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NECESSARY CONDITIONS FOR THE $L^p$-CONVERGENCE
$(0 < p < 1)$ OF SINGLE AND DOUBLE TRIGONOMETRIC SERIES

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Abstract. We give necessary conditions in terms of the coefficients for the convergence of a double trigonometric series in the $L^p$-metric, where $0 < p < 1$. The results and their proofs have been motivated by the recent papers of A.S. Belov (2008) and F. Móricz (2010). Our basic tools in the proofs are the Hardy-Littlewood inequality for functions in $H^p$ and the Bernstein-Zygmund inequalities for the derivatives of trigonometric polynomials and their conjugates in the $L^p$-metric, where $0 < p < 1$.

Keywords: trigonometric series; Hardy-Littlewood inequality for functions in $H^p$; Bernstein-Zygmund inequalities for the derivative of trigonometric polynomials in $L^p$-metric for $0 < p < 1$; necessary conditions for the convergence in $L^p$-metric

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1. INTRODUCTION AND PRELIMINARIES

Let $f_1(x)$ be a complex-valued function, periodic with period $2\pi$, and integrable in Lebesgue’s sense, briefly $f_1 \in L^1(\mathbb{T})$. We consider its Fourier series

$$f_1(x) \sim \sum_{k \in \mathbb{Z}} c_k e^{ikx}, \quad x \in \mathbb{T} = [-\pi, \pi).$$

The symmetric partial sums of the series in (1.1) are defined by

$$s_n(f_1) = s_n(f_1; x) := \sum_{|k| \leq n} c_k e^{ikx}, \quad n \in \mathbb{N},$$

where $\mathbb{N} := \{0, 1, 2, \ldots\}$. The $L^1(\mathbb{T})$-norm of a function $f_1$ is defined by

$$\|f_1\| = \|f_1(x)\|_1 := \frac{1}{2\pi} \int_{\mathbb{T}} |f_1(x)| \, dx.$$
The following theorem was proved by A.S. Belov in [2] (also note that some generalizations of the results of [2] were given by the first author in [5]).

**Theorem 1.1.** Assume \( f_1 \in L^1(\mathbb{T}^2) \) and

\[
\|s_n(f_1) - f_1\| \to 0 \quad \text{as} \; n \to \infty.
\]

Then

(1.2) \[
\sum_{k=\lfloor n/2 \rfloor}^{2n} \frac{|c_k| + |c_{-k}|}{(|k - n| + 1)} \to 0 \quad \text{as} \; n \to \infty.
\]

Now, we shall discuss some known results for double Fourier series. Let us suppose that \( f_2(x, y) \) is a complex-valued function, periodic with period \( 2\pi \) in each variable, and integrable in Lebesgue’s sense, briefly \( f_2 \in L^1(\mathbb{T}^2) \). We consider its double Fourier series

(1.3) \[
f_2(x, y) \sim \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} c_{kl} e^{i(kx + ly)}, \quad (x, y) \in \mathbb{T}^2.
\]

The symmetric rectangular partial sums of the double series in (1.3) are defined by

\[
s_{mn}(f_2) = s_{mn}(f_2; x, y) := \sum_{|k| \leq m} \sum_{|l| \leq n} c_{kl} e^{i(kx + ly)}, \quad (m, n) \in \mathbb{N}^2.
\]

The \( L^1(\mathbb{T}^2) \)-norm of a function \( f_2 \) is defined by

\[
\|f_2\| = \|f_2(x, y)\|_1 := \frac{1}{4\pi^2} \int_{\mathbb{T}^2} |f_2(x, y)| \, dx \, dy.
\]

The next statement is due to the third author [6]; it extends the results of A.S. Belov from single to double Fourier series.

**Theorem 1.2.** Assume \( f_2 \in L^1(\mathbb{T}^2) \) and

\[
\|s_{mn}(f_2) - f_2\| \to 0 \quad \text{as} \; m, n \to \infty
\]

independently of one another. Then

(1.4) \[
\sum_{k=\lfloor m/2 \rfloor}^{2m} \sum_{l=\lfloor n/2 \rfloor}^{2n} \frac{|c_{kl}| + |c_{-k,l}| + |c_{k,-l}| + |c_{-k,-l}|}{(|k - m| + 1)(|l - n| + 1)} \to 0 \quad \text{as} \; m, n \to \infty.
\]
For \( f_1 \in L^p(T) \), \( 0 < p < 1 \), the \( L^p \)-metric is defined by

\[
\| f_1 \|_{L^p} = \| f_1 \|_p = \left( \frac{1}{2\pi} \int_T |f_1(x)|^p \, dx \right)^{1/p}.
\]

Similarly, for \( f_2 \in L^p(T^2) \), \( 0 < p < 1 \), the \( L^p \)-metric is defined by

\[
\| f_2 \|_{L^p} = \| f_2 \|_p = \left( \frac{1}{4\pi^2} \int_{T^2} |f_2(x,y)|^p \, dx \, dy \right)^{1/p}.
\]

We recall that for \( 0 < p < 1 \), \( \| \cdot \|_p \) is not a norm since it does not satisfy the triangle inequality, but it is known as a quasi-norm.

The aim of this paper is to obtain necessary conditions for the convergence of the trigonometric series (1.1) and (1.3) in the \( L^p \)-metric.

Our main tools in proving the main results are the Bernstein-Zygmund inequality and the Hardy-Littlewood theorem in the spaces \( L^p \) \((0 < p < 1)\) and \( H^p \) \((0 < p < 1)\), respectively. We also need the Bernstein-Zygmund inequality for trigonometric polynomials and their conjugates. We recall that

\[
\tilde{T}_n(x) = \sum_{|k| \leq n} (-i \text{sign } k) c_k e^{ikx}
\]
is said to be the conjugate to the polynomial

\[
T_n(x) = \sum_{|k| \leq n} c_k e^{ikx}.
\]

Moreover, the \( r \)th derivative of a function \( f(x) \) is denoted by \( f^{(r)}(x) \).

**Lemma 1.3** ([1] or [3, p. 63]). Let \( T_n(x) \) be a trigonometric polynomial of order \( n \) and \( 0 < p < 1 \). Then the inequality

\[
\| T_n^{(r)} \|_p \leq n^{r} \| T_n \|_p
\]
holds true.

**Lemma 1.4** (see [7]). Let \( T_n(x) \) be a trigonometric polynomial of order \( n \) and \( r > 0 \). Then the inequality

\[
\| \tilde{T}_n^{(r)} \|_p \leq \| K_{p,r} \| T_n \|_p
\]
holds true if and only if \( p > 1/(r + 1) \).
Lemma 1.5 ([4, Theorem 16]). If \( \varphi(z) = \sum_{k=0}^{\infty} c_k z^k \), \( |z| < 1 \) and \( \varphi \in H^p \), \( 0 < p < 1 \), then \( \sum_{k=0}^{\infty} (k+1)^{p-2} |c_k|^p \leq K_p \| \varphi \|^p_p \).

Throughout this paper, \( K_p \), or \( K_{p,r} \), denotes a positive constant which depends only on \( p \), or \( p \) and \( r \), respectively, but not necessarily the same at each occurrence.

2. MAIN RESULTS ON SINGLE TRIGONOMETRIC SERIES

We begin with the following auxiliary statements.

Lemma 2.1. For every \( n \in \mathbb{N} \) and \( 0 < p < 1 \), we have

\[
\min \left\{ \left\| \sum_{k=0}^{n} c_k e^{ikx} \right\|_p^p, \left\| \sum_{k=0}^{n} \bar{c}_k e^{-ikx} \right\|_p^p \right\} \geq K_p \max \left\{ \sum_{k=0}^{n} (k+1)^{p-2} |c_k|^p, \sum_{k=0}^{n} (n-k+1)^{p-2} |c_k|^p \right\}.
\]

Proof. The proof of this lemma is an immediate consequence of Lemma 1.5. Indeed, we have

\[
\left\| \sum_{k=0}^{n} c_k e^{ikx} \right\|_p^p = \frac{1}{2\pi} \int_0^{2\pi} \left\| \sum_{k=0}^{n} c_k e^{ikx} \right\|_p^p dx = \frac{1}{2\pi} \int_0^{2\pi} \left\| \sum_{k=0}^{n} \bar{c}_k e^{-ikx} \right\|_p^p dx
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} e^{inx} \sum_{k=0}^{n} c_k e^{-ikx} \left\| \sum_{k=0}^{n} \bar{c}_k e^{(n-k)x} \right\|_p^p dx = \frac{1}{2\pi} \int_0^{2\pi} \left\| \sum_{k=0}^{n} \bar{c}_k e^{i(n-k)x} \right\|_p^p dx \geq K_p \sum_{k=0}^{n} (n-k+1)^{p-2} |c_k|^p.
\]

An analogous inequality holds true for \( \sum_{k=0}^{n} c_k e^{-ikx} \) since

\[
\left\| \sum_{k=0}^{n} c_k e^{-ikx} \right\|_p^p = \left\| \sum_{k=0}^{n} \bar{c}_k e^{ikx} \right\|_p^p.
\]

\( \square \)

Lemma 2.2. For all \( -1 \leq n < \nu, r = 1, 2, \ldots, \) and \( 1/(r+1) < p < 1 \), we have

\[
\max \left\{ \left\| \sum_{k=n+1}^{\nu} k^r c_k e^{ikx} \right\|_p, \left\| \sum_{k=n+1}^{\nu} (-k)^r c_{-k} e^{-ikx} \right\|_p \right\} \leq K_{p,r} \nu^r \| s_\nu - s_n \|_p.
\]

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Proof. Let us denote

\[
\tilde{s}_n := \sum_{|k| \leq n} (-i \text{sign } k)c_k e^{ikx}, \quad s_n := \sum_{|k| \leq n} c_k e^{ikx}
\]

and

\[
\tilde{s}(n, \nu) := \tilde{s}_\nu - \tilde{s}_n, \quad s(n, \nu) := s_\nu - s_n.
\]

Then it is obvious that

\[
s(n, \nu)(r) + i\tilde{s}(n, \nu)(r) = 2i^r \sum_{k=n+1}^\nu k^r c_k e^{ikx},
\]

\[
s(n, \nu)(r) - i\tilde{s}(n, \nu)(r) = 2i^r \sum_{k=n+1}^\nu (-k)^r c_{-k} e^{-ikx}.
\]

In what follows, we shall use the well-known inequality

\[
|a + b|^\beta \leq |a|^\beta + |b|^\beta \quad \text{if } 0 < \beta < 1.
\]

Since

\[
2^p \left\| \sum_{k=n+1}^\nu k^r c_k e^{ikx} \right\|_p^p = \frac{1}{2\pi} \int_T |s(n, \nu)(r) + i\tilde{s}(n, \nu)(r)|^p \, dx \leq \frac{1}{2\pi} \int_T |s(n, \nu)(r)|^p \, dx + \frac{1}{2\pi} \int_T |i\tilde{s}(n, \nu)(r)|^p \, dx,
\]

we may apply Lemmas 1.3 and 1.4 to obtain the first required inequality, i.e.,

\[
\left\| \sum_{k=n+1}^\nu k^r c_k e^{ikx} \right\|_p^p \leq K_{p, r} \nu^p \|s(n, \nu)\|_p^p = K_{p, r} \nu^p \|s_\nu - s_n\|_p^p.
\]

The other inequality can be proved in a similar way. \(\square\)

Lemma 2.3. For \(0 \leq n < \nu\) and \(0 < p < 1\), we have

\[
\left\| \sum_{k=n+1}^\nu k^r c_k e^{ikx} \right\|_p^p \geq K_p \max \left\{ \sum_{k=n+1}^\nu (k-n)^{p-2} |k^r c_k|^p, \sum_{k=n+1}^\nu (\nu - k + 1)^{p-2} |k^r c_k|^p \right\}
\]

and

\[
\left\| \sum_{k=n+1}^\nu (-k)^r c_{-k} e^{-ikx} \right\|_p^p \geq K_p \max \left\{ \sum_{k=n+1}^\nu (k-n)^{p-2} |k^r c_{-k}|^p, \sum_{k=n+1}^\nu (\nu - k + 1)^{p-2} |k^r c_{-k}|^p \right\}.
\]

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Proof. In a way similar to the proof of [6, Lemma 5], we can prove that
\[ \left\| \sum_{k=n+1}^{\nu} k^r c_k e^{ikx} \right\|_p = \left\| \sum_{k_1=0}^{\nu-n-1} k^r c_k e^{ik_1 x} \right\|_p, \]
where \( k_1 = k - n - 1 \). Applying Lemma 2.1 gives the first inequality. The second inequality can be proved in a similar way.

Now, we establish our main result.

Theorem 2.4. Assume that \( f_1 \in L^p(\mathbb{T}) \), \( 0 < p < 1 \), and
\[ \| s_n(f_1) - f_1 \|_p \to 0 \quad \text{as } n \to \infty. \]
Then
\[ (2.1) \quad \sum_{k=[n/2]}^{2n} \frac{|c_k|^p + |c_{-k}|^p}{(|k-n|+1)^{2-p}} \to 0 \quad \text{as } n \to \infty. \]

Proof. Let \( n \geq 2 \) and set
\[ C_k(p) := |c_k|^p + |c_{-k}|^p \quad \text{for } k \in \mathbb{N} \quad \text{and} \quad r := \left[\frac{1}{p}\right] - 1 + 1, \]
where \([\cdot]\) denotes the integer part of a real number. Applying Lemmas 2.2 and 2.3 with \( \nu := 2n \) we obtain
\[ \sum_{k=n+1}^{2n} \frac{k^r C_k(p)}{(k-n)^{2-p}} \leq K_p \left\{ \left\| \sum_{k=n+1}^{2n} k^r c_k e^{ikx} \right\|_p + \left\| \sum_{k=n+1}^{2n} (-k)^r c_{-k} e^{-ikx} \right\|_p \right\} \leq K_{p,r} n^r \| s_{2n}(f_1) - s_n(f_1) \|_p = K_p n^r \| s_{2n}(f_1) - s_n(f_1) \|_p. \]

Now, repeating the above reasoning with \( \left\lfloor n/2 \right\rfloor - 1 \) in place of \( n \) in the lower limit of the summation and with \( n \) in place of \( 2n \) in the upper limit of the summation, we get
\[ \sum_{k=\lfloor n/2 \rfloor}^{n} \frac{k^r C_k(p)}{(n-k+1)^{2-p}} \leq K_{p,n^r} \| s_n(f_1) - s_{\lfloor n/2 \rfloor - 1}(f_1) \|_p. \]

Using the previous two inequalities yields
\[ \sum_{k=\lfloor n/2 \rfloor}^{2n} \frac{C_k(p)}{(|k-n|+1)^{2-p}} \leq \frac{K_p}{n^r p} \left\{ \sum_{k=n+1}^{2n} \frac{k^r C_k(p)}{(k-n)^{2-p}} + \sum_{k=\lfloor n/2 \rfloor}^{n} \frac{k^r C_k(p)}{(n-k+1)^{2-p}} \right\} \leq K_p \max_{1 \leq \nu_1 < \nu_2 \leq 2n} \| s_{\nu_2}(f_1) - s_{\nu_1}(f_1) \|_p. \]

This proves (2.1). \( \square \)
3. Main results on double trigonometric series

We begin with the following four lemmas.

**Lemma 3.1.** For all \((m, n) \in \mathbb{N}^2\) and \(0 < p < 1\), we have

\[
\min \left\{ \left\| \sum_{k=0}^{m} \sum_{l=0}^{n} c_{kl} e^{i(kx+ly)} \right\|_p^p, \left\| \sum_{k=0}^{m} \sum_{l=0}^{n} c_{kl} e^{i(-kx-ly)} \right\|_p^p, \left\| \sum_{k=0}^{m} \sum_{l=0}^{n} c_{kl} e^{i(kx-ly)} \right\|_p^p, \left\| \sum_{k=0}^{m} \sum_{l=0}^{n} c_{kl} e^{i(-kx+ly)} \right\|_p^p \right\}
\]

\[
\geq K_p \max \left\{ \sum_{k=0}^{m} \sum_{l=0}^{n} ((k+1)(l+1))^{p-2} |c_{kl}|^p, \sum_{k=0}^{m} \sum_{l=0}^{n} ((m-k+1)(l+1))^{p-2} |c_{kl}|^p, \sum_{k=0}^{m} \sum_{l=0}^{n} ((k+1)(n-l+1))^{p-2} |c_{kl}|^p, \sum_{k=0}^{m} \sum_{l=0}^{n} ((m-k+1)(n-l+1))^{p-2} |c_{kl}|^p \right\}.
\]

**Proof.** Applying Lemma 2.1 twice together with Fubini’s theorem, we obtain

\[
4\pi^2 \left\| \sum_{k=0}^{m} \sum_{l=0}^{n} c_{kl} e^{i(kx+ly)} \right\|_p^p = \int_{\mathbb{T}^2} \left| \sum_{k=0}^{m} \sum_{l=0}^{n} c_{kl} e^{i(kx+ly)} \right|_p^p \, dx \, dy
\]

\[
= \int_{\mathbb{T}} \left( \int_{\mathbb{T}} \left| \sum_{k=0}^{m} \left( \sum_{l=0}^{n} c_{kl} e^{ily} \right) e^{ikx} \right|_p^p \, dx \right) \, dy
\]

\[
\geq K_p \int_{\mathbb{T}} \left( \sum_{k=0}^{m} (k+1)^{p-2} \left| \sum_{l=0}^{n} c_{kl} e^{ily} \right|_p^p \right) \, dy = K_p \int_{\mathbb{T}} \left( \sum_{l=0}^{n} (l+1)^{p-2} \left| \sum_{k=0}^{m} c_{kl} e^{ily} \right|_p^p \right) \, dy
\]

\[
\geq K_p \sum_{k=0}^{m} (k+1)^{p-2} \sum_{l=0}^{n} (l+1)^{p-2} |c_{kl}|^p = K_p \sum_{k=0}^{m} \sum_{l=0}^{n} ((k+1)(l+1))^{p-2} |c_{kl}|^p.
\]

The other fifteen inequalities can be proved in an analogous way. We do not enter into further details.

Next, we prove a lemma which is an extension of Lemmas 1.3 and 1.4 from single to double trigonometric polynomials.
Lemma 3.2. For all \((m, n) \in \mathbb{N}^2\), \(r_1, r_2 = 1, 2, \ldots\), and \(\max\{1/r_1, 1/r_2\} < p < 1\), we have

\[
\max \left\{ \left\| \frac{\partial^{r_1 + r_2}}{\partial x^{r_1} \partial y^{r_2}} \sum_{|k| \leq m} \sum_{|l| \leq n} c_{kl} e^{i(kx + ly)} \right\|_p, \left\| \frac{\partial^{r_1 + r_2}}{\partial x^{r_1} \partial y^{r_2}} \sum_{|k| \leq m} \sum_{|l| \leq n} (-i \text{ sign } k) c_{kl} e^{i(kx + ly)} \right\|_p, \left\| \frac{\partial^{r_1 + r_2}}{\partial x^{r_1} \partial y^{r_2}} \sum_{|k| \leq m} \sum_{|l| \leq n} (-i \text{ sign } l) c_{kl} e^{i(kx + ly)} \right\|_p, \right. \\
\left. \left\| \frac{\partial^{r_1 + r_2}}{\partial x^{r_1} \partial y^{r_2}} \sum_{|k| \leq m} \sum_{|l| \leq n} (-i \text{ sign } k)(-i \text{ sign } l) c_{kl} e^{i(kx + ly)} \right\|_p \right\} \\
\leq K_{p, r_1, r_2} m^{r_1} n^{r_2} \left\| \sum_{|k| \leq m} \sum_{|l| \leq n} c_{kl} e^{i(kx + ly)} \right\|_p.
\]

Proof. By Lemmas 1.3 and 1.4, for example, we have

\[
\left\| \frac{\partial^{r_1 + r_2}}{\partial x^{r_1} \partial y^{r_2}} \sum_{|k| \leq m} \sum_{|l| \leq n} (-i \text{ sign } k) c_{kl} e^{i(kx + ly)} \right\|_p = \left\| \frac{\partial^{r_1}}{\partial x^{r_1}} \sum_{|k| \leq m} (-i \text{ sign } k) \left( \frac{\partial^{r_2}}{\partial x^{r_2}} \sum_{|l| \leq n} c_{kl} e^{i ly} \right) e^{ikx} \right\|_p \\
\leq K_{p, r_1} m^{r_1} \left\| \frac{\partial^{r_2}}{\partial x^{r_2}} \sum_{|l| \leq n} \left( \sum_{|k| \leq m} c_{kl} e^{i kx} \right) e^{i ly} \right\|_p \\
\leq K_{p, r_1, r_2} m^{r_1} n^{r_2} \left\| \sum_{|k| \leq m} \sum_{|l| \leq n} c_{kl} e^{i(kx + ly)} \right\|_p.
\]

This proves the second inequality. The other three inequalities can be proved in a similar way.

Lemma 3.3. For all \(-1 \leq m < \mu\), \(-1 \leq n < \nu\), \(r_1, r_2 = 1, 2, \ldots\), and \(\max\{1/r_1, 1/r_2\} < p < 1\), we have

\[
\max \left\{ \left\| \sum_{k=m+1}^{\mu} \sum_{l=n+1}^{\nu} k^{r_1} l^{r_2} c_{kl} e^{i(kx + ly)} \right\|_p, \left\| \sum_{k=m+1}^{\mu} \sum_{l=n+1}^{\nu} (-k)^{r_1} l^{r_2} c_{-k, -l} e^{i(-kx + ly)} \right\|_p, \right. \\
\left. \left\| \sum_{k=m+1}^{\mu} \sum_{l=n+1}^{\nu} k^{r_1} (-l)^{r_2} c_{k, -l} e^{i(kx - ly)} \right\|_p, \left\| \sum_{k=m+1}^{\mu} \sum_{l=n+1}^{\nu} (-k)^{r_1} (-l)^{r_2} c_{-k, -l} e^{i(-kx - ly)} \right\|_p \right\} \\
\leq K_{p, r_1, r_2} \mu^{r_1} \nu^{r_2} \left\| s_{\mu
u} - s_{m\nu} - s_{\mu n} + s_{mn} \right\|_p.
\]

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where
\[ s_{-1,n} = s_{m,-1} = s_{-1,-1} = 0 \quad \text{and} \quad s_{mn} := \sum_{|k| \leq m} \sum_{|l| \leq n} c_{kl}e^{i(kx+ly)}, \quad (m, n) \in \mathbb{N}^2. \]

**Proof.** We introduce the notation
\[
\begin{align*}
\hat{s}_{mn}^{(1,0)} &:= \sum_{|k| \leq m} \sum_{|l| \leq n} (-i \text{sign } k)c_{kl}e^{i(kx+ly)}, \\
\hat{s}_{mn}^{(0,1)} &:= \sum_{|k| \leq m} \sum_{|l| \leq n} (-i \text{sign } l)c_{kl}e^{i(kx+ly)}, \\
\hat{s}_{mn}^{(1,1)} &:= \sum_{|k| \leq m} \sum_{|l| \leq n} (-i \text{sign } k)(-i \text{sign } l)c_{kl}e^{i(kx+ly)},
\end{align*}
\]

and
\[
\begin{align*}
s(m, n; \mu, \nu) &:= s_{\mu\nu} - s_{m\nu} - s_{\mu n} + s_{mn} = \sum_{m < |k| \leq \mu} \sum_{n < |l| \leq \nu} c_{kl}e^{i(kx+ly)}, \\
\hat{s}^{(1,0)}(m, n; \mu, \nu) &:= \hat{s}^{(1,0)}_{\mu\nu} - \hat{s}^{(1,0)}_{m\nu} - \hat{s}^{(1,0)}_{\mu n} + \hat{s}^{(1,0)}_{mn}, \\
\hat{s}^{(0,1)}(m, n; \mu, \nu) &:= \hat{s}^{(0,1)}_{\mu\nu} - \hat{s}^{(0,1)}_{m\nu} - \hat{s}^{(0,1)}_{\mu n} + \hat{s}^{(0,1)}_{mn}, \\
\hat{s}^{(1,1)}(m, n; \mu, \nu) &:= \hat{s}^{(1,1)}_{\mu\nu} - \hat{s}^{(1,1)}_{m\nu} - \hat{s}^{(1,1)}_{\mu n} + \hat{s}^{(1,1)}_{mn}.
\end{align*}
\]

Using the equalities (see in [6])
\[
\begin{align*}
\frac{\partial^2}{\partial x \partial y} \{s(m, n; \mu, \nu) + is^{(1,0)}(m, n; \mu, \nu) + is^{(0,1)}(m, n; \mu, \nu) + i^2\hat{s}^{(1,1)}(m, n; \mu, \nu)\} \\
= -4 \sum_{k=m+1}^\mu \sum_{l=n+1}^{\nu} klc_{kl}e^{i(kx+ly)}, \\
\frac{\partial^2}{\partial x \partial y} \{s(m, n; \mu, \nu) - is^{(1,0)}(m, n; \mu, \nu) + is^{(0,1)}(m, n; \mu, \nu) - i^2\hat{s}^{(1,1)}(m, n; \mu, \nu)\} \\
= -4 \sum_{k=m+1}^\mu \sum_{l=n+1}^{\nu} (-k)lc_{-k}e^{i(kx-ly)}, \\
\frac{\partial^2}{\partial x \partial y} \{s(m, n; \mu, \nu) + is^{(1,0)}(m, n; \mu, \nu) - is^{(0,1)}(m, n; \mu, \nu) - i^2\hat{s}^{(1,1)}(m, n; \mu, \nu)\} \\
= -4 \sum_{k=m+1}^\mu \sum_{l=n+1}^{\nu} k(-l)c_{k}e^{i(kx-ly)}, \\
\frac{\partial^2}{\partial x \partial y} \{s(m, n; \mu, \nu) - is^{(1,0)}(m, n; \mu, \nu) - is^{(0,1)}(m, n; \mu, \nu) + i^2\hat{s}^{(1,1)}(m, n; \mu, \nu)\} \\
= -4 \sum_{k=m+1}^\mu \sum_{l=n+1}^{\nu} (-k)(-l)c_{-k}e^{i(kx-ly)},
\end{align*}
\]
we obtain the following representations:

\[ \frac{\partial^{r_1+r_2}}{\partial x^{r_1} \partial y^{r_2}} \{ s(m, n; \mu, \nu) + i s^{(1,0)}(m, n; \mu, \nu) + i s^{(0,1)}(m, n; \mu, \nu) + i^2 s^{(1,1)}(m, n; \mu, \nu) \} \]

\[ = 4i^{r_1+r_2} \sum_{k=m+1}^\infty \sum_{l=n+1}^\infty k^{r_1} l^{r_2} c_{kl} e^{i(kx+ly)} , \]

\[ \frac{\partial^{r_1+r_2}}{\partial x^{r_1} \partial y^{r_2}} \{ s(m, n; \mu, \nu) - i s^{(1,0)}(m, n; \mu, \nu) + i s^{(0,1)}(m, n; \mu, \nu) - i^2 s^{(1,1)}(m, n; \mu, \nu) \} \]

\[ = 4i^{r_1+r_2} \sum_{k=m+1}^\infty \sum_{l=n+1}^\infty (-k)^{r_1} l^{r_2} c_{-k, l} e^{-i(kx+ly)} , \]

\[ \frac{\partial^{r_1+r_2}}{\partial x^{r_1} \partial y^{r_2}} \{ s(m, n; \mu, \nu) + i s^{(1,0)}(m, n; \mu, \nu) - i s^{(0,1)}(m, n; \mu, \nu) - i^2 s^{(1,1)}(m, n; \mu, \nu) \} \]

\[ = 4i^{r_1+r_2} \sum_{k=m+1}^\infty \sum_{l=n+1}^\infty k^{r_1} (-)^{r_2} c_{k, -l} e^{i(kx-ly)} , \]

\[ \frac{\partial^{r_1+r_2}}{\partial x^{r_1} \partial y^{r_2}} \{ s(m, n; \mu, \nu) - i s^{(1,0)}(m, n; \mu, \nu) - i s^{(0,1)}(m, n; \mu, \nu) + i^2 s^{(1,1)}(m, n; \mu, \nu) \} \]

\[ = 4i^{r_1+r_2} \sum_{k=m+1}^\infty \sum_{l=n+1}^\infty (-k)^{r_1} (-)^{r_2} c_{-k, -l} e^{-i(kx-ly)} . \]

Since

\[ 4^p \left\| \sum_{k=m+1}^\infty \sum_{l=n+1}^\infty k^{r_1} l^{r_2} c_{kl} e^{i(kx+ly)} \right\|^p \]

\[ = \frac{1}{4\pi^2} \int_{\mathbb{T}_2} \left\| \frac{\partial^{r_1+r_2}}{\partial x^{r_1} \partial y^{r_2}} \{ s(m, n; \mu, \nu) + i s^{(1,0)}(m, n; \mu, \nu) + i s^{(0,1)}(m, n; \mu, \nu) + i^2 s^{(1,1)}(m, n; \mu, \nu) \} \right\|^p \, dx \, dy \]

\[ \leq \frac{1}{4\pi^2} \int_{\mathbb{T}_2} \left\| \frac{\partial^{r_1+r_2}}{\partial x^{r_1} \partial y^{r_2}} s(m, n; \mu, \nu) \right\|^p \, dx \, dy \]

\[ + \frac{1}{4\pi^2} \int_{\mathbb{T}_2} \left| i \frac{\partial^{r_1+r_2}}{\partial x^{r_1} \partial y^{r_2}} s^{(1,0)}(m, n; \mu, \nu) \right|^p \, dx \, dy \]

\[ + \frac{1}{4\pi^2} \int_{\mathbb{T}_2} \left| i \frac{\partial^{r_1+r_2}}{\partial x^{r_1} \partial y^{r_2}} s^{(0,1)}(m, n; \mu, \nu) \right|^p \, dx \, dy \]

\[ + \frac{1}{4\pi^2} \int_{\mathbb{T}_2} \left| i^2 \frac{\partial^{r_1+r_2}}{\partial x^{r_1} \partial y^{r_2}} s^{(1,1)}(m, n; \mu, \nu) \right|^p \, dx \, dy , \]
applying Lemma 3.2 four times gives

\[
\left\| \sum_{k=m+1}^{\mu} \sum_{l=n+1}^{\nu} k^{r_1} l^{r_2} C_{kl} e^{i(kx+ly)} \right\|^p_p \leq K_{p,r_1,r_2} \mu^{r_1} \nu^{r_2} \| s(m, n; \mu, \nu) \|^p_p
\]

\[
= K_{p,r_1,r_2} \mu^{r_1} \nu^{r_2} \| s_{\mu} - s_{\nu} - s_{\mu n} + s_{mn} \|^p_p.
\]

This means the first inequality is proved. The other three inequalities can be proved in an analogous way.

\[ \square \]

**Lemma 3.4.** For \( 0 \leq m < \mu, 0 \leq n < \nu, r_1, r_2 = 1, 2, \ldots, \) and \( 0 < p < 1, \) we have

\[
\left\| \sum_{k=m+1}^{\mu} \sum_{l=n+1}^{\nu} k^{r_1} l^{r_2} C_{kl} e^{i(kx+ly)} \right\|^p_p \geq K_{p,r_1,r_2} \max \left\{ \sum_{k=m+1}^{\mu} \sum_{l=n+1}^{\nu} ((k-m)(l-n))^{p-2} |k^{r_1} l^{r_2} c_{kl}|^p, \right. \\
\sum_{k=m+1}^{\mu} \sum_{l=n+1}^{\nu} ((\mu-k+1)(l-n))^{p-2} |k^{r_1} l^{r_2} c_{kl}|^p, \right. \\
\sum_{k=m+1}^{\mu} \sum_{l=n+1}^{\nu} ((k-m)(\nu-l+1))^{p-2} |k^{r_1} l^{r_2} c_{kl}|^p, \right. \\
\sum_{k=m+1}^{\mu} \sum_{l=n+1}^{\nu} ((\mu-k+1)(\nu-l+1))^{p-2} |k^{r_1} l^{r_2} c_{kl}|^p \right\},
\]

and three other analogous inequalities hold involving \( c_{-k,l}, c_{k,-l}, \) and \( c_{-k,-l}, \) respectively, in place of \( c_{kl}. \)

**Proof.** Using the idea of the third author (see [6, in the proof of Lemma 5]), we may write

\[
\left\| \sum_{k=m+1}^{\mu} \sum_{l=n+1}^{\nu} k^{r_1} l^{r_2} C_{kl} e^{i(kx+ly)} \right\|^p_p = \left\| \sum_{k_1=0}^{\mu-m-1} \sum_{l_1=0}^{\nu-n-1} k^{r_1} l^{r_2} C_{kl} e^{i(k_1 x + l_1 y)} \right\|^p_p,
\]

where \( k_1 = k - m - 1 \) and \( l_1 = l - n - 1. \) Applying Lemma 3.1 in the case of the above equality, we immediately obtain the inequality of the lemma which involves \( c_{kl}. \) The other three inequalities which involve \( c_{-k,l}, c_{k,-l} \) and \( c_{-k,-l}, \) respectively, can be proved in a similar way.

\[ \square \]

Now, we pass to the main result. Indeed, the following statement holds true.
\textbf{Theorem 3.5.} Assume $f_2 \in L^p(\mathbb{T}^2), 0 < p < 1$, and
\[
\|s_{mn}(f_2) - f_2\|_p \to 0 \quad \text{as } m, n \to \infty
\]

independently of one another. Then
\[
\sum_{k=[m/2]}^{2m} \sum_{l=[n/2]}^{2n} \frac{|c_{kl}|^p + |c_{-k,l}|^p + |c_{k,-l}|^p + |c_{-k,-l}|^p}{(||k - m|| + 1)(||l - n|| + 1)^{2-p}} \to 0 \quad \text{as } m, n \to \infty.
\]

\textbf{Proof.} Denote
\[
C_{kl}(p) := |c_{kl}|^p + |c_{-k,l}|^p + |c_{k,-l}|^p + |c_{-k,-l}|^p, \quad (k, l) \in \mathbb{N}^2.
\]

Let $m, n \geq 2$ and set $r_1 := r_2 := [1/p - 1] + 1$. Then applying Lemmas 3.3 and 3.4 with $\mu := 2m$ and $\nu := 2n$, we find that
\[
\sum_{k=m+1}^{2m} \sum_{l=n+1}^{2n} \frac{(k^r_1 l^r_2)^p C_{kl}(p)}{(||k - m||)(||l - n||)^{2-p}} \leq K_p \left\{ \left\| \sum_{k=m+1}^{2m} \sum_{l=n+1}^{2n} k^r_1 l^r_2 c_{kl} e^{i(kx+ly)} \right\|_p^p \right. \\
+ \left. \left\| \sum_{k=m+1}^{2m} \sum_{l=n+1}^{2n} (-k)^r_1 l^r_2 c_{-k,l} e^{i(-kx+ly)} \right\|_p^p \right. \\
+ \left. \left\| \sum_{k=m+1}^{2m} \sum_{l=n+1}^{2n} k^r_1 (-l)^r_2 c_{k,-l} e^{i(kx-ly)} \right\|_p^p \right. \\
+ \left. \left\| \sum_{k=m+1}^{2m} \sum_{l=n+1}^{2n} (-k)^r_1 (-l)^r_2 c_{-k,-l} e^{i(-kx-ly)} \right\|_p^p \right\} \\
\leq K_p m^{r_1 p} n^{r_2 p} \left\| s_{2m,2n}(f_2) - s_{m,2n}(f_2) + s_{2m,n}(f_2) + s_{m,n}(f_2) \right\|_p^p.
\]

Repeating the above reasoning with $[m/2] - 1$ in place of $m$ in the lower limit of the summation and with $m$ in place of $2m$ in the upper limit of the summation with respect to $k$ gives
\[
\sum_{k=[m/2]}^{m} \sum_{l=n+1}^{2n} \frac{(k^r_1 l^r_2)^p C_{kl}(p)}{(||m - k + 1||)(||l - n||)^{2-p}} \leq K_p m^{r_1 p} n^{r_2 p} \left\| s_{m,2n}(f_2) - s_{[m/2]-1,2n}(f_2) - s_{m,n}(f_2) + s_{[m/2]-1,n}(f_2) \right\|_p^p.
\]
The symmetric counterpart of this inequality is
\[
\sum_{k=m+1}^{2n} \sum_{l=[n/2]}^{n} \frac{(k^1 l^{r2})^p C_{kl}(p)}{(k-m)(l-m+1)^{2-p}} \leq K_p m^{r1} n^{r2} p \left\| s_{2m,n}(f_2) - s_{m,n}(f_2) - s_{2m,[n/2]-1}(f_2) + s_{m,[n/2]-1}(f_2) \right\|^p.
\]
In a similar way we obtain that
\[
\sum_{k=[m/2]}^{m} \sum_{l=[n/2]}^{n} \frac{(k^1 l^{r2})^p C_{kl}(p)}{(m-k+1)(n-l+1)^{2-p}} \leq K_p m^{r1} n^{r2} p \left\| s_{m,n}(f_2) - s_{[m/2]-1,n}(f_2) - s_{m,[n/2]-1}(f_2) + s_{[m/2]-1,[n/2]-1}(f_2) \right\|^p.
\]
Now, it follows from the last four inequalities that
\[
\sum_{k=[n/2]}^{2n} \sum_{l=[n/2]}^{2n} \frac{C_{kl}(p)}{((|k-m|+1)(|l-n|+1))^{2-p}} \leq \frac{K_p}{m^{r1} n^{r2} p} \left\{ \sum_{k=m+1}^{2m} \sum_{l=n+1}^{2n} \frac{(k^1 l^{r2})^p C_{kl}(p)}{(k-m)(l-n)^{2-p}} + \sum_{k=[m/2]}^{m} \sum_{l=n+1}^{2n} \frac{(k^1 l^{r2})^p C_{kl}(p)}{(m-k+1)(l-n)^{2-p}} + \sum_{k=m+1}^{2m} \sum_{l=[n/2]}^{n} \frac{(k^1 l^{r2})^p C_{kl}(p)}{(k-m)(n-l+1)^{2-p}} + \sum_{k=[m/2]}^{m} \sum_{l=[n/2]}^{n} \frac{(k^1 l^{r2})^p C_{kl}(p)}{(m-k+1)(n-l+1)^{2-p}} \right\}
\leq K_p \max_{[m/2]-1 \leq \mu_1 < \mu_2 \leq 2m} \max_{[n/2]-1 \leq \nu_1 < \nu_2 \leq n} \left\| s_{\mu_2,\nu_2}(f_2) - s_{\mu_1,\nu_1}(f_2) - s_{\mu_2,\nu_1}(f_2) + s_{\mu_1,\nu_2}(f_2) \right\|^p,
\]
and hence (3.1) is proved. \qed

References


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