Liang Bai; Binxiang Dai

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EXISTENCE OF NONZERO SOLUTIONS FOR A CLASS OF DAMPED VIBRATION PROBLEMS WITH IMPULSIVE EFFECTS

Liang Bai, Taiyuan, Binxiang Dai, Changsha

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Abstract. In this paper, a class of damped vibration problems with impulsive effects is considered. An existence result is obtained by using the variational method and the critical point theorem due to Brezis and Nirenberg. The obtained result is also valid and new for the corresponding second-order impulsive Hamiltonian system. Finally, an example is presented to illustrate the feasibility and effectiveness of the result.

Keywords: impulsive problem; damped vibration problem; variational method; critical point

MSC 2010: 34B37, 58E30

1. Introduction

This paper is devoted to proving the existence of nonzero solutions to the following damped vibration problem:

$$\ddot{u}(t) + g(t)\dot{u}(t) = \nabla F(t, u(t)) \quad \text{a.e. } t \in [0, T], \tag{1.1}$$

$$u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \tag{1.2}$$

with the impulsive conditions

$$\Delta(\dot{u}^i(t_j)) = I_{ij}(u^i(t_j)), \quad i = 1, 2, \ldots, N, \ j = 1, 2, \ldots, p, \tag{1.3}$$

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where $u(t) = (u^1(t), u^2(t), \ldots, u^N(t))$, $T > 0$, $t_0 = 0 < t_1 < t_2 < \ldots < t_p < t_{p+1} = T$, $g \in L^1(0, T; \mathbb{R})$, $\int_0^T g(t)\,\mathrm{d}t = 0$, $\Delta(\dot{u}^i(t_j)) = \dot{u}^i(t_j^+) - \dot{u}^i(t_j^-)$, where $\dot{u}^i(t_j^+)$ and $\dot{u}^i(t_j^-)$ denote the right and left limits of $\dot{u}^i(t)$ at $t = t_j$, respectively, impulsive functions $I_{ij}\colon \mathbb{R} \to \mathbb{R}$ $(i = 1, 2, \ldots, N,\ j = 1, 2, \ldots, p)$ are continuous, $\nabla F(t, x)$ is the gradient of $F\colon [0, T] \times \mathbb{R}^N \to \mathbb{R}$ with respect to x and F satisfies the following assumption:

(A) $F(t, x)$ is measurable in t for every $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0, T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1(0, T; \mathbb{R}^+)$ such that

$$|F(t, x)| \leqslant a(|x|)b(t), \quad |\nabla F(t, x)| \leqslant a(|x|)b(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$.

We refer to the impulsive problem (1.1)–(1.3) as (IP).

Impulsive effects exist in many evolution processes in which their states are changed abruptly at certain moments of time. Applications of impulsive problems occur in control theory, biology, population dynamics, chemotherapeutic treatment in medicine and so on (see e.g. [9], [16], [15], [5], [6], [8], [12]). The theory of impulsive problems has been developed by numerous mathematicians (see e.g. [14], [20], [26], [1], [18], [19], [27], [2], [3], [28], [17], [7]). In particular, Zhou and Li [28] obtained some sufficient conditions for the existence of at least one solution for the following second-order impulsive Hamiltonian systems:

$$\begin{cases} \ddot{u}(t) = \nabla F(t, u(t)) \quad \text{a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \\ \Delta(\dot{u}^i(t_j)) = I_{ij}(u^i(t_j)), \quad i = 1, 2, \ldots, N,\ j = 1, 2, \ldots, p. \end{cases} \tag{1.4}$$

By using some critical points theorems of B. Ricceri, Sun et al. [17] got some criteria for guaranteeing the existence of at least three solutions for the following impulsive Hamiltonian systems with a perturbed term:

$$\begin{cases} -\ddot{u} + A(t)u = \lambda\nabla F(t, u) + \mu\nabla G(t, u) \quad \text{a.e. } t \in [0, T], \\ \Delta(\dot{u}^i(t_j)) = I_{ij}(u^i(t_j)), \quad i = 1, 2, \ldots, N,\ j = 1, 2, \ldots, l, \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0. \end{cases}$$

In [7], Han and Zhang studied the periodic and homoclinic solutions generated by impulses for the asymptotically linear and sublinear Hamiltonian system

$$\begin{cases} \ddot{q}(t) = f(t, q(t)) \quad \text{for } t \in (s_{k-1}, s_k), \\ \Delta\dot{q}(s_k) = g_k(q(s_k)). \end{cases}$$

Recently, Nieto [13] introduced the concept of a weak solution for a damped linear equation with Dirichlet boundary conditions and impulses. And the existence and uniqueness of weak solutions is obtained by using the classical Lax-Milgram Theorem. Soon after, Xiao and Nieto [25] used the critical point theory and variational methods to investigate the solutions of a Dirichlet boundary value problem for damped nonlinear impulsive differential equations.

Moreover, by using the variational method, Wu et al. [22], [24], [21], [23], [10] obtained the existence and multiplicity of solutions for some damped vibration problems, such as damped vibration problems with obstacles, damped vibration problems with super-quadratic potentials and forced vibration problems with obstacles. However, the study of solutions for impulsive damped vibration problems using the variational method has received considerably less attention.

Inspired by the above facts, the aim of this paper is to study the existence of at least two nonzero solutions for the impulsive damped vibration problem (IP) via Brezis and Nirenberg's linking theorem. It is worth stressing that the result of this paper is also valid and new even if (IP) is reduced to the second-order impulsive Hamiltonian system (1.4).

For the sake of convenience, in the sequel, we define

$$\mathcal{A} := \{1, 2, \dots, N\}, \quad \mathcal{B} := \{1, 2, \dots, p\}, \quad \omega := \frac{2\pi}{T},$$
$$\zeta := \Big(\frac{1}{T} + T\Big)^{1/2} \quad \text{and} \quad M := \int_0^T |g(t)| \, \mathrm{d}t.$$

The organization of the paper is as follows. Some fundamental facts are given in the next section. In Section 3, the main result of the paper is presented and an example is given to illustrate it.

2. Preliminaries

Let us recall some basic concepts.

$$H_T^1 := \{u\colon [0,T] \to \mathbb{R}^N;\ u \text{ is absolutely continuous,}$$
$$u(0) = u(T) \text{ and } \dot{u} \in L^2(0,T;\mathbb{R}^N)\}$$

is a Hilbert space with the inner product

$$\langle u, v\rangle = \int_0^T (\dot{u}(t), \dot{v}(t)) \, \mathrm{d}t + \int_0^T (u(t), v(t)) \, \mathrm{d}t \quad \text{for any } u, v \in H_T^1,$$

where $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^N$. The corresponding norm is

$$\|u\| = \left(\int_0^T |\dot{u}(t)|^2 \,\mathrm{d}t + \int_0^T |u(t)|^2 \,\mathrm{d}t\right)^{1/2} \quad \text{for any } u \in H_T^1.$$

For $u \in \widetilde{H}_T^1 := \bigl\{u \in H_T^1;\ \int_0^T u(t)\,\mathrm{d}t = 0\bigr\}$ we have Wirtinger's inequality (see Proposition 1.3 in [11])

$$\int_0^T |u(t)|^2 \,\mathrm{d}t \leqslant \frac{1}{\omega^2}\int_0^T |\dot{u}(t)|^2 \,\mathrm{d}t. \tag{2.1}$$

Lemma 2.1. *If $u \in H_T^1$, then*

$$\|u\|_\infty \leqslant \zeta\|u\|,$$

where $\|u\|_\infty = \max\limits_{t\in[0,T]} |u(t)|$.

Proof. For any $i \in \mathcal{A}$, it follows from the mean value theorem that

$$\frac{1}{T}\int_0^T u^i(s)\,\mathrm{d}s = u^i(\tau)$$

for some $\tau \in (0,T)$. Hence, for $t \in [0,T]$, using the Hölder inequality,

$$\begin{aligned}
|u^i(t)| &= \left|u^i(\tau) + \int_\tau^t \dot{u}^i(s)\,\mathrm{d}s\right| \\
&\leqslant |u^i(\tau)| + \int_0^T |\dot{u}^i(s)|\,\mathrm{d}s \\
&\leqslant \frac{1}{T}\int_0^T |u^i(s)|\,\mathrm{d}s + \int_0^T |\dot{u}^i(s)|\,\mathrm{d}s \\
&\leqslant \frac{1}{T}T^{1/2}\left(\int_0^T |u^i(s)|^2\,\mathrm{d}s\right)^{1/2} + T^{1/2}\left(\int_0^T |\dot{u}^i(s)|^2\,\mathrm{d}s\right)^{1/2},
\end{aligned}$$

which combined with the Cauchy-Schwarz inequality yields that

$$|u^i(t)| \leqslant \left(\frac{1}{T}+T\right)^{1/2}\left(\int_0^T |\dot{u}^i(s)|^2 + |u^i(s)|^2\,\mathrm{d}s\right)^{1/2}. \tag{2.2}$$

In view of (2.2), we have that, for $t \in [0,T]$,

$$|u(t)| = \left(\sum_{i=1}^N |u^i(t)|^2\right)^{1/2} \leqslant \left(\sum_{i=1}^N \zeta^2\left(\int_0^T |\dot{u}^i(s)|^2 + |u^i(s)|^2\,\mathrm{d}s\right)\right)^{1/2} = \zeta\|u\|,$$

which implies the conclusion. $\square$

It follows from Lemma 2.1 that

$$|u(t)| \leqslant \|u\|_\infty \leqslant \zeta\|u\| \quad \text{for all } u \in H_T^1 \text{ and } t \in [0,T]. \tag{2.3}$$

In view of (2.3), we have

$$|u^i(t)| \leqslant |u(t)| \leqslant \zeta\|u\| \quad \text{for all } u \in H_T^1,\ t \in [0,T] \text{ and } i \in \mathcal{A}. \tag{2.4}$$

Let

$$G(t) = \int_0^t g(s)\,\mathrm{d}s, \quad t \in [0,T].$$

Since $g \in L^1(0,T;\mathbb{R})$, we have $G'(t) = g(t)$ for a.e. $t \in [0,T]$, $G(t)$ is absolutely continuous, and

$$|G(t)| \leqslant \int_0^t |g(s)|\,\mathrm{d}s \leqslant M \quad \text{for all } t \in [0,T]. \tag{2.5}$$

Following the ideas of [13], multiplying both sides of (1.1) by $\mathrm{e}^{G(t)}$, we have

$$\mathrm{e}^{G(t)}\ddot{u}(t) + \mathrm{e}^{G(t)}g(t)\dot{u}(t) = \mathrm{e}^{G(t)}\nabla F(t,u(t)) \quad \text{a.e. } t \in [0,T]. \tag{2.6}$$

Taking into account that $\dot{u}$ is the classical derivative of u a.e. on $[0,T]$ (see Remarks in [11, p. 7]), (2.6) implies that

$$[\mathrm{e}^{G(t)}\dot{u}(t)]' = \mathrm{e}^{G(t)}\nabla F(t,u(t)) \quad \text{a.e. } t \in [0,T]. \tag{2.7}$$

Now multiplying (2.7) by $v \in H_T^1$ and integrating between 0 and T, we have

$$\int_0^T ([\mathrm{e}^{G(t)}\dot{u}(t)]', v(t))\,\mathrm{d}t - \int_0^T \mathrm{e}^{G(t)}(\nabla F(t,u(t)), v(t))\,\mathrm{d}t = 0. \tag{2.8}$$

Taking into account that $\dot{u}$ is the classical derivative of u a.e. on $[0,T]$, (1.2), (1.3), and $\int_0^T g(t)\,\mathrm{d}t = 0$, the first term of (2.8) is

$$\begin{aligned}
&\int_0^T ([\mathrm{e}^{G(t)}\dot{u}(t)]', v(t))\,\mathrm{d}t \\
&\quad = \sum_{j=0}^{p}\int_{t_j}^{t_{j+1}} ([\mathrm{e}^{G(t)}\dot{u}(t)]', v(t))\,\mathrm{d}t \\
&\quad = (\mathrm{e}^{G(t)}\dot{u}(t), v(t))\big|_0^{t_1^-} + \sum_{j=1}^{p-1}(\mathrm{e}^{G(t)}\dot{u}(t), v(t))\big|_{t_j^+}^{t_{j+1}^-} \\
&\qquad + (\mathrm{e}^{G(t)}\dot{u}(t), v(t))\big|_{t_p^+}^{T} - \sum_{j=0}^{p}\int_{t_j}^{t_{j+1}} (\mathrm{e}^{G(t)}\dot{u}(t), v'(t))\,\mathrm{d}t \\
&\quad = \mathrm{e}^{G(T)}(\dot{u}(T), v(T)) - \mathrm{e}^{G(0)}(\dot{u}(0), v(0)) \\
&\qquad - \sum_{j=1}^{p}(\mathrm{e}^{G(t)}\dot{u}(t), v(t))\big|_{t_j^-}^{t_j^+} - \int_0^T (\mathrm{e}^{G(t)}\dot{u}(t), \dot{v}(t))\,\mathrm{d}t \\
&\quad = -\sum_{j=1}^{p}[(\mathrm{e}^{G(t_j)}\dot{u}(t_j^+), v(t_j)) - (\mathrm{e}^{G(t_j)}\dot{u}(t_j^-), v(t_j))] - \int_0^T \mathrm{e}^{G(t)}(\dot{u}(t), \dot{v}(t))\,\mathrm{d}t \\
&\quad = -\sum_{j=1}^{p}\sum_{i=1}^{N}[\mathrm{e}^{G(t_j)}\dot{u}^i(t_j^+)v^i(t_j) - \mathrm{e}^{G(t_j)}\dot{u}^i(t_j^-)v^i(t_j)] - \int_0^T \mathrm{e}^{G(t)}(\dot{u}(t), \dot{v}(t))\,\mathrm{d}t \\
&\quad = -\sum_{j=1}^{p}\sum_{i=1}^{N}\mathrm{e}^{G(t_j)}\Delta(\dot{u}^i(t_j))v^i(t_j) - \int_0^T \mathrm{e}^{G(t)}(\dot{u}(t), \dot{v}(t))\,\mathrm{d}t \\
&\quad = -\sum_{j=1}^{p}\sum_{i=1}^{N}\mathrm{e}^{G(t_j)}I_{ij}(u^i(t_j))v^i(t_j) - \int_0^T \mathrm{e}^{G(t)}(\dot{u}(t), \dot{v}(t))\,\mathrm{d}t,
\end{aligned}$$

which combined with (2.8) yields that

$$\begin{aligned}
\int_0^T \mathrm{e}^{G(t)}(\dot{u}(t), \dot{v}(t))\,\mathrm{d}t &+ \int_0^T \mathrm{e}^{G(t)}(\nabla F(t,u(t)), v(t))\,\mathrm{d}t \\
&= -\sum_{j=1}^{p}\sum_{i=1}^{N}\mathrm{e}^{G(t_j)}I_{ij}(u^i(t_j))v^i(t_j).
\end{aligned} \tag{2.9}$$

Considering the above equality, we introduce the following concept of the weak solution for (IP).

Definition 2.2. A function $u \in H_T^1$ is a weak solution of (IP) if (2.9) holds for any $v \in H_T^1$.

Definition 2.3. Suppose E is a real Banach space. For $I \in C^1(E, \mathbb{R})$, we say I satisfies the Palais-Smale condition (denoted by PS condition for short) if any sequence $\{u_n\} \subset E$ for which $I(u_n)$ is bounded and $I'(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence.

Consider the functional $\Phi\colon H_T^1 \to \mathbb{R}$ defined by

$$\Phi(u) = \varphi_1(u) + \varphi_2(u),$$

where

$$\varphi_1(u) = \frac{1}{2}\int_0^T \mathrm{e}^{G(t)}|\dot{u}(t)|^2\,\mathrm{d}t + \int_0^T \mathrm{e}^{G(t)}F(t, u(t))\,\mathrm{d}t$$

and

$$\varphi_2(u) = \sum_{j=1}^{p}\sum_{i=1}^{N} \mathrm{e}^{G(t_j)} \int_0^{u^i(t_j)} I_{ij}(s)\,\mathrm{d}s.$$

Let $L(t, x, y) = \mathrm{e}^{G(t)}|y|^2/2 + \mathrm{e}^{G(t)}F(t, x)$ for all $x, y \in \mathbb{R}^N$ and $t \in [0, T]$. It follows from Theorem 1.4 in [11], assumption (A) and (2.5) that φ_1 is continuously differentiable on H_T^1 and

$$\langle \varphi_1'(u), v\rangle = \int_0^T \mathrm{e}^{G(t)}(\dot{u}(t), \dot{v}(t))\,\mathrm{d}t + \int_0^T \mathrm{e}^{G(t)}(\nabla F(t, u(t)), v(t))\,\mathrm{d}t \tag{2.10}$$

for any $u, v \in H_T^1$. Moreover, it follows from the continuity of all I_{ij} that $\varphi_2 \in C^1(H_T^1, \mathbb{R})$ and

$$\langle \varphi_2'(u), v\rangle = \sum_{j=1}^{p}\sum_{i=1}^{N} \mathrm{e}^{G(t_j)} I_{ij}(u^i(t_j))v^i(t_j) \tag{2.11}$$

for any $u, v \in H_T^1$. Thus $\Phi \in C^1(H_T^1, \mathbb{R})$ and it follows from (2.10) and (2.11) that the weak solutions of (IP) correspond to the critical points of Φ.

For the reader's convenience, we now recall the critical point theorem, which is due to Brezis and Nirenberg. It will be our main tool.

Lemma 2.4 [4, Theorem 4]**.** *Let X be a Banach space with a direct sum decomposition $X = X_1 \oplus X_2$ with $k := \dim X_2 < \infty$. Let Φ be a C^1 function on X with $\Phi(0) = 0$, satisfying the* PS *condition. Assume that, for some $\varrho > 0$,*

$$\begin{cases} \Phi(u) \geqslant 0 & \text{for } u \in X_1,\ \|u\| \leqslant \varrho,\\ \Phi(u) \leqslant 0 & \text{for } u \in X_2,\ \|u\| \leqslant \varrho. \end{cases}$$

Assume also that Φ is bounded below and $\inf\limits_X \Phi < 0$. Then Φ has at least two nonzero critical points.

3. Main result

In this section, the main result of this paper is presented. To this end, we first introduce the following assumptions:

(h1) There exists a constant $\alpha > 0$ such that

$$\liminf_{|x|\to\infty} \frac{F(t,x)}{|x|^2} \geqslant \alpha \quad \text{uniformly for a.e. } t \in [0,T].$$

(h2) For any $i \in \mathcal{A}$, $j \in \mathcal{B}$, there exist constants $a_{ij} > 0$, $b_{ij} > 0$ and $\gamma_{ij} \in [0,1]$ (assume that $\gamma_{ij} = 1$ for $(i,j) \in \mathcal{D} \subseteq \mathcal{A} \times \mathcal{B}$ and $\gamma_{ij} \in [0,1)$ for $(i,j) \in (\mathcal{A} \times \mathcal{B})/\mathcal{D}$) such that

$$\limsup_{s\to-\infty} \frac{I_{ij}(s)}{|s|^{\gamma_{ij}}} < a_{ij} \quad \text{and} \quad \liminf_{s\to+\infty} \frac{I_{ij}(s)}{|s|^{\gamma_{ij}}} > -b_{ij}.$$

(h3) There exist constants $\sigma_1 > 0$, $\beta > 0$ and an integer $k \geqslant 1$ such that

$$-\frac{1}{2}\mathrm{e}^{-2M}(k+1)^2\omega^2|x|^2 \leqslant F(t,x) \leqslant -\frac{1}{2}\mathrm{e}^{2M}\beta k^2\omega^2|x|^2 \tag{3.1}$$

for all $|x| \leqslant \sigma_1$ and a.e. $t \in [0,T]$.

(h4) There exist constants $\sigma_2 > 0$ and $\lambda > 0$ such that

$$-\frac{1}{2}\mathrm{e}^{-2M}\omega^2|x|^2 \leqslant F(t,x) \leqslant -\frac{1}{2}\lambda\mathrm{e}^{2M}\omega^2|x|^2 \tag{3.2}$$

for all $|x| \leqslant \sigma_2$ and a.e. $t \in [0,T]$.

(h5) For any $i \in \mathcal{A}$, $j \in \mathcal{B}$, there exist constants $\sigma_3 > 0$ and $\mu_{ij} > 0$ such that

$$\mu_{ij}\omega^2 s \leqslant I_{ij}(s) \leqslant 0 \quad \text{for all } -\sigma_3 \leqslant s < 0 \tag{3.3}$$

and

$$0 \leqslant I_{ij}(s) \leqslant \mu_{ij}\omega^2 s \quad \text{for all } 0 \leqslant s \leqslant \sigma_3. \tag{3.4}$$

The following theorem is the main result of this paper.

Theorem 3.1. *Suppose that assumption* (A) *holds. Assume that one of the following two conditions holds:*

(H1) (h1) *and* (h2) *hold with* $\alpha > 1/2$ *and* $\mathrm{e}^{2M}\zeta^2 \sum\limits_{(i,j)\in\mathcal{D}} c_{ij} < 1$;

(H2) (h1) *and* (h2) *hold with* $\mathrm{e}^{2M}\zeta^2 \sum\limits_{(i,j)\in\mathcal{D}} c_{ij}/2 < \alpha \leqslant 1/2$,

where $c_{ij} = \max\{a_{ij}, b_{ij}\}$ *for all* $i \in \mathcal{A}$, $j \in \mathcal{B}$, *and* $\sum\limits_{(i,j)\in\mathcal{D}} c_{ij} = 0$ *if* $\mathcal{D} = \emptyset$. *Assume also that one of the following two conditions holds:*

(H3) (h3) *and* (h5) *hold with* $\beta \geqslant 1 + p\mu\omega^2\zeta^2 + p\mu\zeta^2$;

(H4) (h4) *and* (h5) *hold with* $\lambda \geqslant \mu p/T$,

where $\mu = \max\limits_{i\in\mathcal{A}, j\in\mathcal{B}}\{\mu_{ij}\}$. *Then* (IP) *has at least two nonzero weak solutions in* H_T^1.

Proof. We complete the proof in three steps.

Step 1. (H1) or (H2) implies that

$$\lim_{\|u\|\to\infty} \Phi(u) = \infty \tag{3.5}$$

and $\Phi(u)$ is bounded below on H_T^1.

In fact, for any $0 < \varepsilon < \alpha$, (h1) implies that there exists $\delta > 0$ such that

$$F(t,x) \geqslant (\alpha - \varepsilon)|x|^2 \quad \text{for all } x \in \mathbb{R}^N \text{ with } |x| \geqslant \delta \text{ and a.e. } t \in [0,T]. \tag{3.6}$$

Let $a_\delta = \max\limits_{|x|\leqslant\delta} a(|x|)$; assumption (A) implies that

$$F(t,x) \geqslant -a(|x|)b(t) \geqslant -a_\delta b(t) \geqslant -a_\delta b(t) + (\alpha-\varepsilon)|x|^2 - (\alpha-\varepsilon)\delta^2 \tag{3.7}$$

for all $x \in \mathbb{R}^N$ with $|x| \leqslant \delta$ and a.e. $t \in [0,T]$. Then it follows from (3.6) and (3.7) that

$$F(t,x) \geqslant (\alpha-\varepsilon)|x|^2 - (\alpha-\varepsilon)\delta^2 - a_\delta b(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$. Thus

$$\begin{aligned} \varphi_1(u) \geqslant \frac{1}{2}\int_0^T \mathrm{e}^{G(t)}|\dot{u}(t)|^2\,\mathrm{d}t + (\alpha-\varepsilon)\int_0^T \mathrm{e}^{G(t)}|u(t)|^2\,\mathrm{d}t \\ - (\alpha-\varepsilon)\delta^2\int_0^T \mathrm{e}^{G(t)}\,\mathrm{d}t - a_\delta\int_0^T \mathrm{e}^{G(t)}b(t)\,\mathrm{d}t \end{aligned} \tag{3.8}$$

for all $u \in H_T^1$.

It follows from (h2) that there exist $\nu_1 > 0$ and $\nu_2 > 0$ such that

$$I_{ij}(s) < a_{ij}|s|^{\gamma_{ij}} \leqslant c_{ij}(-s)^{\gamma_{ij}} \quad \text{for all } s \leqslant -\nu_1 \tag{3.9}$$

and

$$I_{ij}(s) > -b_{ij}|s|^{\gamma_{ij}} \geqslant -c_{ij}s^{\gamma_{ij}} \quad \text{for all } s \geqslant \nu_2. \tag{3.10}$$

Taking into account that $I_{ij}(s) - c_{ij}(-s)^{\gamma_{ij}}$ and $I_{ij}(s) + c_{ij}s^{\gamma_{ij}}$ are continuous, there exists $d_{ij} > 0$ such that

$$I_{ij}(s) - c_{ij}(-s)^{\gamma_{ij}} \leqslant d_{ij} \quad \text{for all } -\nu_1 \leqslant s \leqslant 0 \tag{3.11}$$

and

$$I_{ij}(s) + c_{ij}s^{\gamma_{ij}} \geqslant -d_{ij} \quad \text{for all } 0 \leqslant s \leqslant \nu_2. \tag{3.12}$$

Thus in view of (3.9) and (3.11) we have that

$$I_{ij}(s) \leqslant c_{ij}(-s)^{\gamma_{ij}} + d_{ij} \quad \text{for all } s \leqslant 0.$$

Thus, for all $z < 0$ we have

$$\int_z^0 I_{ij}(s)\,\mathrm{d}s \leqslant -\frac{c_{ij}(-1)^{\gamma_{ij}} z^{\gamma_{ij}+1}}{\gamma_{ij}+1} - d_{ij}z = \frac{c_{ij}}{\gamma_{ij}+1}|z|^{\gamma_{ij}+1} + d_{ij}|z|. \tag{3.13}$$

It follows from (3.10) and (3.12) that

$$I_{ij}(s) \geqslant -c_{ij}s^{\gamma_{ij}} - d_{ij} \quad \text{for all } s \geqslant 0.$$

Then, for all $z \geqslant 0$ we have

$$\int_0^z I_{ij}(s)\,\mathrm{d}s \geqslant -\frac{c_{ij}}{\gamma_{ij}+1}z^{\gamma_{ij}+1} - d_{ij}z = -\frac{c_{ij}}{\gamma_{ij}+1}|z|^{\gamma_{ij}+1} - d_{ij}|z|. \tag{3.14}$$

In view of (3.13) and (3.14), for any $z \in \mathbb{R}$ we have

$$\int_0^z I_{ij}(s)\,\mathrm{d}s \geqslant -\frac{c_{ij}}{\gamma_{ij}+1}|z|^{\gamma_{ij}+1} - d_{ij}|z|,$$

which combined with (2.4) yields that

$$\begin{aligned} \varphi_2(u) &\geqslant \sum_{j=1}^{p}\sum_{i=1}^{N} \mathrm{e}^{G(t_j)}\Big(-\frac{c_{ij}}{\gamma_{ij}+1}|u^i(t_j)|^{\gamma_{ij}+1} - d_{ij}|u^i(t_j)|\Big) \\ &\geqslant \sum_{j=1}^{p}\sum_{i=1}^{N} \mathrm{e}^{G(t_j)}\Big(-\frac{c_{ij}\zeta^{\gamma_{ij}+1}}{\gamma_{ij}+1}\|u\|^{\gamma_{ij}+1} - d_{ij}\zeta\|u\|\Big) \end{aligned} \tag{3.15}$$

for all $u \in H_T^1$.

Thus it follows from (3.8) and (3.15) that

$$\begin{aligned}\Phi(u) \geqslant \frac{1}{2}\int_0^T \mathrm{e}^{G(t)}|\dot{u}(t)|^2\,\mathrm{d}t + (\alpha-\varepsilon)\int_0^T \mathrm{e}^{G(t)}|u(t)|^2\,\mathrm{d}t \\ - (\alpha-\varepsilon)\delta^2\int_0^T \mathrm{e}^{G(t)}\,\mathrm{d}t - a_\delta\int_0^T \mathrm{e}^{G(t)}b(t)\,\mathrm{d}t \\ + \sum_{j=1}^{p}\sum_{i=1}^{N}\mathrm{e}^{G(t_j)}\Big(-\frac{c_{ij}\zeta^{\gamma_{ij}+1}}{\gamma_{ij}+1}\|u\|^{\gamma_{ij}+1} - d_{ij}\zeta\|u\|\Big),\end{aligned}$$

which combined with (2.5) yields that

$$\begin{aligned}\Phi(u) &\geqslant \frac{1}{2}\mathrm{e}^{-M}\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t + (\alpha-\varepsilon)\mathrm{e}^{-M}\int_0^T |u(t)|^2\,\mathrm{d}t \\ &\quad - (\alpha-\varepsilon)\delta^2\mathrm{e}^{M}T - a_\delta\mathrm{e}^{M}\int_0^T b(t)\,\mathrm{d}t \\ &\quad + \mathrm{e}^{M}\sum_{j=1}^{p}\sum_{i=1}^{N}\Big(-\frac{c_{ij}\zeta^{\gamma_{ij}+1}}{\gamma_{ij}+1}\|u\|^{\gamma_{ij}+1} - d_{ij}\zeta\|u\|\Big) \\ &\geqslant \min\Big\{\frac{1}{2},\alpha-\varepsilon\Big\}\mathrm{e}^{-M}\|u\|^2 - \mathrm{e}^{M}\sum_{j=1}^{p}\sum_{i=1}^{N}\Big[\frac{c_{ij}\zeta^{\gamma_{ij}+1}}{\gamma_{ij}+1}\|u\|^{\gamma_{ij}+1}\Big] \\ &\quad - \mathrm{e}^{M}\sum_{j=1}^{p}\sum_{i=1}^{N} d_{ij}\zeta\|u\| - (\alpha-\varepsilon)\delta^2\mathrm{e}^{M}T - a_\delta\mathrm{e}^{M}\int_0^T b(t)\,\mathrm{d}t.\end{aligned} \tag{3.16}$$

If $\mathcal{D}=\emptyset$, then $\sum\limits_{(i,j)\in\mathcal{D}} c_{ij} = 0$ and $\gamma_{ij}\in[0,1)$ for all $i\in\mathcal{A}$, $j\in\mathcal{B}$. Then in view of (3.16), for any $\alpha>0$, choosing $0<\varepsilon<\alpha$, we have (3.5) holds. Thus (H1) or (H2) implies (3.5) when $\mathcal{D}=\emptyset$.

If $\mathcal{D}\neq\emptyset$, the following two cases may occur:

Case 1: $\alpha>1/2$. Choosing $\varepsilon=(\alpha-1/2)/2$, we have $\alpha-\varepsilon>1/2$. Thus it follows from (3.16) that

$$\begin{aligned}\Phi(u) \geqslant \frac{1}{2}\Big(\mathrm{e}^{-M} - \mathrm{e}^{M}\zeta^2\sum_{(i,j)\in\mathcal{D}} c_{ij}\Big)\|u\|^2 \\ - \mathrm{e}^{M}\sum_{(i,j)\in(\mathcal{A}\times\mathcal{B})/\mathcal{D}}\Big[\frac{c_{ij}\zeta^{\gamma_{ij}+1}}{\gamma_{ij}+1}\|u\|^{\gamma_{ij}+1}\Big] - \mathrm{e}^{M}\sum_{j=1}^{p}\sum_{i=1}^{N} d_{ij}\zeta\|u\| \\ - \frac{1}{2}\Big(\alpha+\frac{1}{2}\Big)\delta^2\mathrm{e}^{M}T - a_\delta\mathrm{e}^{M}\int_0^T b(t)\,\mathrm{d}t,\end{aligned}$$

which combined with (H1) yields that (3.5) holds.

Case 2: $\alpha \leqslant 1/2$. Let $\varepsilon = \Big(\alpha - \mathrm{e}^{2M}\zeta^2 \sum\limits_{(i,j)\in\mathcal{D}} c_{ij}/2\Big)/2$. It follows from (H2) that

$$\alpha - \varepsilon = \frac{1}{2}\bigg(\alpha + \frac{1}{2}\mathrm{e}^{2M}\zeta^2 \sum_{(i,j)\in\mathcal{D}} c_{ij}\bigg) > \frac{1}{2}\mathrm{e}^{2M}\zeta^2 \sum_{(i,j)\in\mathcal{D}} c_{ij} > 0$$

and $\varepsilon > 0$. Then $\alpha - \varepsilon \leqslant 1/2 - \varepsilon < 1/2$. Thus it follows from (3.16) that

$$\begin{aligned}\Phi(u) \geqslant{}& \frac{1}{2}\bigg(\alpha\mathrm{e}^{-M} - \frac{1}{2}\mathrm{e}^{M}\zeta^2 \sum_{(i,j)\in\mathcal{D}} c_{ij}\bigg)\|u\|^2 \\ &- \mathrm{e}^{M} \sum_{(i,j)\in(\mathcal{A}\times\mathcal{B})/\mathcal{D}} \Big[\frac{c_{ij}\zeta^{\gamma_{ij}+1}}{\gamma_{ij}+1}\|u\|^{\gamma_{ij}+1}\Big] - \mathrm{e}^{M}\sum_{j=1}^{p}\sum_{i=1}^{N} d_{ij}\zeta\|u\| \\ &- \frac{1}{2}\bigg(\alpha + \frac{1}{2}\mathrm{e}^{2M}\zeta^2 \sum_{(i,j)\in\mathcal{D}} c_{ij}\bigg)\delta^2\mathrm{e}^{M}T - a_\delta\mathrm{e}^{M}\int_0^T b(t)\,\mathrm{d}t,\end{aligned}$$

which combined with (H2) yields that (3.5) holds.

Therefore, for any $\mathcal{D} \subseteq \mathcal{A}\times\mathcal{B}$, (H1) or (H2) implies (3.5). Thus $\Phi(u)$ is bounded below on H_T^1.

Step 2. (H1) or (H2) implies that $\Phi(u)$ satisfies the PS condition.

Suppose that $\{u_n\}$ is a sequence in H_T^1 such that $\Phi(u_n)$ is bounded and $\Phi'(u_n) \to 0$ as $n \to \infty$. Then $\{u_n\}$ is bounded on H_T^1. In fact, if $\{u_n\}$ is an unbounded sequence, without loss of generality we assume that $\|u_n\| \to \infty$ as $n \to \infty$. By Step 1, we know that (H1) or (H2) implies (3.5). Thus $\Phi(u_n) \to \infty$, which contradicts the boundedness of $\Phi(u_n)$.

Since H_T^1 is a reflexive Banach space, there exists $u \in H_T^1$ and a subsequence of $\{u_n\}$ (denoted again by $\{u_n\}$ for simplicity) such that u_n converges weakly to u on H_T^1. By Proposition 1.2 in [11], we know that u_n converges uniformly to u on $[0,T]$. Then

$$\int_0^T |u_n(t) - u(t)|^2\,\mathrm{d}t \to 0 \quad \text{as } n\to\infty, \tag{3.17}$$

and for any $i\in\mathcal{A}$, $j\in\mathcal{B}$, we have that $u_n^i(t_j) \to u^i(t_j)$ as $n\to\infty$. In fact,

$$|u_n^i(t_j) - u^i(t_j)| \leqslant |u_n(t_j) - u(t_j)| \quad \text{for any } i\in\mathcal{A},\ j\in\mathcal{B}.$$

Thus it follows from the continuity of all I_{ij} that

$$\sum_{j=1}^{p}\sum_{i=1}^{N} \mathrm{e}^{G(t_j)}(I_{ij}(u_n^i(t_j)) - I_{ij}(u^i(t_j)))(u_n^i(t_j) - u^i(t_j)) \to 0 \quad \text{as } n\to\infty. \tag{3.18}$$

Taking into account that u_n converges uniformly to u on $[0,T]$ and assumption (A), we have

$$\int_0^T \mathrm{e}^{G(t)}(\nabla F(t,u_n(t))-\nabla F(t,u(t)),u_n(t)-u(t))\,\mathrm{d}t\to 0 \quad \text{as } n\to\infty. \tag{3.19}$$

Since u_n converges weakly to u on H_T^1 and $\Phi'(u_n)\to 0$, we have

$$\langle \Phi'(u_n)-\Phi'(u),u_n-u\rangle \to 0 \quad \text{as } n\to\infty. \tag{3.20}$$

Moveover, it follows from (2.10) and (2.11) that

$$\begin{aligned}\langle \Phi'(u_n)-\Phi'(u),u_n-u\rangle &= \int_0^T \mathrm{e}^{G(t)}|\dot u_n(t)-\dot u(t)|^2\,\mathrm{d}t\\ &+\int_0^T \mathrm{e}^{G(t)}(\nabla F(t,u_n(t))-\nabla F(t,u(t)),u_n(t)-u(t))\,\mathrm{d}t\\ &+\sum_{j=1}^p\sum_{i=1}^N \mathrm{e}^{G(t_j)}(I_{ij}(u_n^i(t_j))-I_{ij}(u^i(t_j)))(u_n^i(t_j)-u^i(t_j)).\end{aligned} \tag{3.21}$$

Thus in view of (3.18)–(3.21) and (2.5), we have that

$$0\leqslant \mathrm{e}^{-M}\int_0^T|\dot u_n(t)-\dot u(t)|^2\,\mathrm{d}t\leqslant \int_0^T \mathrm{e}^{G(t)}|\dot u_n(t)-\dot u(t)|^2\,\mathrm{d}t\to 0 \quad \text{as } n\to\infty,$$

which combined with (3.17) yields that

$$\|u_n-u\|=\Big(\int_0^T|\dot u_n(t)-\dot u(t)|^2\,\mathrm{d}t+\int_0^T|u_n(t)-u(t)|^2\,\mathrm{d}t\Big)^{1/2}\to 0$$

as $n\to\infty$. That is, $\{u_n\}$ strongly converges to u on H_T^1, which means that $\Phi(u)$ satisfies the PS condition.

Step 3. (H3) or (H4) implies that (2.12) holds for some $\varrho>0$.

In fact, owing to (3.3) in (h5), we have that

$$\int_z^0 \mu_{ij}\omega^2 s\,\mathrm{d}s\leqslant \int_z^0 I_{ij}(s)\,\mathrm{d}s\leqslant\int_z^0 0\,\mathrm{d}s \quad \text{for all } -\sigma_3\leqslant z<0,$$

which implies that

$$-\frac{1}{2}\mu_{ij}\omega^2z^2\leqslant \int_z^0 I_{ij}(s)\,\mathrm{d}s\leqslant 0 \quad \text{for all } -\sigma_3\leqslant z<0. \tag{3.22}$$

It follows from (3.4) in (h5) that

$$\int_0^z 0\,\mathrm{d}s \leqslant \int_0^z I_{ij}(s)\,\mathrm{d}s \leqslant \int_0^z \mu_{ij}\omega^2 s\,\mathrm{d}s \quad \text{for all } 0 \leqslant z \leqslant \sigma_3,$$

which combined with (3.22) yields that

$$0 \leqslant \int_0^z I_{ij}(s)\,\mathrm{d}s \leqslant \frac{1}{2}\mu_{ij}\omega^2 z^2 \quad \text{for all } |z| \leqslant \sigma_3. \tag{3.23}$$

By (2.4), $\|u\| \leqslant \sigma_3/\zeta$ implies that $|u^i(t)| \leqslant \sigma_3$ for all $t \in [0,T]$ and $i \in \mathcal{A}$. Thus, it follows from (3.23) that

$$0 \leqslant \varphi_2(u) \leqslant \sum_{j=1}^{p}\sum_{i=1}^{N} \mathrm{e}^{G(t_j)}\frac{1}{2}\mu_{ij}\omega^2|u^i(t_j)|^2 \quad \text{for all } \|u\| \leqslant \frac{\sigma_3}{\zeta}. \tag{3.24}$$

Owing to (2.5), we have

$$\begin{aligned}\sum_{j=1}^{p}\sum_{i=1}^{N} &\mathrm{e}^{G(t_j)}\frac{1}{2}\mu_{ij}\omega^2|u^i(t_j)|^2 \\ &\leqslant \frac{1}{2}\mathrm{e}^{M}\mu\omega^2\sum_{j=1}^{p}\sum_{i=1}^{N}|u^i(t_j)|^2 = \frac{1}{2}\mathrm{e}^{M}\mu\omega^2\sum_{j=1}^{p}|u(t_j)|^2.\end{aligned} \tag{3.25}$$

It follows from (3.24) and (3.25) that

$$0 \leqslant \varphi_2(u) \leqslant \frac{1}{2}\mathrm{e}^{M}\mu\omega^2\sum_{j=1}^{p}|u(t_j)|^2 \quad \text{for all } \|u\| \leqslant \frac{\sigma_3}{\zeta}, \tag{3.26}$$

which combined with (2.3) yields that

$$0 \leqslant \varphi_2(u) \leqslant \frac{1}{2}\mathrm{e}^{M}p\mu\omega^2\zeta^2\|u\|^2 \quad \text{for all } \|u\| \leqslant \frac{\sigma_3}{\zeta}. \tag{3.27}$$

Let

$$X_2 = \left\{\sum_{l=0}^{k}(a_l\cos l\omega t + b_l \sin l\omega t);\ a_l, b_l \in \mathbb{R}^N\right\}$$

and let X_1 be the orthogonal complement of X_2 in H_T^1, where $a_l = (a_l^1, a_l^2, \dots, a_l^N)$ and $b_l = (b_l^1, b_l^2, \dots, b_l^N)$.

If (H3) holds, we will consider X_2 with $k \geqslant 1$. On the one hand, when $u \in X_2$, we have

$$\begin{aligned}\int_0^T |u(t)|^2\,\mathrm{d}t &= \int_0^T \sum_{i=1}^N \Big(\sum_{l=0}^k (a_l^i \cos l\omega t + b_l^i \sin l\omega t)\Big)^2\,\mathrm{d}t \\ &= \frac{T}{2}\sum_{i=1}^N\sum_{l=0}^k ((a_l^i)^2 + (b_l^i)^2)\end{aligned} \tag{3.28}$$

and

$$\begin{aligned}\int_0^T |\dot u(t)|^2\,\mathrm{d}t &= \int_0^T \sum_{i=1}^N \Big(\sum_{l=0}^k (-a_l^i l\omega \sin l\omega t + b_l^i l\omega \cos l\omega t)\Big)^2\,\mathrm{d}t \\ &= \frac{T}{2}\sum_{i=1}^N\sum_{l=0}^k (l^2\omega^2 (a_l^i)^2 + l^2\omega^2 (b_l^i)^2).\end{aligned} \tag{3.29}$$

In view of (2.5) and the right-hand side of (3.1) in (h3), for all $\|u\| \leqslant \sigma_1/\zeta$ we have

$$\begin{aligned}\varphi_1(u) &\leqslant \frac12\int_0^T \mathrm{e}^{G(t)}|\dot u(t)|^2\,\mathrm{d}t - \frac12 \mathrm{e}^{2M}\beta k^2\omega^2\int_0^T \mathrm{e}^{G(t)}|u(t)|^2\,\mathrm{d}t \\ &\leqslant \frac12\mathrm{e}^M\int_0^T |\dot u(t)|^2\,\mathrm{d}t - \frac12\mathrm{e}^M\beta k^2\omega^2\int_0^T |u(t)|^2\,\mathrm{d}t.\end{aligned} \tag{3.30}$$

Let $\varrho_1 = \min\{\sigma_1/\zeta, \sigma_3/\zeta\}$. Owing to (3.30) and the right-hand side of (3.27), for all $\|u\| \leqslant \varrho_1$ we have

$$\Phi(u) \leqslant \frac12\mathrm{e}^M\int_0^T |\dot u(t)|^2\,\mathrm{d}t - \frac12\mathrm{e}^M\beta k^2\omega^2\int_0^T |u(t)|^2\,\mathrm{d}t + \frac12\mathrm{e}^M p\mu\omega^2\zeta^2\|u\|^2,$$

which combined with (3.28) and (3.29) yields that

$$\Phi(u) \leqslant \frac{T}{4}\mathrm{e}^M\omega^2\sum_{i=1}^N\sum_{l=0}^k [(l^2 - \beta k^2 + p\mu\omega^2\zeta^2 l^2 + p\mu\zeta^2)((a_l^i)^2 + (b_l^i)^2)]$$

for all $u \in X_2$ with $\|u\| \leqslant \varrho_1$. Since $l \leqslant k$ and $k \geqslant 1$, we have that

$$l^2 - \beta k^2 + p\mu\omega^2\zeta^2 l^2 + p\mu\zeta^2 \leqslant k^2 - \beta k^2 + p\mu\omega^2\zeta^2 k^2 + p\mu\zeta^2 k^2.$$

Thus $\beta \geqslant 1 + p\mu\omega^2\zeta^2 + p\mu\zeta^2$ implies that $\Phi(u) \leqslant 0$ for all $u \in X_2$ with $\|u\| \leqslant \varrho_1$. On the other hand, in view of (2.5), the left-hand side of (3.1) in (h3) and the left-hand side of (3.27) we have

$$\Phi(u) \geqslant \frac{1}{2}\int_0^T e^{G(t)}|\dot{u}(t)|^2\,dt - \frac{1}{2}e^{-2M}(k+1)^2\omega^2\int_0^T e^{G(t)}|u(t)|^2\,dt$$
$$\geqslant \frac{1}{2}e^{-M}\int_0^T |\dot{u}(t)|^2\,dt - \frac{1}{2}e^{-M}(k+1)^2\omega^2\int_0^T |u(t)|^2\,dt \geqslant 0$$

for all $u \in X_1$ with $\|u\| \leqslant \varrho_1$. Thus (H3) implies that (2.12) holds for ϱ_1.

In the following, it will be shown that (H4) implies that (2.12) holds for $\varrho_2 = \min\{\sigma_2/\zeta, \sigma_3/\zeta\}$. In fact, if (H4) holds, we will consider X_2 with $k = 0$. Then $X_2 = \mathbb{R}^N$ and the orthogonal complement of $\mathbb{R}^N$ in H_T^1 is $\widetilde{H}_T^1$. On the one hand, in view of (2.5), the right-hand side of (3.2) in (h4) and the right-hand side of (3.26), for all $u \in \mathbb{R}^N$ with $\|u\| \leqslant \varrho_2$ we have

$$\Phi(u) \leqslant \int_0^T -\frac{1}{2}\lambda e^{G(t)+2M}\omega^2|u|^2\,dt + \frac{1}{2}e^M\mu\omega^2\sum_{j=1}^p |u|^2$$
$$\leqslant -\frac{1}{2}\lambda T e^M\omega^2|u|^2 + \frac{1}{2}e^M\mu\omega^2 p|u|^2,$$

which combined with $\lambda \geqslant \mu p/T$ yields that $\Phi(u) \leqslant 0$ for all $u \in \mathbb{R}^N$ with $\|u\| \leqslant \varrho_2$. On the other hand, owing to (2.1), (2.5), the left-hand side of (3.2) in (h4) and the left-hand side of (3.26), we have that

$$\Phi(u) \geqslant \frac{1}{2}\int_0^T e^{G(t)}|\dot{u}(t)|^2\,dt + \int_0^T e^{G(t)}F(t,u(t))\,dt$$
$$\geqslant \frac{1}{2}\int_0^T e^{G(t)}|\dot{u}(t)|^2\,dt + \int_0^T -\frac{1}{2}e^{G(t)-2M}\omega^2|u(t)|^2\,dt$$
$$\geqslant \frac{1}{2}e^{-M}\int_0^T |\dot{u}(t)|^2\,dt - \frac{1}{2}e^{-M}\omega^2\int_0^T |u(t)|^2\,dt \geqslant 0$$

for all $u \in \widetilde{H}_T^1$ with $\|u\| \leqslant \varrho_2$.

Moreover, it follows from (h3) or (h4) that $F(t,0) = 0$ for a.e. $t \in [0,T]$. Thus (h3) or (h4) implies $\Phi(0) = 0$.

Now if $\inf\limits_{H_T^1} \Phi \geqslant 0$, by Step 3 we have that all $u \in X_2$ with $\|u\| \leqslant \varrho$ are minima of Φ, which implies that Φ has infinitely many critical points. If $\inf\limits_{H_T^1} \Phi < 0$, then it follows from Lemma 2.4 that Φ has at least two nonzero critical points. Hence (IP) has at least two nonzero weak solutions in H_T^1. $\square$

Remark 3.2. It follows from $k \geqslant 1$ and $1 < 1 + p\mu\omega^2\zeta^2 + p\mu\zeta^2 \leqslant \beta$ that

$$-\frac{1}{2}e^{2M}\beta k^2\omega^2|x|^2 \leqslant -\frac{1}{2}e^{2M}\beta\omega^2|x|^2 \leqslant -\frac{1}{2}e^{-2M}\omega^2|x|^2.$$

Therefore, (H3) and (H4) do not contain each other.

In view of Theorem 3.1, if (h2) holds with $\mathcal{D} = \emptyset$, then $\sum\limits_{(i,j)\in\mathcal{D}} c_{ij} = 0$. Then (H1) combines with (H2) to the following condition:

$\triangleright$ (h1) holds, and (h2) holds with $\mathcal{D} = \emptyset$.

Thus by Theorem 3.1, we can get the following fact.

Corollary 3.3. *Suppose that assumptions* (A) *and* (h1) *hold. Assume that* (h2) *holds with* $\mathcal{D} = \emptyset$. *Assume also that* (H3) *or* (H4) *holds. Then* (IP) *has at least two nonzero weak solutions in* H^1_T.

Example 3.4. Consider the following damped vibration problem with impulsive effects:

$$\begin{cases} \ddot{u}(t) + g(t)\dot{u}(t) = \nabla F(t, u(t)) \quad \text{a.e. } t \in [0, 2\pi], \\ u(0) - u(2\pi) = \dot{u}(0) - \dot{u}(2\pi) = 0, \\ \Delta(\dot{u}^i(t_j)) = I_{ij}(u^i(t_j)), \quad i = 1, 2, 3, \ j = 1, \end{cases} \tag{3.31}$$

where $0 < t_1 < 2\pi$,

$$g(t) = \frac{1}{8\pi^2}(t - \pi)$$

and for any $i \in \{1, 2, 3\}$,

$$I_{i1}(s) = \begin{cases} \frac{1}{17}(s + 16s^3), & |s| \leqslant 1, \\ s^{1/3}, & |s| > 1. \end{cases}$$

Direct computation shows that $M = 1/8$, $\omega = 1$ and (h2) holds with all $\gamma_{i1} = 1/3$. Since for any $i \in \{1, 2, 3\}$,

$$\lim_{s\to 0} \frac{I_{i1}(s)}{s} = \frac{1}{17},$$

choosing $\varepsilon = 1/272$, there exists $\sigma_3 > 0$ such that

$$0 < \frac{15}{272} \leqslant \frac{I_{i1}(s)}{s} \leqslant \frac{1}{16} \quad \text{for all } 0 < |s| \leqslant \sigma_3 \text{ and } i \in \{1, 2, 3\},$$

which combined with the fact that all $I_{i1}(0) = 0$ yields that (h5) holds with all $\mu_{i1} = 1/16$.

In the following, two cases are considered. Two criteria in Corollary 3.3 are employed respectively.

Case 1: F of (3.31) is

$$F(t, x) = \begin{cases} \left(\frac{1}{20}t + \frac{24}{25}e^{1/4}\right)(|x|^4 - |x|^2), & |x| \leqslant 1, \\ \left(\frac{1}{10}t + \frac{48}{25}e^{1/4}\right)(|x|^2 - |x|), & |x| > 1 \end{cases} \tag{3.32}$$

for all $t \in [0, 2\pi]$.

In this case, we have that (A) holds with $a(|x|) = \max\{|x|^4 + |x|^2, 2|x|^2 + 2|x|, 4|x|^3 + 2|x|, 4|x| + 2\}$ and $b(t) = t/20 + 24e^{1/4}/25$. Direct computation shows that (h1) holds. Since

$$\lim_{|x|\to 0} \frac{F(t,x)}{|x|^2} = -\frac{1}{20}t - \frac{24}{25}e^{1/4},$$

choosing $\varepsilon = e^{1/4}/200$, there exists $\sigma_1 > 0$ such that

$$-\Big(\frac{\pi}{10} + \frac{193}{200}e^{1/4}\Big)|x|^2 \leqslant F(t,x) \leqslant -\frac{191}{200}e^{1/4}|x|^2 \tag{3.33}$$

for all $0 < |x| \leqslant \sigma_1$ and $t \in [0, 2\pi]$. It follows from (3.33) and $F(t,0) \equiv 0$ that (h3) holds with $\beta = 1.9$ and $k = 1$. Thus

$$1 + p\mu\omega^2\zeta^2 + p\mu\zeta^2 = 1 + \frac{1}{8}\Big(\frac{1}{2\pi} + 2\pi\Big) \approx 1.8 \leqslant 1.9 = \beta.$$

Thus (H3) holds. Therefore, when F is (3.32), we have that (3.31) has at least two nonzero weak solutions in H^1_T by Corollary 3.3.

Case 2: F of (3.31) is

$$F(t,x) = \begin{cases} \frac{1}{20}(t+1)(|x|^4 - |x|^2), & |x| \leqslant 1, \\ \frac{1}{10}(t+1)(|x|^2 - |x|), & |x| > 1 \end{cases} \tag{3.34}$$

for all $t \in [0, 2\pi]$.

In this case, we have that (A) holds with $a(|x|) = \max\{|x|^4 + |x|^2, 2|x|^2 + 2|x|, 4|x|^3 + 2|x|, 4|x| + 2\}$ and $b(t) = (t+1)/20$. Direct computation shows that (h1) holds. Since

$$\lim_{|x|\to 0} \frac{F(t,x)}{|x|^2} = -\frac{1}{20}(t+1),$$

choosing $\varepsilon = 1/50$, there exists $\sigma_2 > 0$ such that

$$-\Big(\frac{2\pi + 1}{20} + \frac{1}{50}\Big)|x|^2 \leqslant F(t,x) \leqslant -\frac{3}{100}|x|^2 \tag{3.35}$$

for all $0 < |x| \leqslant \sigma_2$ and $t \in [0, 2\pi]$. It follows from (3.35) and $F(t,0) \equiv 0$ that (h4) holds with $\lambda = 0.04$. Then

$$\mu p/T \approx 0.0099 \leqslant 0.04 = \lambda.$$

Thus (H4) holds. Therefore, when F is (3.34), we have that (3.31) has at least two nonzero weak solutions in H^1_T by Corollary 3.3.

In the following, criteria for guaranteeing the existence of nonzero solutions for the second-order impulsive Hamiltonian systems (1.4) will be presented. To the best of our knowledge, the result is new.

When $g(t) \equiv 0$, (IP) is reduced to (1.4), and $M = \int_0^T |g(t)|\,\mathrm{d}t = 0$. Impulsive Hamiltonian systems (1.4) have been considered in [28] by using some critical point theorems. In view of Definition 2.2, the concept of a weak solution for (1.4) is a function $u \in H_T^1$ such that (2.9) holds with $g(t) \equiv 0$ for any $v \in H_T^1$. It is worth stressing that the concept of a weak solution for (1.4) is the same as that in [28]. Moreover, we introduce the following assumptions:

(h3′) There exist constants $\sigma_1 > 0$, $\beta > 0$ and an integer $k \geqslant 1$ such that

$$-\frac{1}{2}(k+1)^2\omega^2|x|^2 \leqslant F(t,x) \leqslant -\frac{1}{2}\beta k^2\omega^2|x|^2$$

for all $|x| \leqslant \sigma_1$ and a.e. $t \in [0,T]$;

(h4′) There exist constants $\sigma_2 > 0$ and $\lambda > 0$ such that

$$-\frac{1}{2}\omega^2|x|^2 \leqslant F(t,x) \leqslant -\frac{1}{2}\lambda\omega^2|x|^2$$

for all $|x| \leqslant \sigma_2$ and a.e. $t \in [0,T]$.

In view of Theorem 3.1, we immediately obtain the following result.

Corollary 3.5. *Suppose that assumption* (A) *holds. Assume that one of the following two conditions holds:*

(H1′) (h1) *and* (h2) *hold with* $\alpha > 1/2$ *and* $\zeta^2 \sum_{(i,j)\in\mathcal{D}} c_{ij} < 1$*;*

(H2′) (h1) *and* (h2) *hold with* $\zeta^2 \sum_{(i,j)\in\mathcal{D}} c_{ij}/2 < \alpha \leqslant 1/2$,

where $c_{ij} = \max\{a_{ij}, b_{ij}\}$ *for all* $i \in \mathcal{A}$, $j \in \mathcal{B}$, *and* $\sum_{(i,j)\in\mathcal{D}} c_{ij} = 0$ *if* $\mathcal{D} = \emptyset$. *Assume also that one of the following two conditions holds:*

(H3′) (h3′) *and* (h5) *hold with* $\beta \geqslant 1 + p\mu\omega^2\zeta^2 + p\mu\zeta^2$*;*

(H4′) (h4′) *and* (h5) *hold with* $\lambda \geqslant \mu p/T$,

where $\mu = \max_{i\in\mathcal{A}, j\in\mathcal{B}}\{\mu_{ij}\}$. *Then the second-order impulsive Hamiltonian system* (1.4) *has at least two nonzero weak solutions in* H_T^1.

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Authors' addresses: *Liang Bai* (corresponding author), College of Mathematics, Taiyuan University of Technology, Taiyuan, Shanxi 030024, People's Republic of China, e-mail: tj_bailiang@126.com; *Binxiang Dai*, School of Mathematics and Statistics, Central South University, Changsha, Hunan 410075, People's Republic of China, e-mail: bxdai@mail.csu.edu.cn.