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CORRECTIONS TO THE PAPER "THE BOUNDEDNESS OF CERTAIN SUBLINEAR OPERATOR IN THE WEIGHTED VARIABLE LEBESGUE SPACES"

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Abstract. In this paper the author proved the boundedness of the multidimensional Hardy type operator in weighted Lebesgue spaces with variable exponent. As an application he proved the boundedness of certain sublinear operators on the weighted variable Lebesgue space. The proof of the boundedness of the multidimensional Hardy type operator in weighted Lebesgue spaces with a variable exponent does not contain any mistakes. But in the proof of the boundedness of certain sublinear operators on the weighted variable Lebesgue space Georgian colleagues discovered a small but significant error in my paper, which was published as R. A. Bandaliev, The boundedness of certain sublinear operator in the weighted variable Lebesgue spaces, Czech. Math. J. 60 (2010), 327–337.

Keywords: variable Lebesgue space; weight; sublinear operator; boundedness

MSC 2010: 46B50, 47B38, 26D15

This result appears as Theorem 5 in the above-mentioned paper. In other words, sufficient conditions for general weights ensuring the validity of the two-weight strong type inequalities for some sublinear operator were found. In this theorem the inequality (9) is not true. In this note we give the details of the correct argument. We presume that the reader is familiar with the contents and notation of our original paper. At the heart of our correction is the following Theorem 1 which replaces Theorem 5 in [1]. The numbering in this note remains as in [1].

Theorem 1. *Let $1 < \underline{p} \leqslant p(x) \leqslant \bar{p} < \infty$ for $x \in \mathbb{R}^n$ and suppose that there exists a constant $\delta \in [0,1)$ such that $\int_{\mathbb{R}^n} \delta^{\underline{p}p(x)/p(x)-\underline{p}}\,\mathrm{d}x < \infty$. Let T be a sublinear operator acting boundedly from $L_{\underline{p}}(\mathbb{R}^n)$ to $L_{p(x)}(\mathbb{R}^n)$ such that, for any $f \in L_1(\mathbb{R}^n)$*

with compact support and $x \notin \operatorname{supp} f$,

$$|Tf(x)| \leqslant C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n}\,\mathrm{d}y, \tag{3}$$

where $C > 0$ *is independent of* f *and* x.

Moreover, let $v(x)$ *and* $w(x)$ *be weight functions on* $\mathbb{R}^n$ *and assume there exist numbers* $0 < \alpha, \beta < 1$ *such that*

$$\begin{aligned} A_1 = \sup_{t>0} &\Big(\int_{|y|<t} [v(y)]^{-\bar{p}'}\,\mathrm{d}y\Big)^{\alpha/\bar{p}'} \\ &\times \Big\| \frac{w(\cdot)}{|\cdot|^n} \Big(\int_{|y|<|\cdot|} [v(y)]^{-\bar{p}'}\,\mathrm{d}y\Big)^{(1-\alpha)/\bar{p}'} \Big\|_{L_{p(\cdot)}(|\cdot|>t)} < \infty, \end{aligned} \tag{4}$$

$$\begin{aligned} B = \sup_{t>0} &\Big(\int_{|y|>t} [v(y)|y|^n]^{-\bar{p}'}\,\mathrm{d}y\Big)^{\beta/\bar{p}'} \\ &\times \Big\| w(\cdot) \Big(\int_{|y|>|\cdot|} [v(y)|y|^n]^{-\bar{p}'}\,\mathrm{d}y\Big)^{(1-\beta)/\bar{p}'} \Big\|_{L_{p(\cdot)}(|\cdot|<t)} < \infty, \end{aligned} \tag{5}$$

and that there exists $M > 0$ *such that for any* $x \in \mathbb{R}^n$ *the inequality*

$$\sup_{|x|/4<|y|\leqslant 4|x|} w(y) \leqslant M \inf_{|x|/4<|y|\leqslant 4|x|} v(y) \tag{6}$$

holds.

Then there exists a positive constant C, *independent of* f, *such that for all* $f \in L_{p(x),v}(\mathbb{R}^n)$

$$\|Tf\|_{L_{p(x),w}(\mathbb{R}^n)} \leqslant C\|f\|_{L_{p(x),v}(\mathbb{R}^n)}.$$

P r o o f. For $k \in \mathbb{Z}$ we define $E_k = \{x \in \mathbb{R}^n \colon 2^k < |x| \leqslant 2^{k+1}\}$, $E_{k,1} = \{x \in \mathbb{R}^n \colon |x| \leqslant 2^{k-1}\}$, $E_{k,2} = \{x \in \mathbb{R}^n \colon 2^{k-1} < |x| \leqslant 2^{k+2}\}$ and $E_{k,3} = \{x \in \mathbb{R}^n \colon |x| > 2^{k+2}\}$. Then $E_{k,2} = E_{k-1} \cup E_k \cup E_{k+1}$ and the multiplicity of the covering $E_{k,2}$ is equal to 3.

Given $f \in L_{p(x),v}(\mathbb{R}^n)$, we have

$$\begin{aligned} |Tf(x)| &= \sum_{k\in\mathbb{Z}} |Tf(x)|\chi_{E_k}(x) \\ &\leqslant \sum_{k\in\mathbb{Z}} |Tf_{k,1}(x)|\chi_{E_k}(x) + \sum_{k\in\mathbb{Z}} |Tf_{k,2}(x)|\chi_{E_k}(x) + \sum_{k\in\mathbb{Z}} |Tf_{k,3}(x)|\chi_{E_k}(x) \\ &= T_1f(x) + T_2f(x) + T_3f(x), \end{aligned}$$

where χ_{E_k} is the characteristic function of the set E_k, $f_{k,i} = f\chi_{E_{k,i}}$, $i = 1,2,3$.

The estimates

$$\|T_1 f\|_{L_{p(x),w}(\mathbb{R}^n)} \leqslant C\|f\|_{L_{p(x),v}(\mathbb{R}^n)} \quad \text{and} \quad \|T_3 f\|_{L_{p(x),w}(\mathbb{R}^n)} \leqslant C\|f\|_{L_{p(x),v}(\mathbb{R}^n)}$$

remain precisely as in Theorem 5 in [1].

The mistake of author was in assuming that the inequality

$$(9) \qquad \|T_2 f\|_{L_{p(x),w}(\mathbb{R}^n)} \leqslant \|T\|_{L_{p(\cdot)}(\mathbb{R}^n)} M \sum_{k\in\mathbb{Z}} \|fv\|_{L_{p(x)}(E_{k,2})} \leqslant C_3 \|fv\|_{L_{p(x)}(\mathbb{R}^n)}$$

holds. But after publication it was discovered that the inequality (9) is not true.

Now we deduce the correct variant of this inequality. Since the operator T is sublinear it suffices to prove that $\|f\|_{L_{\underline{p}v}(\mathbb{R}^n)} \leqslant 1$ implies

$$\int_{\mathbb{R}^n} w^{p(x)}(x) \Big[\sum_{k\in\mathbb{Z}} |Tf_{k,2}|\chi_{E_k}\Big]^{p(x)} \,\mathrm{d}x \leqslant C,$$

where $C > 0$ is independent of $k \in \mathbb{Z}$ (see [2]).

We have

$$\int_{\mathbb{R}^n} w^{p(x)}(x) \Big[\sum_{k\in\mathbb{Z}} |Tf_{k,2}|\chi_{E_k}\Big]^{p(x)} \,\mathrm{d}x = \sum_{k\in\mathbb{Z}} \int_{E_k} (|Tf_{k,2}| w(x))^{p(x)} \,\mathrm{d}x.$$

By virtue of the boundedness of the operator T and the condition (6), we have

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} \int_{E_k} (|Tf_{k,2}| w(x))^{p(x)} \,\mathrm{d}x \\
&\quad = \sum_{k\in\mathbb{Z}} \int_{E_k} \Big(\frac{|Tf_{k,2}|}{C_1\|f_{k,2}\|_{L_{\underline{p}}(\mathbb{R}^n)}}\Big)^{p(x)} \big(C_1 \|f_{k,2}\|_{L_{\underline{p}}} w(x)\big)^{p(x)} \,\mathrm{d}x \\
&\quad \leqslant C_2 \sum_{k\in\mathbb{Z}} \sup_{x\in E_k} (\|f_{k,2}\|_{L_{\underline{p}}(\mathbb{R}^n)} w(x))^{p(x)} \int_{\mathbb{R}^n} \Big(\frac{|Tf_{k,2}|}{C_1\|f_{k,2}\|_{L_{\underline{p}}(\mathbb{R}^n)}}\Big)^{p(x)} \,\mathrm{d}x \\
&\quad \leqslant C_2 \sum_{k\in\mathbb{Z}} \sup_{x\in E_k} (\|f_{k,2}\|_{L_{\underline{p}}(\mathbb{R}^n)} w(x))^{p(x)} = C_2 \sum_{k\in\mathbb{Z}} \sup_{x\in E_k} (\|f\|_{L_{\underline{p}}(E_{k,2})} w(x))^{p(x)} \\
&\quad = C_2 \sum_{k\in\mathbb{Z}} \sup_{x\in E_k} (\|fw(x)\|_{L_{\underline{p}}(E_{k,2})})^{p(x)} \leqslant C_3 \sum_{k\in\mathbb{Z}} \sup_{x\in E_k} (\|f \inf_{x\in E_{k,2}} v(x)\|_{L_{\underline{p}}(E_{k,2})})^{p(x)} \\
&\quad \leqslant C_3 \sum_{k\in\mathbb{Z}} \sup_{x\in E_k} (\|fv\|_{L_{\underline{p}}(E_{k,2})})^{p(x)} = C_3 \sum_{k\in\mathbb{Z}} \sup_{x\in E_k} (\|f\|_{L_{\underline{p},v}(E_{k,2})})^{p(x)} \\
&\quad = C_3 \sum_{k\in\mathbb{Z}} (\|f\|_{L_{\underline{p},v}(E_{k,2})})^{\inf_{x\in E_k} p(x)} \leqslant C_3 \sum_{k\in\mathbb{Z}} (\|f\|_{L_{\underline{p},v}(E_{k,2})})^{\underline{p}}.
\end{aligned}$$

Further, we obtain

$$\begin{aligned}\sum_{k\in\mathbb{Z}}(\|f\|_{L_{\underline{p},v}(E_{k,2})})^{\underline{p}} &= \sum_{k\in\mathbb{Z}}\biggl(\int_{E_{k,2}}|f(x)v(x)|^{\underline{p}}\,\mathrm{d}x\biggr)\\ &= \sum_{k\in\mathbb{Z}}\biggl(\int_{E_{k-1}}+\int_{E_k}+\int_{E_{k+1}}\biggr)|f(x)v(x)|^{\underline{p}}\,\mathrm{d}x\\ &= 3\biggl(\int_{\mathbb{R}^n}|f(x)v(x)|^{\underline{p}}\,\mathrm{d}x\biggr) = 3\|f\|^{\underline{p}}_{L_{\underline{p},v}(\mathbb{R}^n)} \leqslant 3.\end{aligned}$$

Thus $\|Tf_{k,2}\|_{L_{p(\cdot),w}(\mathbb{R}^n)} \leqslant C\|f\|_{L_{\underline{p},v}(\mathbb{R}^n)}$ and taking into account the condition $\int_{\mathbb{R}^n}\delta^{\underline{p}p(x)/(p(x)-\underline{p})}\,\mathrm{d}x<\infty$, which guarantees the embedding $L_{p(x),v}(\mathbb{R}^n)\hookrightarrow L_{\underline{p},v}(\mathbb{R}^n)$ (see [2]), we obtain

$$\|Tf_{k,2}\|_{L_{p(\cdot),w}(\mathbb{R}^n)} \leqslant C\|f\|_{L_{\underline{p},v}(\mathbb{R}^n)} \leqslant C_1\|f\|_{L_{p(\cdot),v}(\mathbb{R}^n)}.$$

The proof of Theorem 1 is complete. □

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References

[1] *R. A. Bandaliev*: The boundedness of certain sublinear operator in the weighted variable Lebesgue spaces. Czech. Math. J. *60* (2010), 327–337.

[2] *L. Diening, P. Harjulehto, P. Hästö, M. Růžička*: Lebesgue and Sobolev Spaces with Variable Exponents. Lecture Notes in Mathematics 2017, Springer, Berlin, 2011.

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