions in Sections 5 and 6, we will need the following lemmas.

Lemma 3.16. *Let $(I_\alpha)_{\alpha\in A}$ be a system of pairwise non-overlapping compact subintervals of the interval $[0,d]$. Let $\sum\limits_{\alpha\in A} \mu_\alpha < \infty$, where $\mu_\alpha > 0$, $\alpha \in A$. Then there exists an interval $[0,d']$ and an increasing homeomorphism $\Psi\colon [0,d'] \to [0,d]$ such that $\lambda(\Psi^{-1}(I_\alpha)) = \mu_\alpha$ and Ψ^{-1} is absolutely continuous.*

Proof. We can define $\Psi := \omega^{-1}$, where $\omega(x) = \int_0^x \varphi$ $(x \in [0,d])$, and $\varphi(t) = \mu_\alpha/\lambda(I_\alpha)$ for $t \in \operatorname{int}(I_\alpha)$ and $\varphi(t) = 1$ for $t \in [0,d'] \setminus \bigcup\{\operatorname{int}(I_\alpha)\colon \alpha \in A\}$. $\square$

Lemma 3.17. *Let $\mathcal{P}$ be a generalized partition of an open set $\emptyset \neq G \subset (0,d)$. Let $g\colon [0,d] \to X$ be such that $V := \sum\limits_{I\in\mathcal{P}} V(g, I\cap G) < \infty$, $g(x) = 0$ for each $x \in F := [0,d]\setminus G$, and let g be continuous at all points of F. Then $V(g,[0,d]) < \infty$.*

Proof. Let $\mathcal{J}$ be the family of all components of G. Using the continuity of g, it is easy to show that, for each $J \in \mathcal{J}$,

$$V(g,\overline{J}) = \sum\{V(g,I)\colon I\in\mathcal{P}, I\subset\overline{J}\} = \sum\{V(g,I\cap G)\colon I\in\mathcal{P}, I\subset\overline{J}\}.$$

Now consider arbitrary points $0 = x_0 < x_1 < \ldots < x_m = d$. It is easy to see that we can choose points $0 = y_0 < y_1 < \ldots < y_n = d$ such that $\{x_0, \ldots, x_m\} \subset \{y_0, \ldots, y_n\}$ and, for each $0 \leqslant k < n$, either $\{y_k, y_{k+1}\} \subset F$ or $(y_k, y_{k+1}) \subset G$. Then clearly

$$\sum_{i=0}^{m-1} \|g(x_{i+1}) - g(x_i)\| \leqslant \sum_{k=0}^{n-1} \|g(y_{k+1}) - g(y_k)\| \leqslant \sum\{V(g, \overline{J})\colon\ J \in \mathcal{J}\} = V.$$

□

Lemma 3.18. *Let G_1, G_2 be bounded open subsets of $\mathbb{R}$ and let $\varphi\colon G_2 \to G_1$ be an increasing differentiable homeomorphism. Let $Z \subset G_2$ be an interval (of an arbitrary type) such that $\varphi'(z) \in \mathbb{R}$ exists for each $z \in Z$ and $V(\varphi', Z) < \infty$. Let $h\colon G_1 \to X$ be such that $h'_+(x)$ exists for each $x \in \varphi(Z)$ and $V(h'_+, \varphi(Z)) < \infty$. Then*

$$V((h \circ \varphi)'_+, Z) \leqslant \sup_{t \in Z} |\varphi'(t)| \cdot V(h'_+, \varphi(Z)) + \sup_{x \in \varphi(Z)} \|h'_+(x)\| \cdot V(\varphi', Z). \tag{3.9}$$

Proof. Let $t_0 < t_1 < \ldots < t_n$ be arbitrary points in Z. Observe that $(h \circ \varphi)'_+(t) = \varphi'(t)\, h'_+(\varphi(t))$ for each $t \in Z$, and thus

$$\begin{aligned} &\|(h \circ \varphi)'_+(t_{i+1}) - (h \circ \varphi)'_+(t_i)\| \\ &\leqslant \|\varphi'(t_{i+1}) h'_+(\varphi(t_{i+1})) - \varphi'(t_i) h'_+(\varphi(t_{i+1}))\| + \|\varphi'(t_i)\, h'_+(\varphi(t_{i+1})) - \varphi'(t_i)\, h'_+(\varphi(t_i))\| \\ &\leqslant \sup_{x \in \varphi(Z)} \|h'_+(x)\| \cdot |\varphi'(t_{i+1}) - \varphi'(t_i)| + \sup_{t \in Z} |\varphi'(t)| \cdot \|h'_+(\varphi(t_{i+1})) - h'_+(\varphi(t_i))\|. \end{aligned}$$

Now the assertion of the lemma follows easily. □

4. The case of real valued functions

As mentioned in Introduction, functions $f\colon\ [0,1] \to \mathbb{R}$ which alow a C^2 parametrization were completely characterized in [14] and [15]. We will show that the same characterization holds if it is required that the parametrization has bounded convexity or a continuous BV derivative. We prove this result easily by using results from [14] and [15]; however, we could obtain this result also from our theorems on vector functions which we prove without using results of [14] and [15].

Lebedev showed that for a continuous $f\colon\ [0,1] \to \mathbb{R}$ there exists a homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ is a C^n function ($n \in \mathbb{N}$) if and only if

$$\lambda(f(M_f)) = 0 \quad \text{and} \quad \sum_{\alpha \in A} (\omega^f_\alpha)^{1/n} < \infty, \tag{4.1}$$

where $I_\alpha (\alpha \in A)$ are all maximal open intervals in $[0,1]$ on which f is constant or strictly monotone, $M_f := [0,1] \setminus \bigcup\limits_{\alpha \in A} I_\alpha$ is the set of points "of varying monotonicity" of f, and ω_α^f is the oscillation of f on I_α.

Laczkovich and Preiss showed that the same (for a continuous f) holds if and only if

$$V_{1/n}(f, M_f) < \infty, \tag{4.2}$$

where

$$V_{1/n}(f, M_f) := \sup\left\{ \sum_{i=1}^{m} |f(d_i) - f(c_i)|^{1/n} \right\},$$

the supremum being taken over all systems $[c_i, d_i]$, $i = 1, \ldots, m$, of pairwise non-overlapping subintervals of $[0,1]$ with $c_i, d_i \in M_f$. Note that M_f is called K_f in [14].

Theorem 4.1. *Let $f\colon [0,1] \to \mathbb{R}$ be continuous. Then the following conditions are equivalent.*

(i) *There exists a homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ has bounded convexity.*
(ii) *There exists a homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ is $C^{1,\mathrm{BV}}$.*
(iii) *There exists a homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ is C^2.*
(iv) *Lebedev condition (4.1) holds for $n = 2$.*
(v) *Laczkovich-Preiss condition (4.2) holds for $n = 2$.*

Proof. By results of [14] and [15] which were mentioned at the beginning of this section, the conditions (iii), (iv) and (v) are equivalent. Since (iii) implies (ii) and (ii) implies (i) (see (2.4)), it is sufficient to prove that (i) implies (v). Thus suppose that h as in (i) is given and denote $g := f \circ h$. Let c_i, d_i be as in the definition of $V_{1/2}(g, M_g)$. We claim that, for each $x \in M_g \cap [0,1)$, we have

$$\text{either} \quad g'_-(x) \leqslant 0 \leqslant g'_+(x) \quad \text{or} \quad g'_+(x) \leqslant 0 \leqslant g'_-(x) \quad (\text{where } g'_-(0) := 0). \tag{4.3}$$

Indeed, otherwise $\min(g'_+(x), g'_-(x)) > 0$ or $\max(g'_+(x), g'_-(x)) < 0$. Provided $\min(g'_+(x), g'_-(x)) > 0$, then by Lemma 2.7 (ii) there exists $\delta > 0$ such that $g'_+(y) > 0$ for all $y \in (x - \delta, x + \delta)$. Since g is Lipschitz by Lemma 2.7, we have $g(t) - g(s) = \int_s^t g'_+(\xi)\,\mathrm{d}\xi > 0$ for all $x - \delta < s < t < x + \delta$, which contradicts $x \in M_g$. If $\max(g'_+(x), g'_-(x)) < 0$, then we obtain a contradiction in an analogous way.

For each $y \in (c_i, d_i)$, (4.3) easily implies

$$|g'_+(y)| \leqslant |g'_+(c_i) - g'_-(c_i)| + V(g'_+, [c_i, d_i)) =: S_i.$$

Therefore, we subsequently obtain

$$|g(d_i) - g(c_i)| \leqslant S_i\,|d_i - c_i|, \quad |g(d_i) - g(c_i)|^{1/2} \leqslant \sqrt{S_i|d_i - c_i|} \leqslant \frac{S_i + (d_i - c_i)}{2},$$

$$\sum_{i=1}^{m} |g(d_i) - g(c_i)|^{1/2} \leqslant \frac{1}{2}\sum_{i=1}^{m}(S_i + (d_i - c_i)) \leqslant C,$$

where C is a constant depending only on g (see (2.14) and (2.3)). Consequently $V_{1/2}(g, M_g) < \infty$, and so also $V_{1/2}(f, M_f) < \infty$, since clearly $M_g = h^{-1}(M_f)$. □

5. Parametrizations with bounded convexity

The problem of a parametrization with bounded convexity and non-zero unilateral derivatives has a very simple solution.

Proposition 5.1. *Let X be a Banach space, and let $f\colon [0,1] \to X$ be continuous. Then the following conditions are equivalent:*

(i) *There exists a homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ has bounded convexity, $(f \circ h)'_+(x) \neq 0$ for $x \in [0,1)$, and $(f \circ h)'_-(x) \neq 0$ for $x \in (0,1]$.*

(ii) *f has finite turn.*

P r o o f. Suppose that (i) holds. Then [4, Proposition 5.11] and (2.8) easily imply that f has finite turn. (Alternatively, we can apply Lemma 3.3 (iii) to $f \circ h$, since Lemma 2.7 (ii) easily implies that $\inf\limits_{x\in[0,1)} \|(f \circ h)'_+(x)\| > 0$.)

Suppose that (ii) holds. Then Lemma 2.13 (ii), (i) imply that $F = f \circ v_f^{-1}$ has bounded convexity and $F'_\pm(x) = \tau_\pm(F, x) \neq 0$, respectively, for all $x \in [0,l)$ or $x \in (0,l]$, where $l = v_f(1)$. Thus we can put $h(t) := v_f^{-1}(l \cdot t)$ for $t \in [0,1]$. □

Lemma 5.2. *Suppose that a continuous $f\colon [0,1] \to X$ has bounded variation, $\emptyset \neq G \subset (0,1)$ is an open set and f has locally finite turn in G. Suppose that $\mathcal{P}$ is an $(f, 0, K)$-partition of G for some $0 < K < \infty$ such that $\sum\limits_{I\in\mathcal{P}} \sqrt{V(f,I)} < \infty$ and $\mathcal{H}^1(f(H)) = 0$, where $H := [0,1] \setminus G$. Then there exists an increasing homeomorphism $h\colon [0,1] \to [0,1]$ such that:*

(i) *$f \circ h$ has bounded convexity and $(f \circ h)'_+(x) \neq 0$, $(f \circ h)'_-(x) \neq 0$ for each $x \in h^{-1}(G)$;*

(ii) *if f is tangentially smooth on G, then $f \circ h$ is C^1;*

(iii) *if f is nonconstant on any interval, then $\lambda(h^{-1}(H)) = 0$.*

P r o o f. Let U be the maximal open set on which f is locally constant. Set $v^*(x) := v_f(x) + \lambda([0,x] \cap U)$, $x \in [0,1]$. It is clear that v^* is continuous and increasing. Put $d_1 := v^*(1)$ and $\xi := (v^*)^{-1}$; clearly $\xi\colon [0,d_1] \to [0,1]$ is an increasing homeomorphism. Denote $f_1 := f \circ \xi$, $G_1 := \xi^{-1}(G)$ and $\mathcal{P}_1 := \{\xi^{-1}(I)\colon I \in \mathcal{P}\}$. Clearly $\sum\{\sqrt{\lambda(J)}\colon J \in \mathcal{P}_1\} = \sum\limits_{I \in \mathcal{P}} \sqrt{V(f,I)} < \infty$. By (2.8) we have that $T(f_1,J) \leqslant K$ for each $J \in \mathcal{P}_1$.

If the interval (γ,δ) is a component of G_1, then $f_1|_{[\gamma,\delta]}$ is clearly an arc-length parametrization of $f|_{[\xi(\gamma),\xi(\delta)]}$. So Lemma 2.13 (i) implies that

$$\|(f_1)'_+(x)\| = \|(f_1)'_-(x)\| = 1 \quad \text{whenever } x \in G_1, \text{ and} \tag{5.1}$$

$$f_1 \text{ is } C^1 \text{ on } G_1 \text{ if } f \text{ is tangentially smooth on } G. \tag{5.2}$$

Using (5.1), Lemma 2.13 (ii) and Lemma 2.7 (iv), we easily get

$$V((f_1)'_+, J \cap G_1) \leqslant K + 2 \quad \text{whenever } J \in \mathcal{P}_1. \tag{5.3}$$

If $J = [\alpha,\beta] \in \mathcal{P}_1$, set $c(J) := \sqrt{\lambda(J)}$. Further, if $\alpha \notin G$, let $b(J) := 0$, and if $\alpha \in G$, set $b(J) := \min(c(J), c(J_l))$, where $J_l \in \mathcal{P}_1$ is the interval whose right endpoint is α. Similarly, if $\beta \notin G$, let $d(J) := 0$, and if $\beta \in G$, set $d(J) := \min(c(J), c(J_r))$, where $J_r \in \mathcal{P}_1$ is the interval whose left endpoint is β. Finally, set $\mu(J) := 6\lambda(J)(b(J) + 4c(J) + d(J))^{-1}$.

Since $\mu(J) \leqslant 2\sqrt{\lambda(J)}$, we can use Lemma 3.16 and choose $0 < d_2 < \infty$ and an increasing homeomorphism $\Psi\colon [0,d_2] \to [0,d_1]$ such that Ψ^{-1} is absolutely continuous and $\lambda(\Psi^{-1}(J)) = \mu(J)$ for each $J \in \mathcal{P}_1$. Set $\mathcal{P}_2 := \{\Psi^{-1}(J)\colon J \in \mathcal{P}_1\}$, $G_2 := \Psi^{-1}(G_1)$ and $H_2 := [0,d_2] \setminus G_2$.

For each interval $I = [p,q] \in \mathcal{P}_2$, let $J := \Psi(I)$ and let g_I be the continuous function on I such that $g_I(p) = b(J)$, $g_I(q) = d(J)$, $g_I(x) = c(J)$ for each $x \in [p + (q-p)/3, p + 2(q-p)/3]$, and g_I is linear on the intervals $[p, p + (q-p)/3]$, $[p + 2(q-p)/3, q]$. Further put $\varphi_I(x) := \Psi(p) + \int_p^x g_I$ for $x \in I$. We see that

$$\varphi_I(q) - \varphi_I(p) = (q-p) \cdot (b(J) + 4c(J) + d(J))/6 = \lambda(J).$$

Thus, setting $\varphi(x) := \varphi_I(x)$ if $x \in I \in \mathcal{P}_2$ and $\varphi(x) := \Psi(x)$ if $x \in [0,d_2] \setminus \bigcup \mathcal{P}_2$, we easily see that $\varphi\colon [0,d_2] \to [0,d_1]$ is an increasing homeomorphism which is C^1 with $\varphi' > 0$ on G_2 and $\varphi'(x) \leqslant c(J)$ whenever $x \in I \cap G_2$, where $I \in \mathcal{P}_2$ and $J = \varphi(I)$.

Put $f_2 := f_1 \circ \varphi$. Consider $I \in \mathcal{P}_2$ and $J = \varphi(I) \in \mathcal{P}_1$. Using (5.1), we obtain that

$$\|(f_2)'_-(x)\| = \|(f_2)'_+(x)\| = |\varphi'(x)| = |g_I(x)| \leqslant c(J) \quad \text{if } x \in I \cap G_2. \tag{5.4}$$

Using (5.3), (5.1), and the obvious inequality $V(\varphi', I \cap G_2) = V(g_I, I \cap G_2) \leqslant 2c(J)$, we apply Lemma 3.18 (with $Z := I \cap G_2$ and $h := f_1$) and obtain that

$$V((f_2)'_+, I \cap G_2) \leqslant (K+2)c(J) + 2c(J),$$

and therefore

$$\sum_{I \in \mathcal{P}_2} V((f_2)'_+, I \cap G_2) < \infty. \tag{5.5}$$

Now we will show that (setting $(f_2)'_+(d_2) := 0$)

$$\text{for each } u \in H_2,\ (f_2)'(u) = 0 \text{ and } (f_2)'_+ \text{ is continuous at } u. \tag{5.6}$$

Let $u \in H_2$ and $\varepsilon > 0$ be given. If u is a left endpoint of some $I \in \mathcal{P}_2$, observe that $g_I(u) = 0$ and g_I is right continuous at u. Thus (5.4) and [2, Chap. I, par. 2, Proposition 3] clearly imply that there exists $v > u$ such that f_2 is ε-Lipschitz on $[u, v]$.

If $u \neq d_2$ and u is not a left endpoint of an $I \in \mathcal{P}_2$, then observe that $c(J) > \varepsilon$ for finitely many $J \in \mathcal{P}_1$ and therefore we can by (5.4) find a $v > u$, $v \in H_2$, such that $\|(f_2)'_+(x)\| \leqslant \varepsilon$ for each $x \in G_2 \cap [u, v]$. So [2, Chap. I, par. 2, Proposition 3] implies that f_2 is ε-Lipschitz on each component of $G_2 \cap (u, v)$. Since clearly $\mathcal{H}^1(f_2(H_2)) = \mathcal{H}^1(f(H)) = 0$, Lemma 3.2 (iii) implies that f_2 is ε-Lipschitz on $[u, v]$.

Quite similarly we get for each $0 \neq u \in H_2$ a $v < u$ such that f_2 is ε-Lipschitz on $[v, u]$. Now it is easy to see that (5.6) holds. Using (5.5) and (5.6), we apply Lemma 3.17 (with $G := G_2$ and $g := (f_2)'_+$, $g(d_2) = 0$) and obtain, also using (2.3), that g has bounded convexity on $[0, d_2]$.

By (5.4) and the definition of g_I we have that $(f_2)'_+(x) \neq 0$ and $(f_2)'_-(x) \neq 0$ for each $x \in G_2$. Moreover, if f is tangentially smooth on G, then (5.2) implies that f_2 is C^1 on G_2. Using (5.6), we obtain that f_2 is C^1 on $[0, d_2]$.

Thus, to complete the proof of (i) and (ii), it is sufficient to define $h := \xi \circ \varphi \circ \pi$, where $\pi(x) = d_2 x$, $x \in [0,1]$. To prove (iii), suppose that f is nonconstant on any interval. Then $v^* = v_f = \xi^{-1}$; so Lemma 3.2 (iv) implies $\lambda(\xi^{-1}(H)) = 0$. Since Ψ^{-1} is absolutely continuous and $\varphi^{-1}(\xi^{-1}(H)) = \Psi^{-1}(\xi^{-1}(H))$, we have $\lambda((\xi \circ \varphi)^{-1}(H)) = 0$, and thus also $\lambda(h^{-1}(H)) = 0$. $\square$

Let $f\colon [0,1] \to X$. We define the set T_f as the set of all points in $[0,1]$ such that there is no open interval U containing x such that f has finite turn on $\overline{U}$. Clearly, T_f is closed, $\{0,1\} \subset T_f$, and f is not constant on any interval $I \subset [0,1] \setminus T_f$.

Note that $G := [0,1] \setminus T_f$ is the maximal open set in which f has locally finite turn.

Further, if h is a homeomorphism of $[0,1]$ onto itself, then (2.8) implies

$$T_{f\circ h} = h^{-1}(T_f). \tag{5.7}$$

We need the following simple lemma.

Lemma 5.3. *Suppose that $f\colon [0,1] \to X$ has bounded convexity, and $x \in T_f$. Then we have that either $x \in \{0,1\}$ or $f'_+(x) = 0$ or $f'_-(x) = 0$.*

Proof. Let $x \in (0,1) \cap T_f$ with $f'_+(x) \neq 0$ and $f'_-(x) \neq 0$. By Lemma 2.7 (ii) there exist $\delta > 0, \eta > 0$ such that $\|f'_+(x)\| \geqslant \eta$ for all $y \in [x-\delta, x+\delta]$. Thus Lemma 3.3 (iii) implies that

$$T(f,[x-\delta,x+\delta]) \leqslant \frac{2}{\eta} K(f,[x-\delta,x+\delta]) < \infty,$$

and we have a contradiction with $x \in T_f$. □

The main result of the present section is the following theorem which solves the bounded convexity parametrization problem. (Observe that condition (i) is clearly equivalent to the existence of a parametrization of f with bounded convexity, and implies that f is BV.)

Theorem 5.4. *Let $f\colon [0,1] \to X$ be BV continuous nonconstant and $G := [0,1] \setminus T_f$. Then the following assertions are equivalent.*

(i) *There exists a homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ has bounded convexity.*
(ii) *There exists a homeomorphism φ of $[0,1]$ onto itself such that $f\circ\varphi$ has bounded convexity, and $(f\circ\varphi)'_\pm(x) \neq 0$ for each $x \in \varphi^{-1}(G)$.*
(iii) *$\mathcal{H}^1(f(T_f)) = 0$ and $W^\delta(f,G) < \infty$ for each $\delta > 0$.*
(iv) *$\mathcal{H}^1(f(T_f)) = 0$ and $W^\delta(f,G) < \infty$ for some $\delta > 0$.*
(v) *$\mathcal{H}^1(f(T_f)) = 0$ and $\sum_{I\in\mathcal{P}} \sqrt{V(f,I)} < \infty$ whenever $\mathcal{P}$ is an (f,δ,∞)-partition of G with $\delta > 0$.*
(vi) *$\mathcal{H}^1(f(T_f)) = 0$ and $\sum_{I\in\mathcal{P}^*} \sqrt{V(f,I)} < \infty$ for some $(f,0,K^*)$-partition $\mathcal{P}^*$ of G with $K^* < \infty$.*
(vii) *There exists a family $\mathcal{S}$ of pairwise non-overlapping compact intervals such that $\operatorname{int}(J) \subset G$ for each $J \in \mathcal{S}$ and*

$$\sum_{J\in\mathcal{S}} V(f,J) = V(f,[0,1]),\ \sum_{J\in\mathcal{S}} \sqrt{V(f,J)} < \infty,\ \sum_{J\in\mathcal{S}} \sqrt{V(f,J)\cdot T(f,J)} < \infty. \tag{5.8}$$

If f is nonconstant on any interval, then in (ii) *we can also assert $\lambda(\varphi^{-1}(G)) = 1$.*

P r o o f. It is sufficient to prove the implications (ii) $\Longrightarrow$ (i) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (vi) $\Longrightarrow$ (vii) $\Longrightarrow$ (iii), (vi) $\Longrightarrow$ (ii), and (iii) $\Longrightarrow$ (v) $\Longrightarrow$ (vi).

Note that (since f is nonconstant) both $\mathcal{H}^1(f(T_f)) = 0$ and the equality of (5.8) imply $G \neq \emptyset$ by Lemma 3.2 (i). The implications (ii) $\Longrightarrow$ (i) and (iii) $\Longrightarrow$ (iv) are trivial; (v) $\Longrightarrow$ (vi) holds on account of Lemma 3.14. Lemma 3.13 implies the obvious implication (iii) $\Longrightarrow$ (v), and (vii) $\Longrightarrow$ (iii) holds by Lemma 3.2 (ii) and Lemma 3.10. If (vi) holds, then Lemma 5.2 gives (ii) (and also $\lambda(\varphi^{-1}(G)) = 1$ if f is nonconstant on any interval).

To show that (iv) $\Longrightarrow$ (vi), let $\delta > 0$ be as in (iv), and put $K^* := \delta + 3$. Lemma 3.14 (i) implies that there exists an (f, δ, K^*)-partition $\mathcal{P}^*$ of G and we have that $\sum\limits_{I \in \mathcal{P}^*} \sqrt{V(f, I)} < \infty$ by Lemma 3.13.

To prove (vi) $\Longrightarrow$ (vii) suppose that (vi) holds and $\mathcal{P}^*$ is given. Put $\mathcal{S} := \mathcal{P}^*$. Then the equality of (5.8) holds by Lemma 3.2 (i) (applied to $G := \bigcup\limits_{J \in \mathcal{S}} \operatorname{int}(J)$). Since $\sqrt{T(f, J)} \leqslant \sqrt{K^*}$ for each $J \in \mathcal{S}$, also both inequalities of (5.8) hold.

To prove that (i) $\Longrightarrow$ (iii), let h be as in (i), put $g := f \circ h$ and $G^* := h^{-1}(G) = [0, 1] \setminus T_g$. Using (5.7), Lemma 5.3 and Lemma 3.1, we easily obtain $\mathcal{H}^1(f(T_f)) = \mathcal{H}^1(g(T_g)) = 0$. By (5.7), Lemma 5.3, and Lemma 3.8 we obtain $W^\delta(g, G^*) < \infty$; so $W^\delta(f, G) < \infty$ by Remark 3.6. $\square$

Remark 5.5.

(i) Conditions (v) and (vi) give an "algorithmic" way how to decide whether (i) holds:

Decide whether $\mathcal{H}^1(f(T_f)) = 0$. If it holds, then choose an (f, δ, K)-partition $\mathcal{P}$ of $G := [0, 1] \setminus T_f$ with $\delta > 0$ and $K < \infty$ (such a partition exists by Lemma 3.14) and decide whether $\sum\limits_{I \in \mathcal{P}} \sqrt{V(f, I)} < \infty$.

(ii) Condition (vii) is elegant and needs no auxiliary notions for its formulation. On the other hand, it is not easily applicable in the case when (i) does not hold.

(iii) For a very simple necessary and sufficient condition in an interesting, very special case see Proposition 6.10.

Remark 5.6. Let $f\colon [0, 1] \to X$ be BV continuous and $G := [0, 1] \setminus T_f$. Then the following assertions are equivalent.

(a) There exists a homeomorphism h of $[0, 1]$ onto itself such that $f \circ h$ has bounded convexity and $(f \circ h)'(x) \neq 0$ almost everywhere.

(b) f is nonconstant on any interval and any of conditions (iii)–(vii) holds.

It follows immediately from Theorem 5.4 and the simple observation that (a) implies that f is nonconstant on any interval.

Proposition 5.7. *Let a nonconstant function $f\colon [0,1] \to X$ allow an equivalent parametrization with bounded convexity. Set $G := [0,1] \setminus T_f$. Suppose that $F := \mathcal{A}_f$ (see Definition 2.1) is C^2 on $v_f(G)$. Then $\int_{v_f(G)} \sqrt{\|F''\|} < \infty$.*

P r o o f. For each $\delta > 0$, we have $W^\delta(f,G) < \infty$ by Theorem 5.4. Thus the assertion follows from Lemma 3.15. □

Concerning natural questions about the strength of the condition from the preceding proposition, see Example 7.3, and Proposition 6.10.

6. $C^{1,\mathrm{BV}}$-PARAMETRIZATIONS

By Theorem 4.1, a real function $f\colon [0,1] \to \mathbb{R}$ allows a $C^{1,\mathrm{BV}}$-parametrization if and only it allows a parametrization with bounded convexity. The following elementary example shows that it is not true for vector-valued functions.

Example 6.1. There exists a function $f\colon [0,1] \to \mathbb{R}^2$ with bounded convexity which does not admit a C^1 parametrization.

P r o o f. Take $\{a_n\colon n \in \mathbb{N}\}$ to be a dense subset of $(0,1)$, and $g\colon [0,1] \to \mathbb{R}$ a convex Lipschitz function with $g'_+(a_n) \neq g'_-(a_n)$ for all n. Let $f(x) = (x, g(x))$, $x \in [0,1]$. Using Lemma 2.5, it is easy to show that f has bounded convexity. Suppose that f admits a C^1 parametrization. Then there exists an increasing homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ is C^1 on $[0,1]$. Then h is C^1 on $[0,1]$ (since it is the first component of $f \circ h$) and thus there exists an open interval $I \subset [0,1]$ such that $h' > 0$ on I and thus h^{-1} is differentiable on $h(I)$. Choose $j \in \mathbb{N}$ with $a_j \in h(I)$. Since $g = (g \circ h) \circ h^{-1}$, we obtain that g is differentiable at a_j, a contradiction. □

Further, recall (see (2.4)) that $C^{1,\mathrm{BV}}$ functions coincide with C^1 functions with bounded convexity. So, it is not surprising that the characterization in the case of $C^{1,\mathrm{BV}}$ parametrizations is very similar to the characterization in the case of parametrizations with bounded convexity (and the proofs of these characterizations are almost identical). On the other hand, it seems that the results on $C^{1,\mathrm{BV}}$ parametrizations cannnot be easily deduced from the results on parametrizations with bounded convexity (see Example 6.7).

Now we show that the problem of a $C^{1,\mathrm{BV}}$ parametrization with non-zero derivative has a very simple solution.

Proposition 6.2. *Let X be a Banach space, and let $f\colon [0,1]\to X$ be continuous. Then the following conditions are equivalent:*

(i) *There exists a homeomorphism h of $[0,1]$ onto itself such that $f\circ h$ is $C^{1,\mathrm{BV}}$ and $(f\circ h)'(x)\neq 0$ for all $x\in[0,1]$.*
(ii) *f has finite turn and $\tau(f,x)$ exists for all $x\in[0,1]$.*
(iii) *f has finite turn and is tangentially smooth.*

Proof. Suppose that (i) holds. By Proposition 5.1, f has finite turn. Since $(f\circ h)'(x)\neq 0$, we have that $\tau(f\circ h,x)$ exists for each $x\in[0,1]$, and thus $\tau(f,y)$ exists for each $y\in[0,1]$. So (ii) holds.

Now suppose that (ii) holds. Then Lemma 2.13 (ii) implies that $F:=f\circ v_f^{-1}$ has bounded convexity, and Lemma 2.13 (i) implies that $F'(x)=\tau(F,x)\neq 0$ at all points $x\in[0,l]$ (where $l=v_f(1)$); F is C^1 by Lemma 2.7 (ii). So, setting $h(t):=v_f^{-1}(l\cdot t)$, $t\in[0,1]$, we obtain (i).

Since (ii) is equivalent to (iii) by Lemma 2.13 (v), the proof is complete. □

For $f\colon [0,1]\to X$ we define S_f as the set of all points $x\in[0,1]$ such that there is no neighbourhood U of x such that f is either constant or tangentially smooth on U. Clearly S_f is closed and $\{0,1\}\subset S_f$. Further, if h is a homeomorphism of $[0,1]$ onto itself, then clearly

$$S_{f\circ h}=h^{-1}(S_f). \tag{6.1}$$

The following theorem was proved in [7].

Theorem 6.3 ([7])**.** *Let X be a Banach space, and let $f\colon [0,1]\to X$. Then there is a homeomorphism h of $[0,1]$ onto itself such that $f\circ h$ is C^1 if and only if f is continuous, BV, and $\mathcal{H}^1(f(S_f))=0$.*

Lemma 6.4. *Let $g\colon [0,1]\to X$ be C^1 and let $x\in S_g$. Then either $g'(x)=0$ or $x\in\{0,1\}$.*

Proof. For a contradiction, suppose that $x\in(0,1)$ and $g'(x)\neq 0$. Since g is C^1, there exist $\eta,\delta>0$ such that $\|g'(y)\|>\eta$ for all $y\in[x-\delta,x+\delta]$. But then $\tau(g,y)=g'(y)/\|g'(y)\|$ is continuous on $[x-\delta,x+\delta]$, and we have a contradiction with $x\in S_g$. □

The main result of the present section is the following.

Theorem 6.5. *Let $f\colon [0,1] \to X$ be BV continuous nonconstant and $G := [0,1] \setminus (T_f \cup S_f)$. Then the following conditions are equivalent.*

(i) *There exists a homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ is a $C^{1,\mathrm{BV}}$ function.*

(ii) *There exists a homeomorphism φ of $[0,1]$ onto itself such that $f \circ \varphi$ is $C^{1,\mathrm{BV}}$, and $(f \circ \varphi)'(x) \neq 0$ for each $x \in \varphi^{-1}(G)$.*

(iii) *$\mathcal{H}^1(f(T_f \cup S_f)) = 0$ and $W^\delta(f, G) < \infty$ for each $\delta > 0$.*

(iv) *$\mathcal{H}^1(f(T_f \cup S_f)) = 0$ and $W^\delta(f, G) < \infty$ for some $\delta > 0$.*

(v) *$\mathcal{H}^1(f(T_f \cup S_f)) = 0$ and $\sum_{I \in \mathcal{P}} \sqrt{V(f, I)} < \infty$ for each $\mathcal{P}$ which is an (f, δ, ∞)-partition of G with $\delta > 0$.*

(vi) *$\mathcal{H}^1(f(T_f \cup S_f)) = 0$ and $\sum_{I \in \mathcal{P}^*} \sqrt{V(f, I)} < \infty$ for some $(f, 0, K^*)$-partition $\mathcal{P}^*$ of G with $K^* < \infty$.*

(vii) *There exists a family $\mathcal{S}$ of pairwise non-overlapping compact intervals such that $\mathrm{int}(J) \subset G$ for each $J \in \mathcal{S}$ and*

$$\sum_{J \in \mathcal{S}} V(f, J) = V(f, [0,1]), \quad \sum_{J \in \mathcal{S}} \sqrt{V(f, J)} < \infty, \quad \sum_{J \in \mathcal{S}} \sqrt{V(f, J) \cdot T(f, J)} < \infty. \tag{6.2}$$

If f is nonconstant on any interval, then in (ii) *we can also assert $\lambda(\varphi^{-1}(G)) = 1$.*

P r o o f. The proof is literally the same as the proof of Theorem 5.4, except for the implication (i) $\Longrightarrow$ (iii).

To prove that (i) $\Longrightarrow$ (iii), let h be as in (i), put $g := f \circ h$ and $G^* := h^{-1}(G)$. Using (5.7), (6.1), Lemma 5.3, Lemma 6.4, and Lemma 3.1, we easily obtain $\mathcal{H}^1(f(T_f \cup S_f)) = \mathcal{H}^1(g(T_g \cup S_g)) = 0$. Note that $G^* = [0,1] \setminus (T_g \cup S_g)$ by (5.7) and (6.1). Lemma 5.3 and Lemma 6.4 give that we can apply Lemma 3.8 (with $G := G^*$ and $f := g$) and obtain $W^\delta(g, G^*) < \infty$ for each $\delta > 0$. So $W^\delta(f, G) < \infty$ for each $\delta > 0$ by Remark 3.6. $\square$

Remark 6.6. Let $f\colon [0,1] \to X$ be BV continuous and $G := [0,1] \setminus (T_f \cup S_f)$. Then the following conditions are equivalent.

(a) *There exists a homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ is $C^{1,\mathrm{BV}}$ and $(f \circ h)'(x) \neq 0$ for a.e. $x \in [0,1]$.*

(b) *f is nonconstant on any interval and any of conditions* (iii)–(vii) *holds.*

This follows immediately from Theorem 6.5 and the simple observation that (a) implies that f is nonconstant on any interval.

The following example suggests that Theorem 6.5 cannot be easily deduced from Theorem 5.4.

Example 6.7. There exists a function $f\colon [0,1] \to \mathbb{R}^2$ with bounded convexity, such that $f \circ h$ is a C^1 function for some homeomorphism h of $[0,1]$ onto itself, but $f \circ \varphi$ is not a $C^{1,\mathrm{BV}}$ function for any homeomorphism φ of $[0,1]$ onto itself.

Proof. Find $1 > x_1 > x_2 > \ldots > 0$ such that $\lim x_n = 0$ and $x_j - x_{j+1} = cj^{-2}$ for some $c > 0$. Choose a Lipschitz convex function g on $[0,1]$ for which $\{x_1, x_2, \ldots\}$ is the set of all points $x \in (0,1)$ at which g is not differentiable. Let $f(x) := (x, g(x)),\ x \in [0,1]$. Using Lemma 2.5, it is easy to show that f has bounded convexity. It is easy to check that $S_f = \{0, 1, x_1, x_2, \ldots\}$. Since f has finite turn by Lemma 3.3, we have $T_f = \{0,1\}$. Using Theorem 6.3, we obtain that there exists a homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ is a C^1 function. On the other hand, since $\sum\limits_{j=1}^{\infty} \sqrt{V(f, [x_{j+1}, x_j])} \geqslant \sum\limits_{j=1}^{\infty} \sqrt{cj^{-2}} = \infty$, $f \circ \varphi$ is not a $C^{1,\mathrm{BV}}$ function for any homeomorphism φ of $[0,1]$ onto itself. Indeed, otherwise condition (iii) of Theorem 6.5 holds and so Remark 3.7 gives a contradiction. □

Remark 6.8. Example 6.7 shows that the problems of a parametrization with bounded convexity and of a $C^{1,\mathrm{BV}}$ parametrization are not necessarily equivalent even when $T_f = \{0,1\}$ and S_f is countable with only one point of accumulation. The following proposition shows that these two parametrization problems are equivalent if $S_f \setminus T_f$ is finite.

Proposition 6.9. *Let $f\colon [0,1] \to X$ be continuous BV. Suppose that $S_f \setminus T_f$ is finite. Then f admits a $C^{1,\mathrm{BV}}$ parametrization if and only if f admits a parametrization with bounded convexity.*

Proof. The "only if" implication is obvious. So, suppose that f admits a parametrization with bounded convexity, and choose a family $\mathcal{S}$ from condition (vii) of Theorem 5.4. Dividing members of $\mathcal{S}$ by points of $S_f \setminus T_f$, we obtain a nonoverlapping system $\mathcal{S}^*$ of compact intervals (such that $\bigcup\limits_{I \in \mathcal{S}^*} \mathrm{int}(I) \subset \bigcup\limits_{J \in \mathcal{S}} \mathrm{int}(J)$ and $\bigcup\limits_{J \in \mathcal{S}} \mathrm{int}(J) \setminus \bigcup\limits_{I \in \mathcal{S}^*} \mathrm{int}(I) = (S_f \setminus T_f) \cap \bigcup\limits_{J \in \mathcal{S}} \mathrm{int}(J)$). Since $\mathcal{S}^*$ clearly witnesses that condition (vii) of Theorem 6.5 holds, f admits a $C^{1,\mathrm{BV}}$ parametrization by Theorem 6.5. □

The following proposition shows additional assumptions under which Proposition 5.7 can be reversed.

Proposition 6.10. *Let X be a Banach space with a Fréchet smooth norm. Assume that $f\colon [0,1] \to X$ is continuous, BV and nonconstant on any interval. Let $F := f \circ v_f^{-1}$ and $l := v_f(1)$. Suppose that F'' is continuous on $(0,l)$ and for some $\delta > 0$ we have that $\|F''\|$ is monotone on $(0,\delta)$ and on $(l-\delta, l)$.*

Then the following assertions are equivalent.

(i) $\int_0^l \sqrt{\|F''(t)\|}\,\mathrm{d}t < \infty$.

(ii) f *admits a* C^2 *parametrization.*

(iii) f *admits a* $C^{1,\mathrm{BV}}$ *parametrization.*

(iv) f *admits a parametrization with bounded convexity.*

P r o o f. The implication (i) $\Longrightarrow$ (ii) is part of [10, Proposition 4.8]. The implications (ii) $\Longrightarrow$ (iii) and (iii) $\Longrightarrow$ (iv) follow from (2.5) and (2.4). The implication (iv) $\Longrightarrow$ (i) follows from Proposition 5.7. □

Example 7.3 below shows that the implication (i) $\Longrightarrow$ (iv) does not hold without the assumption on the monotonicity of $\|F''\|$.

7. Examples

Example 7.1. For $s > 0$, consider the spiral $f\colon [0,1] \to \mathbb{R}^2$ defined by $f(0) = 0$ and

$$f(t) = (x(t), y(t)) = (t^s \cos(1/t), t^s \sin(1/t)), \quad 0 < t \leqslant 1.$$

By [10, Example 6.3], f is BV if and only if $s > 1$ and f allows a C^2 parametrization if and only if $s > 2$. Since it is shown in [10, Example 6.3] that (if $s > 1$) the assumptions of Proposition 6.10 are satisfied, we obtain that f allows a parametrization with bounded convexity (a $C^{1,\mathrm{BV}}$ parametrization) if and only if $s > 2$.

In the following two examples, we need the following well-known fact.

Lemma 7.2. *Let* $k\colon (0,1) \to \mathbb{R}$ *be positive and* C^∞. *Then there is a continuous* $f\colon [0,1] \to \mathbb{R}^2$ *parametrized by the arc-length,* C^∞ *on* $(0,1)$, *and such that* $\|f''(x)\| = k(x)$ *for* $x \in (0,1)$.

P r o o f. By the Fundamental Theorem of the local theory of curves (see i.e. [13, Theorem 2.15]), there exists $g\colon (0,1) \to \mathbb{R}^2$ parametrized by the arc-length, which is C^∞, and $\|g''(x)\| = k(x)$ for $x \in (0,1)$. Since g is 1-Lipschitz, it has a continuous extension f to $[0,1]$, which has all the desired properties. □

Example 7.3. There exists a continuous $f\colon [0,1] \to \mathbb{R}^2$ which is parametrized by the arc-length, is C^∞ on $(0,1)$, does not allow a parametrization with bounded convexity, but $\int_0^1 \sqrt{\|f''\|} < \infty$.

P r o o f. Let $\mathcal{P} = \{I_n\colon n \in \mathbb{N}\}$ be a generalized partition of $(0,1)$ such that $I_n \subset (0,1)$ and $\lambda(I_n) = c/n^2$ for some $c > 0$. Choose closed intervals $J_n \subset I_n$ with $\lambda(J_n) = c/n^4$. We can clearly choose a positive C^∞ function $k\colon (0,1) \to \mathbb{R}$

such that, for each $n \in \mathbb{N}$, we have $\max_{x\in J_n} k(x) = n^4$, $k(x) \leqslant 1$ for $x \in I_n \setminus J_n$, and $\int_{J_n} k \geqslant c/2$. Choose f corresponding to k by Lemma 7.2. By Lemma 2.12 we see that $\mathcal{P}$ is an $(f, c/2, \infty)$-partition of $(0,1)$, but $\sum_{I\in\mathcal{P}} \sqrt{\lambda(I)} = \infty$, and therefore f does not allow a parametrization with bounded convexity (see Theorem 5.4 (v)). On the other hand, $\int_0^1 \sqrt{\|f''\|} \leqslant 1 + \sum(c/n^4)\sqrt{n^4} < \infty$. $\square$

Example 7.4. There exists a continuous $f\colon [0,1] \to \mathbb{R}^2$ parametrized by the arc-length such that f is C^∞ on $(0,1)$, f allows a $C^{1,\mathrm{BV}}$ parametrization but f does not allow a $C^{1,1}$ parametrization (and the less so a C^2 parametrization).

Proof. Let $\mathcal{P} = \{I_n\colon n \in \mathbb{N}\}$ be a generalized partition of $(0,1)$ such that $I_n \subset (0,1)$ and $\lambda(I_n) = c/n^4$ for some $c > 0$. Divide each interval I_n into closed subintervals J_n^j $(j = 1,\dots,j_n)$ so that $\sum_j \sqrt{\lambda(J_n^j)} \geqslant 1$ and put $\mathcal{P}^* := \{J_n^j\colon n \in \mathbb{N}, 1 \leqslant j \leqslant j_n\}$. We can clearly choose a positive C^∞ function $k\colon (0,1) \to \mathbb{R}$ such that $\max_{x\in J} k(x) = 1/\lambda(J)$ for each $J \in \mathcal{P}^*$ and $\int_I k \leqslant 1$ for each $I \in \mathcal{P}$. Choose f corresponding to k by Lemma 7.2. By Lemma 2.12, $\mathcal{P}$ is an $(f,0,1)$-partition of $(0,1)$ with $\sum_{I\in\mathcal{P}} \sqrt{\lambda(I)} < \infty$, and thus f allows a $C^{1,\mathrm{BV}}$ parametrization by Theorem 6.5 (vi).

Since the arc-length parametrization of f (which equals f) is C^2 on the interior of its domain, [9, Proposition 6.5] implies that f allows a $C^{1,1}$ parametrization if and only if f allows a C^2 parametrization.

Now observe that $\mathcal{P}^*$ is a generalized partition of $(0,1)$ such that

$$\sup_{x\in J} \|f''(x)\| \cdot V(f,J) = \max_{x\in J} k(x) \cdot \lambda(J) = 1 \quad \text{for each } J \in \mathcal{P}^*$$

and

$$\sum_{J\in\mathcal{P}^*} \sqrt{V(f,J)} = \sum_{J\in\mathcal{P}^*} \sqrt{\lambda(J)} = \infty.$$

Therefore condition (v) from [10, Theorem 4.5] does not hold (since f is parametrized by the arc-length, we have shown that $\mathcal{P}^*$ is an $(f,1,\infty)$-partition of $(0,1)$ in the sense of [10, Definition 3.8]), and so [10, Theorem 4.5] implies that f does not allow a C^2 parametrization. $\square$

References

[1] *A. D. Alexandrov, Yu. G. Reshetnyak*: General Theory of Irregular Curves. Transl. from the Russian by L. Ya. Yuzina. Mathematics and Its Applications: Soviet Series 29, Kluwer Academic Publishers, Dordrecht, 1989.

[2] *N. Bourbaki*: Éléments de Mathématique. I: Les structures fondamentales de l'analyse. Livre IV: Fonctions d'une variable réelle (théorie élémentaire). Chapitres 1, 2 et 3:

Dérievées. Primitives et intégrales. Fonctions élémentaires. Second ed. Actualés Sci. Indust. 1074, Hermann, Paris, 1958. (In French.)
[3] *V. V. Chistyakov*: On mappings of bounded variation. J. Dyn. Control Sys. *3* (1997), 261–289.
[4] *J. Duda*: Curves with finite turn. Czech. Math. J. *58* (2008), 23–49.
[5] *J. Duda*: Second order differentiability of paths via a generalized $\frac{1}{2}$-variation. J. Math. Anal. Appl. *338* (2008), 628–638.
[6] *J. Duda*: Generalized α-variation and Lebesgue equivalence to differentiable functions. Fundam. Math. *205* (2009), 191–217.
[7] *J. Duda, L. Zajíček*: Curves in Banach spaces—differentiability via homeomorphisms. Rocky Mt. J. Math. *37* (2007), 1493–1525.
[8] *J. Duda, L. Zajíček*: Curves in Banach spaces which allow a C^2-parametrization or a parametrization with finite convexity. J. London Math. Soc., II. Ser. *83* (2011), 733–754.
[9] *J. Duda, L. Zajíček*: On vector-valued curves that allow a $C^{1,\alpha}$-parametrization. Acta Math. Hung. *127* (2010), 85–111.
[10] *J. Duda, L. Zajíček*: Curves in Banach spaces which allow a C^2-parametrization. J. Lond. Math. Soc., II. Ser. *83* (2011), 733–754.
[11] *H. Federer*: Geometric Measure Theory. Springer, New York, 1969.
[12] *B. Kirchheim*: Rectifiable metric spaces: Local structure and regularity of the Hausdorff measure. Proc. Am. Math. Soc. *121* (1994), 113–123.
[13] *W. Kühnel*: Differential Geometry. Curves-Surfaces-Manifolds. Transl. from the German by Bruce Hunt. Student Mathematical Library 16. AMS, Providence, RI, 2002.
[14] *M. Laczkovich, D. Preiss*: α-variation and transformation into C^n functions. Indiana Univ. Math. J. *34* (1985), 405–424.
[15] *V. V. Lebedev*: Homeomorphisms of an interval and smoothness of a function. Math. Notes *40* (1986), 713–719; translation from Mat. Zametki *40* (1986), 364–373. (In Russian.)
[16] *J. L. Massera, J. J. Schäffer*: Linear differential equations and functional analysis I. Ann. Math. (2) *67* (1958), 517–573.
[17] *A. V. Pogorelov*: Extrinsic Geometry of Convex Surfaces. Translated from the Russian by Israel Program for Scientific Translations. Translations of Mathematical Monographs 35, AMS, Providence, RI, 1973.
[18] *A. W. Roberts, D. E. Varberg*: Convex Functions. Pure and Applied Mathematics 57, Academic Press, a subsidiary of Harcourt Brace Jovanovich, New York, 1973.
[19] *L. Veselý*: On the multiplicity points of monotone operators on separable Banach spaces. Commentat. Math. Univ. Carol. *27* (1986), 551–570.
[20] *L. Veselý, L. Zajíček*: Delta-convex mappings between Banach spaces and applications. Dissertationes Math. (Rozprawy Mat.) *289* (1989), 1–48.
[21] *L. Veselý, L. Zajíček*: On vector functions of bounded convexity. Math. Bohem. *133* (2008), 321–335.

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