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A POSTERIORI UPPER AND LOWER ERROR BOUND OF THE HIGH-ORDER DISCONTINUOUS GALERKIN METHOD FOR THE HEAT CONDUCTION EQUATION

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Abstract. We deal with the numerical solution of the nonstationary heat conduction equation with mixed Dirichlet/Neumann boundary conditions. The backward Euler method is employed for the time discretization and the interior penalty discontinuous Galerkin method for the space discretization. Assuming shape regularity, local quasi-uniformity, and transition conditions, we derive both a posteriori upper and lower error bounds. The analysis is based on the Helmholtz decomposition, the averaging interpolation operator, and on the use of cut-off functions. Numerical experiments are presented.

Keywords: discontinuous Galerkin method; Helmholtz decomposition; averaging interpolation operator; Euler backward scheme; residual-based a posteriori error estimate; local cut-off function

MSC 2010: 65M15

1. Introduction

Our aim is to develop a sufficiently accurate and efficient numerical method for simulations of unsteady flows. A promising technique is a combination of the discontinuous Galerkin method (DGM) for the space discretization and the backward difference formula for the time discretization, see [6]. In order to both apply an adaptive algorithm and assess the discretization error, a posteriori error estimates have to be developed.

In this paper, we focus on a simplified model problem, represented by the heat equation, which is discretized by the high-order DGM and the backward Euler method. Our aim is to derive a posteriori error estimate of the discretization error.

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This topic was already studied in [16], where conforming space discretization was considered. In our case, the main difficulty is to overcome the nonconformity of the space discretization as the approximate solution does not belong to the function space associated with the problem (2.1). There have been various approaches introduced so far. Functional-type a posteriori error estimates were employed on the heat equation in [13]. The derivation of functionals giving the upper bound on the error is based on certain integral equation in terms of deviation from the exact solution. Finally, minimization techniques are used. A different approach based on the flux reconstruction from the Raviart-Thomas-Nédélec (RTN) finite element space was presented in [5]. This technique gives a fully computable upper bound on the error (that is, containing no undetermined constants) and also provides the efficiency of error estimators, however, it requires a reconstruction of the flux in the RTN space.

We were inspired by [12], where Crouzeix-Raviart finite element method is employed for spatial discretization of the heat equation. The derived a posteriori error estimates are based on the Helmholtz decomposition of the gradient of the error.

In this paper, first, we extend the results from [15] applying the approach from [12] to the high-order discontinuous Galerkin discretization. Then, we derive a lower error bound using a technique based on testing with suitable cut-off functions. The a posteriori error estimates are simply computable but they suffer from a presence of undetermined constants. The article is concluded with numerical experiments showing the behavior of derived estimates.

2. Problem definition

Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) be a bounded multiply connected polyhedral Lipschitz domain with boundary $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$, $T > 0$, and $Q_T = \Omega \times (0, T)$. Let us consider the problem:

$$
\begin{aligned}
\frac{\partial u}{\partial t} - \Delta u &= f \text{ in } Q_T, \\
u &= u_D \text{ on } \partial\Omega_D \times (0,T), \\
\nabla u \cdot \mathbf{n} &= g_N \text{ on } \partial\Omega_N \times (0,T), \\
u(x,0) &= u^0(x) \text{ in } \Omega.
\end{aligned}
\tag{2.1}
$$

We use the standard notation for the Lebesgue, Sobolev, and Bochner spaces (see [11]). In particular, for a function v in the appropriate space, we will use the following notation: $\|v\|_{k,\omega} = \|v\|_{H^k(\omega)}$, $\|v\|_{\omega} = \|v\|_{L^2(\omega)}$, $\|v\|_{\partial\omega} = \|v\|_{L^2(\partial\omega)}$, $\|v\|_{1/2,\partial\omega} =$

$\|v\|_{H^{1/2}(\partial\omega)}$, $\|v\|_{-1/2,\partial\omega} = \|v\|_{H^{-1/2}(\partial\omega)}$, $|v|_{k,\omega} = |v|_{H^k(\omega)}$, where $\omega \subseteq \Omega$. Recall that

$$\|v\|_{H^{1/2}(\partial\omega)} := \inf_{\substack{\varphi\in H^1(\omega)\\ \varphi=v \text{ on } \partial\omega}} \|\varphi\|_{1,\omega} \quad \text{and} \quad \|v\|_{H^{-1/2}(\partial\omega)} := \sup_{\substack{\varphi\in H^{1/2}(\partial\omega)\\ \varphi\neq 0}} \frac{((v,\varphi))}{\|\varphi\|_{1/2,\partial\omega}},$$

where $((\cdot,\cdot))$ denotes duality pairing between the spaces $H^{1/2}(\partial\omega)$ and $H^{-1/2}(\partial\omega)$. Moreover, $H^1_D(\Omega) \equiv \{v \in H^1(\Omega);\ v = 0 \text{ on } \partial\Omega_D\}$, $H^1_{z,D}(\Omega) \equiv \{v \in H^1(\Omega);\ v = z$ on $\partial\Omega_D\}$ for a function $z\colon \partial\Omega_D \to \mathbb{R}$.

3. Discretization

3.1. Time semidiscretization. Let $0 = t_0 < t_1 < \ldots < t_{\overline{N}} = T$ be a partition of the time interval $[0,T]$ and set $\tau_n = t_n - t_{n-1}$, $\tau = \max\{\tau_n;\ 1 \leqslant n \leqslant \overline{N}\}$. We use the backward Euler scheme to get the semi-discrete problem: Find a sequence $\{u^n\}_{1\leqslant n\leqslant \overline{N}}$, $u^n - u^*(t_n) \in H^1_D(\Omega)$ such that

$$\int_\Omega \frac{u^n - u^{n-1}}{\tau_n} v \,\mathrm{d}\mathbf{x} + \int_\Omega \nabla u^n \cdot \nabla v \,\mathrm{d}\mathbf{x} = \int_\Omega f^n v \,\mathrm{d}\mathbf{x} + \int_{\partial\Omega_N} g^n_N v \,\mathrm{d}S \quad \forall v \in H^1_D(\Omega), \tag{3.1}$$

where $u^*(t_n) \in H^1(\Omega)$ has the trace $u^n_D := u_D(\cdot,t_n)$ on $\partial\Omega_D$, $f^n := f(\cdot,t_n)$, and $g^n_N := g_N(\cdot,t_n)$. For simplicity, we assume that the functions u^n_D, f^n, and g^n_N are piecewise polynomial. Otherwise, we would have oscillation terms of the form $h_\Gamma^{-1/2}\|u^n_D - u^n_{D,h}\|_\Gamma$, $h_K\|f^n - f^n_h\|_K$, and $h_\Gamma^{1/2}\|g^n_N - g^n_{N,h}\|_\Gamma$ in the error indicator (5.2). See, e.g., [12] on how to handle the right-hand side oscillation. The solution of (3.1) is called the semi-discrete solution.

3.2. Space discretization. As we mentioned above, we will carry out the space discretization with the aid of the high-order DGM. On each time level t_n, $n = 1,\ldots,\overline{N}$, we consider a family $\{\mathcal{T}^n_h\}_{h>0}$ of partitions of the closure of Ω into a finite number of closed simplices with mutually disjoint interiors, possibly containing hanging nodes. These partitions are called triangulations hereafter. We assume that the following conditions are satisfied:

$$\text{shape regularity:} \quad \exists C_s > 0\ \forall h > 0\ \forall K \in \mathcal{T}^n_h\colon \frac{h_K}{\varrho_K} \leqslant C_s, \tag{3.2}$$

$$\text{local quasi-uniformity:} \quad \exists C_H > 0\ \forall h > 0\ \forall K, K' \in \mathcal{T}^n_h \text{ sharing a face:} \quad h_K \leqslant C_H h_{K'}, \tag{3.3}$$

where $h_K = \mathrm{diam}(K)$ for $K \in \mathcal{T}_h^n$, ϱ_K denotes the diameter of the largest d-dimensional ball inscribed into K, and ∂K denotes the boundary of the element K. Moreover, we assume that there exists a triangulation $\widetilde{\mathcal{T}}_h^n$ satisfying (3.2) and (3.3) which is a refinement of both $\mathcal{T}_h^{n-1}$ and $\mathcal{T}_h^n$, $1 \leqslant n \leqslant \overline{N}$, and such that

$$\exists C_{HT} > 0\ \forall h > 0\ \forall 1 \leqslant n \leqslant \overline{N}\ \forall K \in \widetilde{\mathcal{T}}_h^n\ \forall K' \in \mathcal{T}_h^n,\ K \subset K' \colon\ \frac{h_{K'}}{h_K} < C_{HT}.$$

This condition reflects simultaneous presence of finite element functions defined on different triangulations and restricts the refinement and the coarsening rate.

By $\widetilde{\mathcal{F}}_h^{n,\mathrm{I}}$, $\widetilde{\mathcal{F}}_h^{n,\mathrm{D}}$, and $\widetilde{\mathcal{F}}_h^{n,\mathrm{N}}$ we denote the set of all interior faces, faces on $\partial\Omega_D$ and faces on $\partial\Omega_N$ corresponding to $\widetilde{\mathcal{T}}_h^n$, respectively (for $d = 2$ faces are replaced by edges). For simplicity, we put $\widetilde{\mathcal{F}}_h^{n,\mathrm{ID}} := \widetilde{\mathcal{F}}_h^{n,\mathrm{I}} \cup \widetilde{\mathcal{F}}_h^{n,\mathrm{D}}$, $\widetilde{\mathcal{F}}_h^{n,\mathrm{IN}} := \widetilde{\mathcal{F}}_h^{n,\mathrm{I}} \cup \widetilde{\mathcal{F}}_h^{n,\mathrm{N}}$, $\widetilde{\mathcal{F}}_h^{n,\mathrm{DN}} := \widetilde{\mathcal{F}}_h^{n,\mathrm{D}} \cup \widetilde{\mathcal{F}}_h^{n,\mathrm{N}}$, and $\widetilde{\mathcal{F}}_h^{n} := \widetilde{\mathcal{F}}_h^{n,\mathrm{I}} \cup \widetilde{\mathcal{F}}_h^{n,\mathrm{D}} \cup \widetilde{\mathcal{F}}_h^{n,\mathrm{N}}$. For $\Gamma \in \widetilde{\mathcal{F}}_h^{n,\mathrm{I}}$, let K_Γ^{L} and K_Γ^{R} denote elements sharing the face Γ. We set $h_\Gamma := \max\{h_{K_\Gamma^{\mathrm{L}}}, h_{K_\Gamma^{\mathrm{R}}}\}$, where $\Gamma \subset K_\Gamma^{\mathrm{L}} \cap K_\Gamma^{\mathrm{R}}$. We define the unit normal vector $\mathbf{n}_\Gamma$ so that it points out of K_Γ^{L}. For $\Gamma \in \widetilde{\mathcal{F}}_h^{n,\mathrm{DN}}$, we assume that $\mathbf{n}_\Gamma$ has the same orientation as the outward normal to $\partial\Omega$ and we put $h_\Gamma := h_{K_\Gamma^{\mathrm{L}}}$, $\Gamma \subset \partial K_\Gamma^{\mathrm{L}}$.

Over the triangulation $\widetilde{\mathcal{T}}_h^n$ we define the so-called broken Sobolev space

$$H^s(\Omega, \widetilde{\mathcal{T}}_h^n) = \{v;\ v|_K \in H^s(K)\ \forall K \in \widetilde{\mathcal{T}}_h^n\}, \quad s \geqslant 1,$$

equipped with the norm $\|v\|^2_{H^s(\Omega,\widetilde{\mathcal{T}}_h^n)} = \sum_{K \in \widetilde{\mathcal{T}}_h^n} \|v\|^2_{H^s(K)}$. For $v \in H^1(\Omega, \widetilde{\mathcal{T}}_h^n)$ we define the broken gradient $\nabla_h v$ of v by $(\nabla_h v)|_K := \nabla(v|_K)$ for all $K \in \widetilde{\mathcal{T}}_h^n$. Further, v_Γ^{L} stands for the trace of $v|_{K_\Gamma^{\mathrm{L}}}$ on Γ, v_Γ^{R} is the trace of $v|_{K_\Gamma^{\mathrm{R}}}$ on Γ, $\langle v\rangle_\Gamma := (1/2)(v_\Gamma^{\mathrm{L}} + v_\Gamma^{\mathrm{R}})$, $[v]_\Gamma := v_\Gamma^{\mathrm{L}} - v_\Gamma^{\mathrm{R}}$, $\Gamma \in \widetilde{\mathcal{F}}_h^{n,\mathrm{I}}$. Further, for $\Gamma \in \widetilde{\mathcal{F}}_h^{n,\mathrm{D}}$, we denote by v_Γ^{L} the trace of $v|_{K_\Gamma^{\mathrm{L}}}$ on Γ, and set $\langle v\rangle_\Gamma := [v]_\Gamma := v_\Gamma^{\mathrm{L}}$. If $\mathbf{n}_\Gamma$, $[\cdot]_\Gamma$, and $\langle\cdot\rangle_\Gamma$ appear in an integral of the form $\int_\Gamma \ldots \,\mathrm{d}S$, we will omit the subscript Γ and write, respectively, $\mathbf{n}$, $[\cdot]$, and $\langle\cdot\rangle$ instead. Finally, we define the space of discontinuous piecewise polynomial functions

$$S_{hp}^n = \{v;\ v \in L^2(\Omega),\ v|_K \in P^p(K)\ \forall K \in \widetilde{\mathcal{T}}_h^n\},$$

where $P^p(K)$ is the space of all polynomials on K of degree at most $p \in \{0, 1, 2, \ldots\}$. For $u_h^n, v_h^n \in H^2(\Omega, \widetilde{\mathcal{T}}_h^n)$, $1 \leqslant n \leqslant \overline{N}$, we define the forms

$$\begin{aligned} (3.4)\quad a_h^n(u_h^n, v_h^n) := &\sum_{K \in \widetilde{\mathcal{T}}_h^n} \int_K \nabla u_h^n \cdot \nabla v_h^n \,\mathrm{d}\mathbf{x} - \sum_{\Gamma \in \widetilde{\mathcal{F}}_h^{n,\mathrm{ID}}} \int_\Gamma \langle \nabla u_h^n \cdot \mathbf{n}\rangle [v_h^n] \,\mathrm{d}S \\ &+ \theta \sum_{\Gamma \in \widetilde{\mathcal{F}}_h^{n,\mathrm{ID}}} \int_\Gamma \langle \nabla v_h^n \cdot \mathbf{n}\rangle [u_h^n] \,\mathrm{d}S + \sum_{\Gamma \in \widetilde{\mathcal{F}}_h^{n,\mathrm{ID}}} \int_\Gamma \sigma [u_h^n][v_h^n] \,\mathrm{d}S, \end{aligned}$$

$$l_h^n(v_h^n) := \int_\Omega f^n v_h^n \,\mathrm{d}\mathbf{x} + \sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{N}}} \int_\Gamma g_N^n v_h^n \,\mathrm{d}S + \theta \sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{D}}} \int_\Gamma \nabla v_h^n \cdot \mathbf{n} u_D^n \,\mathrm{d}S$$
$$+ \sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{D}}} \int_\Gamma \sigma u_D^n v_h^n \,\mathrm{d}S,$$

where $u_D^n = u_D(\cdot, t_n)$, σ is an appropriate penalty parameter, and the parameter $\theta = -1$, $\theta = 1$, and $\theta = 0$ corresponds to the symmetric, nonsymmetric, and incomplete variants of the DGM, respectively.

Now, we can formulate the discrete problem: For a given approximation $u_h^0 \in S_{hp}^0$ of an initial condition u^0, find a sequence $\{u_h^n\}_{1\leqslant n\leqslant \overline{N}}$, $u_h^n \in S_{hp}^n$, such that

$$\int_\Omega \frac{u_h^n - u_h^{n-1}}{\tau_n} v_h^n \,\mathrm{d}\mathbf{x} + a_h^n(u_h^n, v_h^n) = l_h^n(v_h^n) \quad \forall v_h^n \in S_{hp}^n. \tag{3.5}$$

We call the solution of (3.5) the approximate solution. The reader is referred to [2] for the derivation of discontinuous Galerkin formulation. Let $\{u^n\}_{1\leqslant n\leqslant \overline{N}}$ be the semi-discrete solution given by (3.1) and $\{u_h^n\}_{1\leqslant n\leqslant \overline{N}}$ be the approximate solution given by (3.5). We set $\{e^n\}_{1\leqslant n\leqslant \overline{N}} = \{u^n - u_h^n\}_{1\leqslant n\leqslant \overline{N}}$.

4. Tools for a posteriori error analysis

In this section we state some results of the finite element theory that will be used in the analysis. Further, we prove some auxiliary assertions extending the results from [15]. We also introduce Helmholtz decomposition and an appropriate interpolation operator, as they form the basis of the presented approach. It was developed in [12], where the heat equation was solved with the aid of the combination of the Crouzeix-Raviart nonconforming finite elements in space and the backward Euler scheme in time. However, the idea of using Helmholtz decomposition for splitting the error into conforming and nonconforming parts goes back to the paper [4].

In the analysis, we will need the following results of the finite element theory: the *multiplicative trace inequality*

$$\|v\|_{\partial K}^2 \leqslant C_M(|v|_{1,K}\|v\|_K + h_K^{-1}\|v\|_K^2) \quad \forall v \in H^1(K),\ K \in \widetilde{\mathcal{T}}_h^n, \tag{4.1}$$

the *inverse inequality*

$$|v|_{1,K} \leqslant C_I h_K^{-1}\|v\|_K \quad \forall v \in P^p(K),\ K \in \widetilde{\mathcal{T}}_h^n, \tag{4.2}$$

the *trace inequality*

$$\|\mathbf{n}\cdot \operatorname{curl} v\|_{-1/2,\partial K} \leqslant C_T\|\operatorname{curl} v\|_K \quad \forall v \in (H^1(K))^k,\ K \in \widetilde{\mathcal{T}}_h^n, \tag{4.3}$$

and the *approximation property* of the L^2-projection operator Π_{hp} on S^n_{hp}

$$|v-\Pi_{hp}v|_{i,K}\leqslant C_A h_K^{1-i}|v|_{1,K}\quad \forall v\in H^1(K),\ K\in\widetilde{\mathcal{T}}_h^n,\ i=0,1, \tag{4.4}$$

where C_M, C_I, C_T, and C_A are constants independent of K, h, and n and $k=1$ for $d=2$ and $k=3$ for $d=3$. Let us recall that the curl operator is defined by

$$\operatorname{curl} v := \Big(\frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1}\Big),\ v\colon\ \Omega\to\mathbb{R},\ d=2,$$
$$\operatorname{curl} v := \Big(\frac{\partial v_3}{\partial x_2}-\frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3}-\frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1}-\frac{\partial v_1}{\partial x_2}\Big),\ v=(v_1,v_2,v_3)\colon\ \Omega\to\mathbb{R}^3,\ d=3.$$

We introduce the space $\mathbf{H}(\mathrm{curl},\Omega):=\{v\in(L^2(\Omega))^k;\ \operatorname{curl} v\in(L^2(\Omega))^d\}$, where $k=1$ for $d=2$ and $k=3$ for $d=3$. Moreover, $\operatorname{div}\operatorname{curl}\chi=0$ for $\chi\in(H^1(K))^k$, meaning the operator div in the sense of distributions. Finally, $\operatorname{curl}\chi\cdot\mathbf{n}$ is meant in the following sense (for the proof see [14], for more complex relations see Section 2.2 and Section 2.3 of Chapter I in [7]):

Lemma 4.1. *Let $\Omega\in\mathbb{R}^d$ ($d=2$ or 3) be a bounded domain with Lipschitz-continuous boundary. Then there exists a unique continuous linear operator*

$$T_{\mathbf{n}}\colon\ \mathbf{H}(\mathrm{curl},\Omega)\to H^{-1/2}(\partial\Omega), \tag{4.5}$$

such that

$$\forall v\in(C^\infty(\overline{\Omega}))^k\quad T_{\mathbf{n}}v=\mathbf{n}\cdot\operatorname{curl} v|_{\partial\Omega}, \tag{4.6}$$

where $k=1$ for $d=2$ and $k=3$ for $d=3$.

Now, in order to derive a posteriori error estimates, we introduce the interpolation operator that maps $H^1(\Omega,\widetilde{\mathcal{T}}_h^n)$ into $S^n_{hp}\cap H^1_D(\Omega)$ and the Helmholtz decomposition.

4.1. Averaging interpolation operator. Let $\mathcal{N}_{h,n}$ be the set of all Lagrangian vertices of the elements of $\widetilde{\mathcal{T}}_h^n$ such that functions from $S^n_{hp}\cap H^1_D(\Omega)$ are uniquely determined by their values at the nodes from $\mathcal{N}_{h,n}$. It means that all hanging nodes are excluded from $\mathcal{N}_{h,n}$. The averaging interpolation operator $\mathcal{I}^{\mathrm{D}}_{\mathrm{Av}}\colon\ S^n_{hp}\to S^n_{hp}\cap H^1_D(\Omega)$ is defined by (see, e.g., [8])

$$\mathcal{I}^{\mathrm{D}}_{\mathrm{Av}}(v_h)(V)=\begin{cases}\dfrac{1}{\operatorname{card}(\mathcal{T}_V)}\displaystyle\sum_{K\in\mathcal{T}_V} v_h|_K(V), & V\in\mathcal{N}_{h,n}\setminus\mathcal{N}^D_{h,n},\\ 0, & V\in\mathcal{N}^D_{h,n},\end{cases}$$

where $\mathcal{T}_V = \{K \in \widetilde{\mathcal{T}}_h^n;\ V \in K\}$, $\mathcal{N}_{h,n}^D = \{V \in \mathcal{N}_{h,n};\ V \in \partial\Omega_D\}$. Now, we define the interpolation operator $I_h^{n,D}\colon H^1(\Omega, \widetilde{\mathcal{T}}_h^n) \to S_{hp}^n \cap H_D^1(\Omega)$ by

$$I_h^{n,D}(v) = \mathcal{I}_{\mathrm{Av}}^{\mathrm{D}}(\Pi_{hp}(v)) \quad \forall v \in H^1(\Omega, \widetilde{\mathcal{T}}_h^n), \tag{4.7}$$

where Π_{hp} denotes the L^2-projection of v on the space S_{hp}^n. The proof of the following theorem can be found in [9].

Theorem 4.1. *Let $\widetilde{\mathcal{T}}_h^n$ be a triangulation possibly containing hanging nodes that satisfies* (3.2) *and* (3.3). *Then*

$$\sum_{K\in\widetilde{\mathcal{T}}_h^n} \|v_h - \mathcal{I}_{\mathrm{Av}}^{\mathrm{D}}(v_h)\|_{i,K}^2 \leqslant C_O^2 \sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{ID}}} h_\Gamma^{1-2i} \|[v_h]\|_\Gamma^2, \quad \forall v_h \in S_{hp}^n,\ i = 0, 1, \tag{4.8}$$

where the constant C_O is independent of h and v_h.

Remark 4.1. In the case of the averaging operator satisfying nonhomogeneous Dirichlet boundary condition g_D, the estimate (analogous to (4.8)) differs just in the term on the Dirichlet boundary. It has the form: $\sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{D}}} h_\Gamma^{1-2i}\|v_h - g_D\|_\Gamma^2$.

4.2. Helmholtz decomposition. Since we are dealing with the nonconforming method, u_h^n does not belong to $H^1(\Omega)$. Therefore, techniques known from a posteriori error analysis for conforming methods cannot be used in a straightforward manner. There are basically two ways how one can get over this issue. The first and rather natural possibility is to decompose the error into the conforming part and the remainder. The second possibility is to decompose the gradient of the error using the Helmholtz decomposition, as it was done in [4], which we employ in the following:

Theorem 4.2. *There exists the decomposition*

$$\nabla_h e^n = \nabla\varphi^n + \operatorname{curl}\chi^n, \tag{4.9}$$

where $\varphi^n \in H_D^1(\Omega)$ is the solution of the problem

$$\int_\Omega \nabla\varphi^n \cdot \nabla v \,\mathrm{d}\mathbf{x} = \int_\Omega \nabla_h e^n \cdot \nabla v \,\mathrm{d}\mathbf{x} \quad \forall v \in H_D^1(\Omega),$$

and $\chi^n \in (H^1(\Omega))^k$ ($k = 1$ for $d = 2$ and $k = 3$ for $d = 3$) is such that $\mathbf{n}\cdot\operatorname{curl}\chi^n = 0$ on $\partial\Omega_N$. Moreover, the following holds: $\|\nabla_h e^n\|_\Omega^2 = \|\nabla\varphi^n\|_\Omega^2 + \|\operatorname{curl}\chi^n\|_\Omega^2$.

The orthogonality of the splitting is crucial because it suffices to estimate each part of the error independently. A proof of the above theorem can be found in [4]. Now, we state several relations for the error e^n. The following lemma can be proved similarly to Lemma 2 in [15] (see also [14]).

Lemma 4.2. *Let $v_h \in S_{hp}^n \cap H_D^1(\Omega)$, $\varphi \in H_D^1(\Omega)$, and $\chi \in (H^1(\Omega))^k$ ($k = 1$ for $d = 2$ and $k = 3$ for $d = 3$) such that $\mathbf{n} \cdot \operatorname{curl} \chi = 0$ on $\partial\Omega_N$. Then the error e^n satisfies*

$$\sum_{K \in \widetilde{\mathcal{T}}_h^n} \int_K \nabla e^n \cdot \nabla v_h \,\mathrm{d}\mathbf{x} = \int_\Omega \frac{e^{n-1} - e^n}{\tau_n} v_h \,\mathrm{d}\mathbf{x} + \theta \sum_{\Gamma \in \widetilde{\mathcal{F}}_h^{n,\mathrm{I}}} \int_\Gamma \langle \nabla v_h \cdot \mathbf{n} \rangle [u_h^n] \,\mathrm{d}S, \tag{4.10}$$

$$\begin{aligned} \sum_{K \in \widetilde{\mathcal{T}}_h^n} \int_K \nabla e^n \cdot \nabla \varphi \,\mathrm{d}\mathbf{x} = \int_\Omega \Big(f^n - \frac{u^n - u^{n-1}}{\tau_n} \Big) \varphi \,\mathrm{d}\mathbf{x} - \sum_{K \in \widetilde{\mathcal{T}}_h^n} \int_{\partial K} \nabla u_h^n \cdot \mathbf{n} \varphi \,\mathrm{d}S \\ + \sum_{K \in \widetilde{\mathcal{T}}_h^n} \int_K \Delta u_h^n \varphi \,\mathrm{d}\mathbf{x} + \int_{\partial\Omega_N} g_N^n \varphi \,\mathrm{d}S, \end{aligned} \tag{4.11}$$

$$\sum_{K \in \widetilde{\mathcal{T}}_h^n} \int_K \nabla (e^n - \varphi) \cdot \operatorname{curl} \chi \,\mathrm{d}\mathbf{x} = \sum_{K \in \widetilde{\mathcal{T}}_h^n} \int_{\partial K \setminus \partial\Omega_N} (e^n - \varphi) \operatorname{curl} \chi \cdot \mathbf{n} \,\mathrm{d}S. \tag{4.12}$$

4.3. Auxiliary results. In this section we state some auxiliary assertions. Their versions for $p = 1$ have been proved in [15]. Let $z \in H^s(\Omega, \widetilde{\mathcal{T}}_h^n)$ and $g\colon \partial\Omega_D \to \mathbb{R}$, we introduce the following notation:

$$\Big(J(z)^g_{\pm\frac{1}{2}, \widetilde{\mathcal{F}}_h^n} \Big)^2 := \sum_{\Gamma \in \widetilde{\mathcal{F}}_h^{n,\mathrm{I}}} h_\Gamma^{\pm 1} \| [z] \|_\Gamma^2 + \sum_{\Gamma \in \widetilde{\mathcal{F}}_h^{n,\mathrm{D}}} h_\Gamma^{\pm 1} \| z - g \|_\Gamma^2. \tag{4.13}$$

The constant c occurring in the estimates hereafter is a generic positive constant which can differ from formula to formula and is independent of h and τ.

Lemma 4.3. *Let $w \in H^1(\Omega, \widetilde{\mathcal{T}}_h^n)$ and Π_{hp} be given by* (4.4)*, then, for $i = 0, 1$,*

$$\sum_{\Gamma \in \widetilde{\mathcal{F}}_h^{n,\mathrm{ID}}} h_\Gamma^{1-2i} \| [\Pi_{hp} w] \|_\Gamma^2 \leqslant c \Big(\sum_{K \in \widetilde{\mathcal{T}}_h^n} h_K^{-2i+2} |w|_{1,K}^2 + \Big(J(w)^0_{\frac{1}{2}-i, \widetilde{\mathcal{F}}_h^n} \Big)^2 \Big). \tag{4.14}$$

P r o o f. The following sequence of inequalities holds:

$$\begin{aligned} \sum_{\Gamma \in \widetilde{\mathcal{F}}_h^{n,\mathrm{ID}}} h_\Gamma^{1-2i} \| [\Pi_{hp} w] \|_\Gamma^2 &\leqslant c \sum_{K \in \widetilde{\mathcal{T}}_h^n} h_K^{1-2i} \| \Pi_{hp} w - w \|_{\partial K}^2 + c \sum_{\Gamma \in \widetilde{\mathcal{F}}_h^{n,\mathrm{ID}}} h_\Gamma^{1-2i} \| [w] \|_\Gamma^2 \\ &\leqslant c \sum_{K \in \widetilde{\mathcal{T}}_h^n} h_K^{-2i+2} |w|_{1,K}^2 + c \sum_{\Gamma \in \widetilde{\mathcal{F}}_h^{n,\mathrm{I}}} h_\Gamma^{1-2i} \| [w] \|_\Gamma^2 + c \sum_{\Gamma \in \widetilde{\mathcal{F}}_h^{n,\mathrm{D}}} h_\Gamma^{1-2i} \| w \|_\Gamma^2, \end{aligned}$$

where the first inequality follows from the triangle inequality and the local quasi-uniformity and the second one from (4.1) and (4.4). Hence, due to the definition (4.13), we have the assertion. □

Lemma 4.4. *Let* $w \in H^1(\Omega, \widetilde{\mathcal{T}}_h^n)$ *and* $I_h^{n,D}$ *be given by* (4.7), *then*

$$\sum_{\Gamma \in \widetilde{\mathcal{F}}_h^{n,\mathrm{I}}} h_\Gamma \|\langle \nabla I_h^{n,D}(w) \cdot \mathbf{n} \rangle\|_\Gamma^2 \leqslant c\Big(|w|^2_{H^1(\Omega,\widetilde{\mathcal{T}}_h^n)} + \Big(J(w)^0_{-\frac{1}{2},\widetilde{\mathcal{F}}_h^n}\Big)^2\Big). \tag{4.15}$$

P r o o f. See [15, Lemma 4]. □

Corollary 4.1. *Let* g_D *be the restriction of a function from* $S_{hp}^n \cap H^1(\Omega)$ *to* $\partial\Omega_D$. *Further, let* $v \in H^1_{g_D,D}(\Omega)$ *and* $z \in S_{hp}^n$ *be arbitrary. Let* e^n *and* φ^n *be from* (4.9). *Then*

$$\sum_{\Gamma \in \widetilde{\mathcal{F}}_h^{n,\mathrm{ID}}} h_\Gamma^{1-2i} \|[\Pi_{hp}(v-z)]\|_\Gamma^2 \leqslant c\Big(\sum_{K \in \widetilde{\mathcal{T}}_h^n} h_K^{2-2i} |v-z|^2_{1,K} + \Big(J(z)^{g_D}_{\frac{1}{2}-i,\widetilde{\mathcal{F}}_h^n}\Big)^2\Big), \tag{4.16}$$

$$\sum_{\Gamma \in \widetilde{\mathcal{F}}_h^{n,\mathrm{I}}} h_\Gamma \|\langle \nabla I_h^{n,D}(v-z) \cdot \mathbf{n} \rangle\|_\Gamma^2 \leqslant c\Big(|v-z|^2_{H^1(\Omega,\widetilde{\mathcal{T}}_h^n)} + \Big(J(z)^{g_D}_{-\frac{1}{2},\widetilde{\mathcal{F}}_h^n}\Big)^2\Big), \tag{4.17}$$

$$\sum_{K \in \widetilde{\mathcal{T}}_h^n} \|\nabla I_h^{n,D}(e^n - \varphi^n)\|_K^2 \leqslant c\Big(|e^n - \varphi^n|^2_{H^1(\Omega,\widetilde{\mathcal{T}}_h^n)} + \Big(J(u_h^n)^{u_D^n}_{-\frac{1}{2},\widetilde{\mathcal{F}}_h^n}\Big)^2\Big). \tag{4.18}$$

P r o o f. The estimates (4.16) and (4.17) follow directly from (4.14) and (4.15) where we put $w := v - z$ and use that fact that $J(v-z)^0_{-\frac{1}{2},\widetilde{\mathcal{F}}_h^n} = J(z)^{g_D}_{-\frac{1}{2},\widetilde{\mathcal{F}}_h^n}$ since $v = g_D$ on $\partial\Omega_D$. The estimate (4.18) follows by adding and subtracting $\nabla\Pi_{hp}(e^n - \varphi^n)$ in the norm on the left-hand side of (4.18), applying approximation properties of the averaging operator $\mathcal{I}_{\mathrm{Av}}^{\mathrm{D}}$, and using the fact that $J(e^n - \varphi^n)^0_{-\frac{1}{2},\widetilde{\mathcal{F}}_h^n} = J(u_h^n)^{u_D^n}_{-\frac{1}{2},\widetilde{\mathcal{F}}_h^n}$ since $e^n - \varphi^n = u_D^n - u_h^n$ on $\partial\Omega_D$. □

Lemma 4.5. *Let* $w^n \in H^1(\Omega, \widetilde{\mathcal{T}}_h^n)$ *and* $\varphi \in H^1_D(\Omega)$ *be arbitrary. Then*

$$\sum_{K \in \widetilde{\mathcal{T}}_h^n} \|w^n - I_h^{n,D}(w^n)\|_K^2 \leqslant c\Big(\sum_{K \in \widetilde{\mathcal{T}}_h^n} h_K^2 |w^n|^2_{1,K} + \Big(J(w^n)^0_{\frac{1}{2},\widetilde{\mathcal{F}}_h^n}\Big)^2\Big), \tag{4.19}$$

$$\sum_{K \in \widetilde{\mathcal{T}}_h^n} h_K^{-2} \|\varphi - I_h^{n,D}(\varphi)\|_K^2 \leqslant c|\varphi|^2_{1,\Omega}, \tag{4.20}$$

$$\sum_{\Gamma \in \widetilde{\mathcal{F}}_h^{n,\mathrm{IN}}} h_\Gamma^{-1} \|\varphi - I_h^{n,D}\varphi\|_\Gamma^2 \leqslant c|\varphi|^2_{1,\Omega}. \tag{4.21}$$

P r o o f. For the proof of (4.19) and (4.20) see [15, Lemma 5]. Further, we can write

$$(4.22)\quad \sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{IN}}} h_\Gamma^{-1}\|\varphi - I_h^{n,D}\varphi\|_\Gamma^2 \leqslant c\sum_{K\in\widetilde{\mathcal{T}}_h^n} h_K^{-1}\|\varphi - I_h^{n,D}\varphi\|_{\partial K}^2$$
$$\leqslant c\sum_{K\in\widetilde{\mathcal{T}}_h^n} h_K^{-1}(\|\varphi - I_h^{n,D}\varphi\|_K|\varphi - I_h^{n,D}\varphi|_{1,K} + h_K^{-1}\|\varphi - I_h^{n,D}\varphi\|_K^2)$$
$$\leqslant c\Big(\sum_{K\in\widetilde{\mathcal{T}}_h^n} h_K^{-2}\|\varphi - I_h^{n,D}\varphi\|_K^2\Big)^{1/2}\Big(\sum_{K\in\widetilde{\mathcal{T}}_h^n}|\varphi - I_h^{n,D}\varphi|_{1,K}^2\Big)^{1/2}$$
$$+ c\sum_{K\in\widetilde{\mathcal{T}}_h^n} h_K^{-2}\|\varphi - I_h^{n,D}\varphi\|_K^2,$$

where the first inequality follows from the local quasi-uniformity, the second one from (4.1), and the third one from the Cauchy-Schwarz inequality. The use of the triangle inequality, (4.4), (4.1) with $i := 1$, Lemma 4.3 with $i := 1$, and the fact that $J(\varphi)^0_{-\frac12,\widetilde{\mathcal{F}}_h^n} = 0$ yields

$$(4.23)\quad \sum_{K\in\widetilde{\mathcal{T}}_h^n}|\varphi - I_h^{n,D}\varphi|_{1,K}^2 \leqslant \sum_{K\in\widetilde{\mathcal{T}}_h^n} 2|\varphi - \Pi_{hp}(\varphi)|_{1,K}^2 + \sum_{K\in\widetilde{\mathcal{T}}_h^n} 2|\Pi_{hp}(\varphi) - I_h^{n,D}\varphi|_{1,K}^2$$
$$\leqslant c|\varphi|_{1,\Omega}^2 + c\sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{ID}}} h_\Gamma^{-1}\|[\Pi_{hp}(\varphi)]\|_\Gamma^2$$
$$\leqslant c|\varphi|_{1,\Omega}^2 + c\Big(J(\varphi)^0_{-\frac12,\widetilde{\mathcal{F}}_h^n}\Big)^2 \leqslant c|\varphi|_{1,\Omega}^2.$$

Now, the estimate (4.23) together with (4.20) applied in (4.22) gives (4.21). □

Lemma 4.6. *Let $z \in S_{hp}^n$. Then the following holds*

$$(4.24)\qquad \inf_{v\in H^1_{u_D^n,D}(\Omega)}\sum_{K\in\widetilde{\mathcal{T}}_h^n}\|v - z\|^2_{1/2,\partial K\cap\widetilde{\mathcal{F}}_h^{n,\mathrm{ID}}} \leqslant c\Big(J(z)^{u_D^n}_{-\frac12,\widetilde{\mathcal{F}}_h^n}\Big)^2,$$

$$(4.25)\qquad \inf_{v\in H^1_{u_D^n,D}(\Omega)}\sum_{K\in\widetilde{\mathcal{T}}_h^n} h_K^2\|v - z\|^2_{1/2,\partial K\cap\widetilde{\mathcal{F}}_h^{n,\mathrm{ID}}} \leqslant c\Big(J(z)^{u_D^n}_{\frac12,\widetilde{\mathcal{F}}_h^n}\Big)^2,$$

where c is independent of h and $H^1_{u_D^n,D}(\Omega) = \{v \in H^1(\Omega);\ v = u_D^n$ on $\partial\Omega_D\}$.

P r o o f. For the proof of (4.24), see [3, Lemma 4]. The inequality (4.25) can be proved in the same way. □

Lemma 4.7. *Let φ^n and χ^n be the functions involved in the Helmholtz decomposition* (4.9). *Then there exists a constant $c > 0$ such that*

$$[\operatorname{curl}\chi^n \cdot \mathbf{n}]_\Gamma = 0 \quad \forall \Gamma \in \widetilde{\mathcal{F}}_h^{n,\mathrm{I}}, \tag{4.26}$$

$$\sum_{K\in\widetilde{\mathcal{T}}_h^n} h_K^{2i}|e^n - \varphi^n|_{1,K}^2 = \sum_{K\in\widetilde{\mathcal{T}}_h^n} h_K^{2i}\|\operatorname{curl}\chi^n\|_K^2 \leqslant c\Big(J(u_h^n)^{u_D^n}_{i-\frac12,\widetilde{\mathcal{F}}_h^n}\Big)^2, \quad i = 0,1, \tag{4.27}$$

where the trace $\operatorname{curl}\chi^n \cdot \mathbf{n}$ *on* Γ *is meant in the sense of Lemma* 4.1.

P r o o f. See [15, Lemma 8] and [15, Lemma 9]. □

Corollary 4.2. *The combination of relations* (4.19) *and* (4.27) *together with* $J(e^n - \varphi^n)^0_{\frac12,\widetilde{\mathcal{F}}_h^n} = J(u_h^n)^{u_D^n}_{\frac12,\widetilde{\mathcal{F}}_h^n}$ *gives*

$$\sum_{K\in\widetilde{\mathcal{T}}_h^n} \|(e^n - \varphi^n) - I_h^{n,D}(e^n - \varphi^n)\|_K^2 \leqslant c\Big(J(u_h^n)^{u_D^n}_{\frac12,\widetilde{\mathcal{F}}_h^n}\Big)^2. \tag{4.28}$$

5. A posteriori error estimates

5.1. Upper error bound. In this section we will state a theorem providing the upper error bound. The error is measured in the norm combining the L^2-norm on the last time level and H^1-seminorm on all time levels (except the initial one). First, we introduce some additional notation:

$$\mathrm{R}_K^n := \Big(f^n + \Delta u_h^n - \frac{u_h^n - u_h^{n-1}}{\tau_n}\Big)|_K, \quad K \in \widetilde{\mathcal{T}}_h^n, \quad (\eta_{\mathrm{R}}^n)^2 := \sum_{K\in\widetilde{\mathcal{T}}_h^n} h_K^2\|\mathrm{R}_K^n\|_K^2, \tag{5.1}$$

$$(\eta_{\mathrm{J}}^n)^2 := \sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{I}}} h_\Gamma^{-1}\|[u_h^n]\|_\Gamma^2, \quad (\eta_{\mathrm{JD}}^n)^2 := \sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{D}}} h_\Gamma^{-1}\|u_h^n - u_D^n\|_\Gamma^2,$$

$$(\eta_{\mathrm{Jd}}^n)^2 := \sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{I}}} h_\Gamma\|[\mathbf{n}\cdot\nabla u_h^n]\|_\Gamma^2, \quad (\eta_{\mathrm{JdN}}^n)^2 := \sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{N}}} h_\Gamma\|g_N^n - \mathbf{n}\cdot\nabla u_h^n\|_\Gamma^2.$$

For time level $n \geqslant 1$ we define the *local error indicators*

$$\begin{aligned}\eta_{K,1}^n = {}& h_K\|\mathrm{R}_K^n\|_K + \sum_{\Gamma\in\widetilde{\mathcal{F}}_K^{n,\mathrm{N}}} h_\Gamma^{1/2}\|g_N^n - \mathbf{n}\cdot\nabla u_h^n\|_\Gamma + \sum_{\Gamma\in\widetilde{\mathcal{F}}_K^{n,\mathrm{I}}} h_\Gamma^{1/2}\|[\mathbf{n}\cdot\nabla u_h^n]\|_\Gamma \\ &+ \sum_{\Gamma\in\widetilde{\mathcal{F}}_K^{n,\mathrm{I}}} h_\Gamma^{-1/2}\|[u_h^n]\|_\Gamma + \sum_{\Gamma\in\widetilde{\mathcal{F}}_K^{n,\mathrm{D}}} h_\Gamma^{-1/2}\|u_D^n - u_h^n\|_\Gamma, \\ \eta_{K,2}^n = {}& \sum_{\Gamma\in\widetilde{\mathcal{F}}_K^{n,\mathrm{I}}} h_\Gamma^{1/2}\|[u_h^n]\|_\Gamma + \sum_{\Gamma\in\widetilde{\mathcal{F}}_K^{n,\mathrm{D}}} h_\Gamma^{1/2}\|u_D^n - u_h^n\|_\Gamma,\end{aligned} \tag{5.2}$$

where $\widetilde{\mathcal{F}}_K^{n,\mathrm{I}}$, $\widetilde{\mathcal{F}}_K^{n,\mathrm{N}}$, and $\widetilde{\mathcal{F}}_K^{n,\mathrm{D}}$ denote the set of all interior faces of an element K, faces on $\partial\Omega_N \cap \partial K$, and faces on $\partial\Omega_D \cap \partial K$, respectively (for $d = 2$ faces are replaced by edges). The indicators reflect the residual of the equation, the jump in the boundary conditions, the interelement jumps of the approximate solution and the jump of its normal component of the gradient.

Now, we state an upper estimate on the error.

Theorem 5.1. *Let* $\{u^n\}_{1\leqslant n\leqslant \overline{N}}$ *and* $\{u_h^n\}_{1\leqslant n\leqslant \overline{N}}$ *be the semi-discrete and approximate solutions given by* (3.1) *and* (3.5), *respectively. Let* $1 \leqslant N \leqslant \overline{N}$. *Then the error* e^n *satisfies*

$$\|e^N\|_\Omega^2 + \sum_{n=1}^{N} \tau_n \sum_{K\in\widetilde{\mathcal{T}}_h^n} \|\nabla e^n\|_K^2 \leqslant \|e^0\|_\Omega^2 + \sum_{n=1}^{N} C\Big(\tau_n \sum_{K\in\widetilde{\mathcal{T}}_h^n} (\eta_{K,1}^n)^2 + \sum_{K\in\widetilde{\mathcal{T}}_h^n} (\eta_{K,2}^n)^2\Big),$$

where the constant C *is independent of the mesh parameter and the time step.*

P r o o f. According to (4.9), we can write

$$(5.3)\qquad \tau_n \sum_{K\in\widetilde{\mathcal{T}}_h^n} \|\nabla e^n\|_K^2 = \tau_n \sum_{K\in\widetilde{\mathcal{T}}_h^n} \int_K \nabla e^n \cdot \nabla\varphi^n \,\mathrm{d}\mathbf{x} + \tau_n \sum_{K\in\widetilde{\mathcal{T}}_h^n} \int_K \nabla e^n \operatorname{curl}\chi^n \,\mathrm{d}\mathbf{x}.$$

We denote ψ_1 and ψ_2 the two terms on the right-hand side of (5.3). Setting $\varphi := \varphi^n$ in (4.11) multiplied by τ_n and adding a τ_n-multiple of (4.10) with $v_h := I_h^{n,D}\varphi^n$ yields

$$\begin{aligned}(5.4)\qquad \psi_1 = {} & \tau_n \int_\Omega \Big(f^n - \frac{u^n - u^{n-1}}{\tau_n}\Big)\varphi^n \,\mathrm{d}\mathbf{x} - \tau_n \sum_{K\in\widetilde{\mathcal{T}}_h^n} \int_{\partial K} \nabla u_h^n \cdot \mathbf{n}\varphi^n \,\mathrm{d}S \\ & + \tau_n \sum_{K\in\widetilde{\mathcal{T}}_h^n} \int_K \Delta u_h^n \varphi^n \,\mathrm{d}\mathbf{x} + \tau_n \int_{\partial\Omega_N} g_N^n \varphi^n \,\mathrm{d}S \\ & - \tau_n \sum_{K\in\widetilde{\mathcal{T}}_h^n} \int_K \nabla e^n \cdot \nabla I_h^{n,D}\varphi^n \,\mathrm{d}\mathbf{x} + \tau_n \int_\Omega \frac{e^{n-1} - e^n}{\tau_n} I_h^{n,D}\varphi^n \,\mathrm{d}\mathbf{x} \\ & + \tau_n\theta \sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{I}}} \int_\Gamma \langle \nabla I_h^{n,D}\varphi^n \cdot \mathbf{n}\rangle [u_h^n] \,\mathrm{d}S.\end{aligned}$$

By expressing the term $-\tau_n \sum\limits_{K\in\widetilde{\mathcal{T}}_h^n} \int_K \nabla e^n \cdot \nabla I_h^{n,D}\varphi^n \,\mathrm{d}\mathbf{x}$ according to the identity (4.11), adding and subtracting the term $\tau_n \sum\limits_{K\in\widetilde{\mathcal{T}}_h^n} \int_K (f^n - (u_h^n - u_h^{n-1})/\tau_n)\varphi^n \,\mathrm{d}\mathbf{x}$, and reordering the terms we have

$$\begin{aligned}(5.5)\qquad \psi_1 = \tau_n \sum_{K\in\widetilde{\mathcal{T}}_h^n} \int_K \mathrm{R}_K^n(\varphi^n - I_h^{n,D}\varphi^n)\,\mathrm{d}\mathbf{x} - \sum_{K\in\widetilde{\mathcal{T}}_h^n} \int_K (e^n - e^{n-1})\varphi^n\,\mathrm{d}\mathbf{x}\\ - \tau_n \sum_{K\in\widetilde{\mathcal{T}}_h^n} \int_{\partial K} \nabla u_h^n\cdot\mathbf{n}(\varphi^n - I_h^{n,D}\varphi^n)\,\mathrm{d}S\\ + \tau_n \int_{\partial\Omega_N} g_N^n(\varphi^n - I_h^{n,D}\varphi^n)\,\mathrm{d}S\\ + \tau_n\theta \sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{I}}} \int_\Gamma \langle\nabla I_h^{n,D}\varphi^n\cdot\mathbf{n}\rangle[u_h^n]\,\mathrm{d}S.\end{aligned}$$

Putting (5.5) into (5.3), expressing ψ_2 with the aid of (4.12), and adding the terms $\pm \sum_{K\in\widetilde{\mathcal{T}}_h^n} \int_K (e^n - e^{n-1})(e^n - I_h^{n,D}(e^n-\varphi^n))\,\mathrm{d}\mathbf{x}$, we obtain

$$\begin{aligned}(5.6)\quad \tau_n \sum_{K\in\widetilde{\mathcal{T}}_h^n} \|\nabla e^n\|_K^2\\ = \sum_{K\in\widetilde{\mathcal{T}}_h^n} \int_K (e^n - e^{n-1})((e^n-\varphi^n) - I_h^{n,D}(e^n-\varphi^n))\,\mathrm{d}\mathbf{x} \quad (=: \xi_1)\\ + \sum_{K\in\widetilde{\mathcal{T}}_h^n} \int_K (e^n - e^{n-1}) I_h^{n,D}(e^n-\varphi^n)\,\mathrm{d}\mathbf{x} \quad (=: \xi_2)\\ + \sum_{K\in\widetilde{\mathcal{T}}_h^n} \int_K e^n e^{n-1}\,\mathrm{d}\mathbf{x} - \sum_{K\in\widetilde{\mathcal{T}}_h^n} \|e^n\|_K^2\,\mathrm{d}\mathbf{x}\\ + \tau_n \sum_{K\in\widetilde{\mathcal{T}}_h^n} \int_K \mathrm{R}_K^n(\varphi^n - I_h^{n,D}\varphi^n)\,\mathrm{d}\mathbf{x} \quad (=: \xi_3)\\ - \tau_n \sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{I}}} \int_\Gamma [\nabla u_h^n\cdot\mathbf{n}](\varphi^n - I_h^{n,D}\varphi^n)\,\mathrm{d}S \quad (=: \xi_4)\\ + \tau_n\theta \sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{I}}} \int_\Gamma \langle\nabla I_h^{n,D}\varphi^n\cdot\mathbf{n}\rangle[u_h^n]\,\mathrm{d}S \quad (=: \xi_5)\\ + \tau_n \sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{N}}} \int_\Gamma (g_N^n - \nabla u_h^n\cdot\mathbf{n})(\varphi^n - I_h^{n,D}\varphi^n)\,\mathrm{d}S \quad (=: \xi_6)\\ + \tau_n \sum_{K\in\widetilde{\mathcal{T}}_h^n} \int_{\partial K\setminus\partial\Omega_N} e^n \operatorname{curl}\chi^n\cdot\mathbf{n}\,\mathrm{d}S \quad (=: \xi_7).\end{aligned}$$

Now, we have to estimate all the terms in (5.6). The Cauchy-Schwarz inequality and (4.28) yield

$$(5.7)\qquad \begin{aligned}|\xi_1|^2 &\leqslant \sum_{K\in\widetilde{\mathcal{T}}_h^n}\|e^n-e^{n-1}\|_K^2 \sum_{K\in\widetilde{\mathcal{T}}_h^n}\|(e^n-\varphi^n)-I_h^{n,D}(e^n-\varphi^n)\|_K^2\\ &\leqslant c\|e^n-e^{n-1}\|_\Omega^2\Big(J(u_h^n)^{u_D^n}_{\frac{1}{2},\widetilde{\mathcal{F}}_h^n}\Big)^2.\end{aligned}$$

We express ξ_2 with the aid of (4.10).

$$\begin{aligned}\xi_2 = \tau_n\theta &\sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{I}}}\int_\Gamma \langle\nabla I_h^{n,D}(e^n-\varphi^n)\cdot\mathbf{n}\rangle[u_h^n]\,\mathrm{d}S \quad (=:\xi_{2a})\\ &-\tau_n\sum_{K\in\widetilde{\mathcal{T}}_h^n}\int_K \nabla e^n\nabla I_h^{n,D}(e^n-\varphi^n)\,\mathrm{d}\mathbf{x}\quad (=:\xi_{2b})\end{aligned}$$

An application of the Cauchy-Schwarz inequality, (5.1), and (4.17) with settings $v:=u^n-\varphi^n$, $z:=u_h^n$, $g_D:=u_D^n$ yield

$$(5.8)\quad \begin{aligned}|\xi_{2a}|^2 &\leqslant \tau_n^2\Big(\sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{I}}} h_\Gamma\|\langle\nabla I_h^{n,D}(e^n-\varphi^n)\cdot\mathbf{n}\rangle\|_\Gamma^2\Big)\Big(\sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{I}}} h_\Gamma^{-1}\|[u_h^n]\|_\Gamma^2\Big)\\ &\leqslant \tau_n^2\eta_{\mathrm{J}}^n c\Big(|e^n-\varphi^n|^2_{H^1(\Omega,\widetilde{\mathcal{T}}_h^n)}+\Big(J(u_h^n)^{u_D^n}_{-\frac{1}{2},\widetilde{\mathcal{F}}_h^n}\Big)^2\Big).\end{aligned}$$

Furthermore, the Cauchy-Schwarz inequality and (4.18) give

$$(5.9)\qquad \begin{aligned}|\xi_{2b}|^2 &\leqslant \tau_n^2|e^n|^2_{H^1(\Omega,\widetilde{\mathcal{T}}_h^n)}\sum_{K\in\widetilde{\mathcal{T}}_h^n}\|\nabla I_h^{n,D}(e^n-\varphi^n)\|_K^2\\ &\leqslant c\tau_n^2|e^n|^2_{H^1(\Omega,\widetilde{\mathcal{T}}_h^n)}\Big(|e^n-\varphi^n|^2_{H^1(\Omega,\widetilde{\mathcal{T}}_h^n)}+\Big(J(u_h^n)^{u_D^n}_{-\frac{1}{2},\widetilde{\mathcal{F}}_h^n}\Big)^2\Big).\end{aligned}$$

Further, the Cauchy-Schwarz inequality and (4.20) give

$$(5.10)\qquad |\xi_3|^2\leqslant \tau_n^2(\eta_{\mathrm{R}}^n)^2\sum_{K\in\widetilde{\mathcal{T}}_h^n}h_K^{-2}\|\varphi^n-I_h^{n,D}\varphi^n\|_K^2\leqslant c\tau_n^2(\eta_{\mathrm{R}}^n)^2|\varphi^n|^2_{1,\Omega}.$$

Furthermore, the Cauchy-Schwarz inequality and (4.21) yield

$$(5.11)\qquad |\xi_4|^2\leqslant \tau_n^2(\eta_{\mathrm{Jd}}^n)^2\sum_{\widetilde{\mathcal{F}}_h^{n,\mathrm{I}}}h_\Gamma^{-1}\|\varphi^n-I_h^{n,D}\varphi^n\|_\Gamma^2\leqslant c\tau_n^2(\eta_{\mathrm{Jd}}^n)^2|\varphi^n|^2_{1,\Omega}.$$

Similarly, the Cauchy-Schwarz inequality, the estimate $|\theta| \leqslant 1$, and (4.17) with settings $v := \varphi^n$, $z := 0$, $g_D := 0$ give

$$(5.12)\qquad \begin{aligned} |\xi_5|^2 &\leqslant \tau_n^2 \Big(\sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{I}}} h_\Gamma \| \langle \nabla I_h^{n,D} \varphi^n \cdot \mathbf{n} \rangle \|_\Gamma^2 \Big) \Big(\sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{I}}} h_\Gamma^{-1} \| [u_h^n] \|_\Gamma^2 \Big) \\ &\leqslant c\tau_n^2 (\eta_\mathrm{J}^n)^2 |\varphi^n|_{1,\Omega}^2. \end{aligned}$$

Again, the Cauchy-Schwarz inequality and (4.21) imply

$$(5.13)\qquad |\xi_6|^2 \leqslant \tau_n^2 (\eta_\mathrm{JdN}^n)^2 \sum_{\widetilde{\mathcal{F}}_h^{n,\mathrm{N}}} h_\Gamma^{-1} \| \varphi^n - I_h^{n,D} \varphi^n \|_\Gamma^2 \leqslant c\tau_n^2 (\eta_\mathrm{JdN}^n)^2 |\varphi^n|_{1,\Omega}^2.$$

Due to Lemma 4.7, u^n can be substituted for any function $v \in H^1_{u_D^n,D}(\Omega)$ in ξ_7 as follows:

$$|\xi_7| = \tau_n \Big| \sum_{K\in\widetilde{\mathcal{T}}_h^n} \int_{\partial K\setminus\partial\Omega_N} (v - u_h^n) \operatorname{curl} \chi^n \cdot \mathbf{n} \, \mathrm{d}S \Big|,$$

which together with the Cauchy-Schwarz inequality and (4.24) yield

$$(5.14)\qquad |\xi_7|^2 \leqslant c\tau_n^2 \Big(J(u_h^n)^{u_D^n}_{-\frac{1}{2},\widetilde{\mathcal{F}}_h^n} \Big)^2 \sum_{K\in\widetilde{\mathcal{T}}_h^n} \| \operatorname{curl} \chi^n \cdot \mathbf{n} \|^2_{-1/2,\partial K\setminus\partial\Omega_N}.$$

Finally, using the trace inequality (4.3) in (5.14) leads to

$$(5.15)\qquad |\xi_7|^2 \leqslant c\tau_n^2 \Big(J(u_h^n)^{u_D^n}_{-\frac{1}{2},\widetilde{\mathcal{F}}_h^n} \Big)^2 \| \operatorname{curl} \chi^n \|_\Omega^2.$$

Now, the relation (5.6) with the particular estimates of ξ_l, $l = 1, \ldots, 7$, given in (5.7)–(5.15) gives

$$(5.16)\qquad \begin{aligned} &\sum_{K\in\widetilde{\mathcal{T}}_h^n} \|e^n\|_K^2 + \tau_n \sum_{K\in\widetilde{\mathcal{T}}_h^n} \|\nabla e^n\|_K^2 \\ &\quad\leqslant c\tau_n |\varphi^n|_{1,\Omega} (\eta_\mathrm{R}^n + \eta_\mathrm{JdN}^n + \eta_\mathrm{Jd}^n + \eta_\mathrm{J}^n) + c\|e^n - e^{n-1}\|_\Omega J(u_h^n)^{u_D^n}_{\frac{1}{2},\widetilde{\mathcal{F}}_h^n} \\ &\quad\quad + c(\tau_n |e^n|_{H^1(\Omega,\widetilde{\mathcal{T}}_h^n)} + \eta_\mathrm{J}^n) \Big(J(u_h^n)^{u_D^n}_{-\frac{1}{2},\widetilde{\mathcal{F}}_h^n} + |e^n - \varphi^n|_{H^1(\Omega,\widetilde{\mathcal{T}}_h^n)} \Big) \\ &\quad\quad + c\tau_n J(u_h^n)^{u_D^n}_{-\frac{1}{2},\widetilde{\mathcal{F}}_h^n} \|\operatorname{curl}\chi^n\|_\Omega + \sum_{K\in\widetilde{\mathcal{T}}_h^n} \int_K e^n e^{n-1} \, \mathrm{d}\mathbf{x}. \end{aligned}$$

Multiplying (5.16) by 2, an application of Young's inequality, and the relation

$$\|e^n - e^{n-1}\|_\Omega^2 = \|e^n\|_\Omega^2 - 2\int_\Omega e^n e^{n-1} \, \mathrm{d}\mathbf{x} + \|e^{n-1}\|_\Omega^2,$$

give

$$(5.17)\quad 2\sum_{K\in\widetilde{\mathcal{T}}_h^n}\|e^n\|_K^2+2\tau_n\sum_{K\in\widetilde{\mathcal{T}}_h^n}\|\nabla e^n\|_K^2$$
$$\leqslant \sum_{K\in\widetilde{\mathcal{T}}_h^n}\|e^n\|_K^2+\sum_{K\in\widetilde{\mathcal{T}}_h^n}\|e^{n-1}\|_K^2$$
$$+c\tau_n\Big((\eta_{\mathrm{R}}^n)^2+(\eta_{\mathrm{JdN}}^n)^2+(\eta_{\mathrm{Jd}}^n)^2+(\eta_{\mathrm{J}}^n)^2+\Big(J(u_h^n)^{u_D^n}_{-\frac12,\widetilde{\mathcal{F}}_h^n}\Big)^2\Big)$$
$$+\frac{\tau_n}{4}|\varphi^n|_{1,\Omega}^2+c\Big(\Big(J(u_h^n)^{u_D^n}_{-\frac12,\widetilde{\mathcal{F}}_h^n}\Big)^2+\Big(J(u_h^n)^{u_D^n}_{\frac12,\widetilde{\mathcal{F}}_h^n}\Big)^2+(\eta_{\mathrm{J}}^n)^2\Big)$$
$$+\frac{\tau_n}{2}|e^n|^2_{H^1(\Omega,\widetilde{\mathcal{T}}_h^n)}+c\tau_n(|e^n-\varphi^n|^2_{H^1(\Omega,\widetilde{\mathcal{T}}_h^n)}+\|\operatorname{curl}\chi^n\|_\Omega^2).$$

Moving some terms from the right-hand side of (5.17), using (4.27), and

$$|\varphi^n|^2_{H^1(\Omega,\widetilde{\mathcal{T}}_h^n)}\leqslant 2|\varphi^n-e^n|^2_{H^1(\Omega,\widetilde{\mathcal{T}}_h^n)}+2|e^n|^2_{H^1(\Omega,\widetilde{\mathcal{T}}_h^n)},$$

we derive

$$\sum_{K\in\widetilde{\mathcal{T}}_h^n}\|e^n\|_K^2+\tau_n\sum_{K\in\widetilde{\mathcal{T}}_h^n}\|\nabla e^n\|_K^2$$
$$\leqslant\sum_{K\in\widetilde{\mathcal{T}}_h^n}\|e^{n-1}\|_K^2+c\tau_n\Big((\eta_{\mathrm{R}}^n)^2+(\eta_{\mathrm{JdN}}^n)^2+(\eta_{\mathrm{Jd}}^n)^2+(\eta_{\mathrm{J}}^n)^2+\Big(J(u_h^n)^{u_D^n}_{-\frac12,\widetilde{\mathcal{F}}_h^n}\Big)^2\Big)$$
$$+c\Big(\Big(J(u_h^n)^{u_D^n}_{-\frac12,\widetilde{\mathcal{F}}_h^n}\Big)^2+\Big(J(u_h^n)^{u_D^n}_{\frac12,\widetilde{\mathcal{F}}_h^n}\Big)^2+(\eta_{\mathrm{J}}^n)^2\Big),$$

which together with the definitions (5.2), (4.13), and (5.1) finally yields

$$\|e^n\|_\Omega^2+\tau_n\sum_{K\in\widetilde{\mathcal{T}}_h^n}\|\nabla e^n\|_K^2\leqslant\|e^{n-1}\|_\Omega^2+c\Big(\tau_n\sum_{K\in\widetilde{\mathcal{T}}_h^n}(\eta_{K,1}^n)^2+\sum_{K\in\widetilde{\mathcal{T}}_h^n}(\eta_{K,2}^n)^2\Big).$$

Summing over $n=1,\dots,\overline{N}$, we come to the assertion of the theorem. □

5.2. Lower error bound. What is different in comparison with the conforming methods are the jump terms appearing in the indicator (5.2). Several techniques how to deal with this issue have been developed. The article [5] estimates the jump terms using only the discrete scheme itself. Another way how to do that has been developed in [1]. There it has been proved that if the penalty parameter is sufficiently large, the interelement jumps of the approximate solution are subordinated to the error in the broken H^1-seminorm. At last, a completely different approach has been

carried out in [10], where the continuous Galerkin approximation is compared to the discontinuous Galerkin approximation to derive an estimate of the jump terms.
The remaining terms in local error indicators (5.2), i.e. the residual and the normal jumps of the approximate solution, can be estimated in a standard way using suitable cut-off functions (see, e.g., [8], [12], [17]).

In this section we will derive the lower error bound using techniques as in [5] and [12]. Recall that f, g_N, and u_D are assumed to be polynomial functions at each time step t^n.

Theorem 5.2. *Let* $\{u^n\}_{1\leqslant n\leqslant \overline{N}}$ *and* $\{u_h^n\}_{1\leqslant n\leqslant \overline{N}}$ *be the semi-discrete and approximate solutions given by* (3.1) *and* (3.5), *respectively. Then*

$$\begin{aligned} h_K\|\mathrm{R}_K^n\|_K &\leqslant c(h_K\tau_n^{-1}\|e^n-e^{n-1}\|_K+|e^n|_{1,K}), \quad K\in\widetilde{\mathcal{T}}_h^n,\\ h_\Gamma^{1/2}\|[\mathbf{n}\cdot\nabla u_h^n]\|_\Gamma &\leqslant c(|e^n|_{1,K_\Gamma^{\mathrm{L}}\cup K_\Gamma^{\mathrm{R}}}+h_\Gamma\tau_n^{-1}\|e^n-e^{n-1}\|_{K_\Gamma^{\mathrm{L}}\cup K_\Gamma^{\mathrm{R}}}), \quad \Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{I}},\\ h_\Gamma^{1/2}\|g_N^n-\nabla u_h^n\cdot\mathbf{n}\|_\Gamma &\leqslant c(|e^n|_{1,K}+h_\Gamma\tau_n^{-1}\|e^n-e^{n-1}\|_K), \quad \Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{N}},\\ C_W J(u_h^n)^2_{-\frac12,\widetilde{\mathcal{F}}_h^n} &\leqslant c((\eta_{\mathrm{R}}^n)^2+(\eta_{\mathrm{JdN}}^n)^2+(\eta_{\mathrm{Jd}}^n)^2), \end{aligned}$$

where C_W *is a sufficiently large constant, specified later, involved in the formula for the penalty parameter* σ, *and the constant* c *is independent of the mesh parameter and the time step.*

Proof. First, according to (4.11) and (4.12) we can write

$$\begin{aligned} \sum_{K\in\widetilde{\mathcal{T}}_h^n}\int_K\nabla e^n(\nabla\varphi+\operatorname{curl}\chi)\,\mathrm{d}\mathbf{x}+\int_\Omega\frac{e^n-e^{n-1}}{\tau_n}\varphi\,\mathrm{d}\mathbf{x} \\ =\sum_{K\in\widetilde{\mathcal{T}}_h^n}\int_K\mathrm{R}_K^n\varphi\,\mathrm{d}\mathbf{x}-\sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{I}}}\int_\Gamma[\nabla u_h^n\cdot\mathbf{n}]\varphi\,\mathrm{d}S \\ +\int_{\partial\Omega_N}(g_N^n-\nabla u_h^n\cdot\mathbf{n})\varphi\,\mathrm{d}S+\sum_{K\in\widetilde{\mathcal{T}}_h^n}\int_{\partial K\setminus\partial\Omega_N}e^n\operatorname{curl}\chi\cdot\mathbf{n}\,\mathrm{d}S \end{aligned} \tag{5.18}$$

for $\varphi\in H_D^1(\Omega)$ and $\chi\in(H^1(\Omega))^k$ ($k=1$ for $d=2$ and $k=3$ for $d=3$).

Let b_K be a standard interior bubble function supported on the element K (see, e.g., [17]). There exists a constant $c>0$ such that the inequality $\int_K(\mathrm{R}_K^n)^2\,\mathrm{d}\mathbf{x}\leqslant c\int_K\mathrm{R}_K^n b_K\mathrm{R}_K^n\,\mathrm{d}\mathbf{x}$ holds, because $\bigl(\int_K(\cdot)^2 b_K\,\mathrm{d}\mathbf{x}\bigr)^{1/2}$ is a norm on $L^2(K)$ ($b_K>0$ on the interior of K), equivalent to the L^2 norm on $P^p(K)$. Let us fix an arbitrary $K\in\widetilde{\mathcal{T}}_h^n$. Setting $\varphi|_K:=b_K\mathrm{R}_K^n$, $\varphi:=0$ outside of K, and $\chi:=0$ in (5.18), yields

$$\int_K\mathrm{R}_K^n b_K\mathrm{R}_K^n\,\mathrm{d}\mathbf{x}=\int_K\frac{e^n-e^{n-1}}{\tau_n}b_K\mathrm{R}_K^n\,\mathrm{d}\mathbf{x}+\int_K\nabla e^n\cdot\nabla(b_K\mathrm{R}_K^n)\,\mathrm{d}\mathbf{x}. \tag{5.19}$$

Now, applying the Cauchy-Schwarz inequality together with the inverse inequality $|b_K \mathrm{R}_K^n|_{1,K} \leqslant h_K^{-1} \|\mathrm{R}_K^n\|_K$ in (5.19), we obtain

$$\|\mathrm{R}_K^n\|_K^2 \leqslant c(\tau_n^{-1}\|e^n - e^{n-1}\|_K + |e^n|_{1,K} h_K^{-1})\|\mathrm{R}_K^n\|_K. \tag{5.20}$$

Dividing (5.20) by $\|\mathrm{R}_K^n\|_K$ and multiplying it by h_K finally yields

$$h_K\|\mathrm{R}_K^n\|_K \leqslant c(h_K\tau_n^{-1}\|e^n - e^{n-1}\|_K + |e^n|_{1,K}). \tag{5.21}$$

Let us fix an arbitrary $\Gamma \in \widetilde{\mathcal{F}}_h^{n,\mathrm{I}}$. Let b_Γ be a standard edge bubble function supported on the elements K_Γ^L and K_Γ^R (see, e.g., [17]). Let $\mathbf{x}_\Gamma$, V_Γ^L, and V_Γ^R denote the barycenter of Γ, the vertex of K_Γ^L opposite to Γ, and the vertex of K_Γ^R opposite to Γ, respectively. Then, v_Γ^n is defined as the extension of $[\mathbf{n}\cdot\nabla u_h^n]|_\Gamma$ to $K_\Gamma^\mathrm{L} \cup K_\Gamma^\mathrm{R}$ such that it is constant along the lines $\overrightarrow{\mathbf{x}_\Gamma V_\Gamma^\mathrm{L}}$ and $\overrightarrow{\mathbf{x}_\Gamma V_\Gamma^\mathrm{R}}$, respectively, see Fig. 1 for the two-dimensional case. Setting $\varphi|_{K_\Gamma^\mathrm{L}\cup K_\Gamma^\mathrm{R}} := b_\Gamma v_\Gamma^n$, $\varphi := 0$ outside of $K_\Gamma^\mathrm{L}\cup K_\Gamma^\mathrm{R}$, and $\chi := 0$ in (5.18), yields

$$\begin{aligned}\int_\Gamma [\mathbf{n}\cdot\nabla u_h^n] b_\Gamma [\mathbf{n}\cdot\nabla u_h^n]\,\mathrm{d}S \\ = \int_{K_\Gamma^\mathrm{L}\cup K_\Gamma^\mathrm{R}} \mathrm{R}_K^n b_\Gamma v_\Gamma^n\,\mathrm{d}\mathbf{x} - \int_{K_\Gamma^\mathrm{L}\cup K_\Gamma^\mathrm{R}} \nabla e^n\cdot\nabla(b_\Gamma v_\Gamma^n)\,\mathrm{d}\mathbf{x} \\ - \int_{K_\Gamma^\mathrm{L}\cup K_\Gamma^\mathrm{R}} \frac{e^n - e^{n-1}}{\tau_n} b_\Gamma v_\Gamma^n\,\mathrm{d}\mathbf{x}.\end{aligned} \tag{5.22}$$

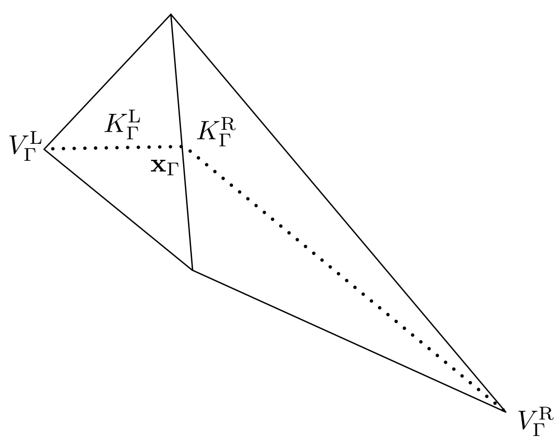

Figure 1. The extension of $[\mathbf{n}\cdot\nabla u_h^n]|_\Gamma$ on elements sharing Γ.

Notice that the use of the inverse inequality and the definition of the extension v_Γ^n yield

$$\sum_{K\in\{K_\Gamma^{\rm L},K_\Gamma^{\rm R}\}} |b_\Gamma v_\Gamma^n|_{1,K} \leqslant c, \quad \sum_{K\in\{K_\Gamma^{\rm L},K_\Gamma^{\rm R}\}} h_K^{-1}\|b_\Gamma v_\Gamma^n\|_K \leqslant c h_\Gamma^{-1/2}\|[\mathbf{n}\cdot\nabla u_h^n]\|_\Gamma. \tag{5.23}$$

Now, applying the Cauchy-Schwarz inequality and (5.23) in (5.22) and using similar arguments as above for the validity of $\int_\Gamma[\mathbf{n}\cdot\nabla u_h^n]^2\,\mathrm{d}S \leqslant c\int_\Gamma b_\Gamma[\mathbf{n}\cdot\nabla u_h^n]^2\,\mathrm{d}S$ leads to

$$\begin{aligned}\|[\mathbf{n}\cdot\nabla u_h^n]\|_\Gamma^2 \leqslant c\|[\mathbf{n}\cdot\nabla u_h^n]\|_\Gamma(&\|\mathrm{R}_K^n\|_{K_\Gamma^{\rm L}\cup K_\Gamma^{\rm R}} h_\Gamma^{1/2} \\ &+ |e^n|_{1,K_\Gamma^{\rm L}\cup K_\Gamma^{\rm R}} h_\Gamma^{-1/2} + \tau_n^{-1}\|e^n-e^{n-1}\|_{K_\Gamma^{\rm L}\cup K_\Gamma^{\rm R}} h_\Gamma^{1/2}).\end{aligned} \tag{5.24}$$

Dividing (5.24) by $\|[\mathbf{n}\cdot\nabla u_h^n]\|_\Gamma$ and multiplying it by $h_\Gamma^{1/2}$ finally yields

$$\begin{aligned}h_\Gamma^{1/2}\|[\mathbf{n}\cdot\nabla u_h^n]\|_\Gamma \leqslant c(&\|\mathrm{R}_K^n\|_{K_\Gamma^{\rm L}\cup K_\Gamma^{\rm R}} h_\Gamma + |e^n|_{1,K_\Gamma^{\rm L}\cup K_\Gamma^{\rm R}} \\ &+ \tau_n^{-1}\|e^n-e^{n-1}\|_{K_\Gamma^{\rm L}\cup K_\Gamma^{\rm R}} h_\Gamma).\end{aligned} \tag{5.25}$$

Analogously, it can be proved that for an arbitrary $\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{N}}$

$$h_\Gamma^{1/2}\|g_N^n - \nabla u_h^n\cdot\mathbf{n}\|_\Gamma \leqslant c(\|\mathrm{R}_K^n\|_K h_\Gamma + |e^n|_{1,K} + \tau_n^{-1}\|e^n-e^{n-1}\|_K h_\Gamma). \tag{5.26}$$

According to (5.2), it remains to estimate $\sum\limits_{\Gamma\in\widetilde{\mathcal{F}}_K^{n,\mathrm{I}}} h_\Gamma^{-1/2}\|[u_h^n]\|_\Gamma + \sum\limits_{\Gamma\in\widetilde{\mathcal{F}}_K^{n,\mathrm{D}}} h_\Gamma^{-1/2}\|u_D^n - u_h^n\|_\Gamma$. This was done in [5] as follows. Integrating by parts in the first term of a_h^n in (3.4), we have

$$\sum_{K\in\widetilde{\mathcal{T}}_h^n}\int_K \nabla u_h^n\cdot\nabla v_h\,\mathrm{d}\mathbf{x} = -\sum_{K\in\widetilde{\mathcal{T}}_h^n}\int_K \Delta u_h^n v_h\,\mathrm{d}\mathbf{x} + \sum_{K\in\widetilde{\mathcal{T}}_h^n}\int_{\partial K}\nabla u_h^n\cdot\mathbf{n}v_h\,\mathrm{d}S. \tag{5.27}$$

We can express the second term on the right-hand side of (5.27) as follows:

$$\begin{aligned}&\sum_{K\in\widetilde{\mathcal{T}}_h^n}\int_{\partial K}\nabla u_h^n\cdot\mathbf{n}v_h\,\mathrm{d}S \\ &= \sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{I}}}\int_\Gamma(\nabla u_h^n\cdot\mathbf{n}|_\Gamma^L v_h|_\Gamma^L - \nabla u_h^n\cdot\mathbf{n}|_\Gamma^R v_h|_\Gamma^R)\,\mathrm{d}S + \sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{DN}}}\int_\Gamma\nabla u_h^n\cdot\mathbf{n}v_h\,\mathrm{d}S \\ &= \sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{I}}}\int_\Gamma([\nabla u_h^n\cdot\mathbf{n}]\langle v_h\rangle + \langle\nabla u_h^n\cdot\mathbf{n}\rangle[v_h])\,\mathrm{d}S + \sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{DN}}}\int_\Gamma\nabla u_h^n\cdot\mathbf{n}v_h\,\mathrm{d}S.\end{aligned} \tag{5.28}$$

Keep the terms with jumps of the approximate solution on the left-hand side and move all the other terms to the right-hand side in (3.5), moreover take into account (5.27) and (5.28), to obtain

$$
\begin{aligned}
(5.29)\quad & \sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{ID}}} \int_\Gamma \sigma[u_h^n][v_h]\,\mathrm{d}S \\
&= \sum_{K\in\widetilde{\mathcal{T}}_h^n} \int_K \mathrm{R}_K^n v_h\,\mathrm{d}\mathbf{x} + \sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{D}}} \int_\Gamma \sigma u_D^n v_h\,\mathrm{d}S - \sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{I}}} \int_\Gamma [\nabla u_h^n\cdot\mathbf{n}]\langle v_h\rangle\,\mathrm{d}S \\
&\quad + \sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{N}}} \int_\Gamma (g_N^n - \nabla u_h^n\cdot\mathbf{n})v_h\,\mathrm{d}S - \theta \sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{I}}} \int_\Gamma \langle\nabla v_h\cdot\mathbf{n}\rangle[u_h^n]\,\mathrm{d}S \\
&\quad + \theta \sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{D}}} \int_\Gamma \nabla v_h\cdot\mathbf{n}(u_D^n - u_h^n)\,\mathrm{d}S.
\end{aligned}
$$

Setting $v_h := u_h^n - \mathcal{I}_{\mathrm{Av}}^{\mathrm{u_D^n}}(v_h)$ ($\mathcal{I}_{\mathrm{Av}}^{\mathrm{u_D^n}}(v_h)$ is the averaging interpolation operator satisfying the boundary condition given by u_D^n on $\partial\Omega_D$) in (5.29) and applying the Cauchy-Schwarz inequality yields

$$
\begin{aligned}
(5.30)\quad & \sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{I}}} \int_\Gamma \sigma[u_h^n]^2\,\mathrm{d}S + \sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{D}}} \int_\Gamma \sigma u_h^n(u_h^n - u_D^n)\,\mathrm{d}S \\
&\leqslant ((\eta_{\mathrm{R}}^n)^2 + (\eta_{\mathrm{JdN}}^n)^2 + (\eta_{\mathrm{Jd}}^n)^2)^{1/2} \Bigg(\sum_{K\in\widetilde{\mathcal{T}}_h^n} h_K^{-2}\|u_h^n - \mathcal{I}_{\mathrm{Av}}^{\mathrm{u_D^n}}(u_h^n)\|_K^2 \\
&\quad + \sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{N}}} h_\Gamma^{-1}\|u_h^n - \mathcal{I}_{\mathrm{Av}}^{\mathrm{u_D^n}}(u_h^n)\|_\Gamma^2 + \sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{I}}} h_\Gamma^{-1}\|\langle u_h^n - \mathcal{I}_{\mathrm{Av}}^{\mathrm{u_D^n}}(u_h^n)\rangle\|_\Gamma^2 \Bigg)^{1/2} \\
&\quad + \Bigg(\sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{I}}} h_\Gamma\|\langle\mathbf{n}\cdot\nabla(u_h^n - \mathcal{I}_{\mathrm{Av}}^{\mathrm{u_D^n}}(u_h^n))\rangle\|_\Gamma^2 \Bigg)^{1/2} ((\eta_{\mathrm{J}}^n)^2)^{1/2} \\
&\quad + \Bigg(\sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{D}}} h_\Gamma\|\mathbf{n}\cdot\nabla(u_h^n - \mathcal{I}_{\mathrm{Av}}^{\mathrm{u_D^n}}(u_h^n))\|_\Gamma^2 \Bigg)^{1/2} ((\eta_{\mathrm{JD}}^n)^2)^{1/2} \\
&\quad + \sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{D}}} \int_\Gamma \sigma u_D^n(u_h^n - \mathcal{I}_{\mathrm{Av}}^{\mathrm{u_D^n}}(u_h^n))\,\mathrm{d}S.
\end{aligned}
$$

Let us estimate the terms containing the averaging operator. First, analogously to the estimates for ξ_4 and ξ_6 in (5.11) and (5.13), respectively, we obtain

$$
(5.31)\qquad \sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{IN}}} h_\Gamma^{-1}\|\langle u_h^n - \mathcal{I}_{\mathrm{Av}}^{\mathrm{u_D^n}}(u_h^n)\rangle\|_\Gamma^2 \leqslant c \sum_{K\in\widetilde{\mathcal{T}}_h^n} h_K^{-2}\|u_h^n - \mathcal{I}_{\mathrm{Av}}^{\mathrm{u_D^n}}(u_h^n)\|_K^2.
$$

Further, analogously to the estimate for ξ_5 in (5.12), we get

$$\sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{I}}} h_\Gamma \|\langle \mathbf{n}\cdot\nabla(u_h^n - \mathcal{I}_{\mathrm{Av}}^{u_{\mathrm{D}}^n}(u_h^n))\rangle\|_\Gamma^2 \leqslant c \sum_{K\in\widetilde{\mathcal{T}}_h^n} \|\nabla(u_h^n - \mathcal{I}_{\mathrm{Av}}^{u_{\mathrm{D}}^n}(u_h^n))\|_K^2. \tag{5.32}$$

By putting the last term of the right-hand side of (5.30) on the left-hand side, and using (5.31) and (5.32) together with (4.4), we obtain

$$\begin{aligned}\sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{I}}}\int_\Gamma \sigma[u_h^n]^2\,\mathrm{d}S + \sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{D}}}\int_\Gamma \sigma(u_h^n-u_D^n)^2\,\mathrm{d}S \\ \leqslant ((\eta_{\mathrm{R}}^n)^2 + (\eta_{\mathrm{JdN}}^n)^2 + (\eta_{\mathrm{Jd}}^n)^2)^{1/2} C_O((\eta_{\mathrm{J}}^n)^2 + (\eta_{\mathrm{JD}}^n)^2)^{1/2} \\ + C_O((\eta_{\mathrm{J}}^n)^2 + (\eta_{\mathrm{JD}}^n)^2)^{1/2}((\eta_{\mathrm{J}}^n)^2)^{1/2} + C_O((\eta_{\mathrm{J}}^n)^2 + (\eta_{\mathrm{JD}}^n)^2)^{1/2}((\eta_{\mathrm{JD}}^n)^2)^{1/2}.\end{aligned} \tag{5.33}$$

Now, the use of Young's inequality, the notation (5.1), and provided that the penalty parameter is sufficiently large to be able to subtract all the terms on the right-hand side except for those contained in (5.1) from the left-hand side in (5.33), we finally obtain the desired estimate

$$\sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{I}}}\int_\Gamma \sigma[u_h^n]^2\,\mathrm{d}S + \sum_{\Gamma\in\widetilde{\mathcal{F}}_h^{n,\mathrm{D}}}\int_\Gamma \sigma(u_h^n-u_D^n)^2\,\mathrm{d}S \leqslant c((\eta_{\mathrm{R}}^n)^2 + (\eta_{\mathrm{JdN}}^n)^2 + (\eta_{\mathrm{Jd}}^n)^2). \tag{5.34}$$

□

6. Numerical example

In this section, we present numerical experiments illustrating the a posteriori error estimates of this paper. We consider the problem (2.1) where $T = 1$, $\Omega = (0,1)\times(0,1)$, $\partial\Omega_N = \emptyset$, and the initial and boundary conditions are chosen in such a way that the exact solution is

$$u(x_1,x_2,t) = \exp[x_1 + x_2 + 2t]. \tag{6.1}$$

We simply observe that the right-hand side f of (2.1) vanishes. We performed a set of numerical experiments with the aid of the DGM (3.5) for $p = 1,2,3$ polynomial approximations.

We consider a uniform space-time discretizations characterized by the space and time steps h_m and τ_m, $m = 1,\ldots,4$, respectively. We choose $\{h_1,\tau_1\} = (1/8, 1/100)$ and then set $h_{m+1} = h_m/2$, $\tau_{m+1} = \tau_m/2^p$ for $m = 1,2,3$. The space grids are triangulations of right triangles resulting from diagonal cuttings of squares with edges of length $h_l = h_K/\sqrt{2}$. We evaluate the experimental order of convergence

$$\mathrm{EOC} := \frac{\log(E_m/E_{m-1})}{\log(h_m/h_{m-1})}, \quad m = 2, 3, 4,$$

where E_m is either the error, or the error estimator on the space-time discretization $\{h_m, \tau_m\}$. Table 1 shows the values from (5.3), namely

$$\begin{aligned} \|e_h\|_Y &:= \|e^N\|_\Omega^2 + \sum_{n=1}^{N} \tau_n \sum_{K\in\widetilde{\mathcal{T}}_h^n} \|\nabla e^n\|_K^2, & \eta_1 &:= \sum_{n=1}^{N} \tau_n \sum_{K\in\widetilde{\mathcal{T}}_h^n} (\eta_{K,1}^n)^2, \\ \eta_2 &:= \sum_{n=1}^{N} \sum_{K\in\widetilde{\mathcal{T}}_h^n} (\eta_{K,2}^n)^2, & \eta_{\mathrm{IC}} &:= \|e^0\|_\Omega^2, \\ \eta_{\mathrm{tot}} &:= \eta_1 + \eta_2 + \eta_{\mathrm{IC}}, & i_{\mathrm{eff}} &:= \eta_{\mathrm{tot}}/\|e_h\|_Y. \end{aligned}$$

The value i_{eff} corresponds to the *effectivity index*. However, since our estimate (5.3) contains an undetermined constant C, this value may be lower than one. Our aim is to show that the presented a posteriori error estimate is independent of the discretization parameters h and τ. Table 1 shows that this is really the case but a posteriori error estimate depends on the degree of polynomial approximation p. This is caused by the fact that we have not considered the dependence of generic constants on p in our analysis.

7. Conclusion

We extended the results from [15] for the upper bound as the high-order DGM is used. Further, we derived the lower bound. The heat conduction equation was discretized by the high-order DGM in space and the backward Euler scheme in time. Analogously to [12], the Helmholtz decomposition was used to overcome difficulties arising due to the nonconformity of the DGM. Finally, notice that the presented estimators estimate the discretization error only, i.e., we assume that the linear system resulting from discretization is solved exactly.

P_k	h_m	τ_m	$\|e_h\|_Y$	η_1	η_2	$\eta_{\rm IC}$	$\eta_{\rm tot}$	$i_{\rm eff}$
1	1.25E-01	1.00E-02	1.22E+00	1.43E+01	3.01E-01	4.31E-03	1.43E+01	11.7360
1	6.25E-02	5.00E-03	6.10E-01	7.39E+00	1.00E-01	1.08E-03	7.39E+00	12.1190
	(EOC)		(1.00)	(0.95)	(1.59)	(2.00)	(0.95)	
1	3.12E-02	2.50E-03	3.05E-01	3.76E+00	3.40E-02	2.70E-04	3.76E+00	12.3178
	(EOC)		(1.00)	(0.97)	(1.56)	(2.00)	(0.97)	
1	1.56E-02	1.25E-03	1.53E-01	1.90E+00	1.18E-02	6.75E-05	1.90E+00	12.4193
	(EOC)		(1.00)	(0.99)	(1.53)	(2.00)	(0.99)	
2	1.25E-01	1.00E-02	2.02E-01	4.53E-01	9.96E-03	7.89E-05	4.53E-01	2.2479
2	6.25E-02	2.50E-03	5.04E-02	1.15E-01	2.46E-03	9.88E-06	1.15E-01	2.2898
	(EOC)		(2.00)	(1.97)	(2.02)	(3.00)	(1.97)	
2	3.12E-02	6.25E-04	1.26E-02	2.92E-02	6.10E-04	1.24E-06	2.92E-02	2.3152
	(EOC)		(2.00)	(1.98)	(2.01)	(3.00)	(1.98)	
2	1.56E-02	1.56E-04	3.15E-03	7.34E-03	1.52E-04	1.54E-07	7.35E-03	2.3338
	(EOC)		(2.00)	(1.99)	(2.00)	(3.00)	(1.99)	
3	1.25E-01	1.00E-02	1.99E-01	1.21E-02	1.66E-04	1.11E-06	1.21E-02	0.0609
3	6.25E-02	1.25E-03	2.49E-02	1.46E-03	2.43E-05	6.95E-08	1.49E-03	0.0598
	(EOC)		(3.00)	(3.04)	(2.77)	(4.00)	(3.03)	
3	3.12E-02	1.56E-04	3.11E-03	1.84E-04	4.08E-06	4.34E-09	1.96E-04	0.0630
	(EOC)		(3.00)	(2.99)	(2.58)	(4.00)	(2.93)	

Table 1. The computed errors, error estimators, and effectivity indices

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