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contain these Terms of use.
UNIQUENESS OF ENTIRE FUNCTIONS CONCERNING
DIFFERENCE POLYNOMIALS

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Abstract. In this paper, we investigate the uniqueness problem of difference polynomials sharing a small function. With the notions of weakly weighted sharing and relaxed weighted sharing we prove the following: Let \( f(z) \) and \( g(z) \) be two transcendental entire functions of finite order, and \( \alpha(z) \) a small function with respect to both \( f(z) \) and \( g(z) \). Suppose that \( c \) is a non-zero complex constant and \( n \geq 7 \) (or \( n \geq 10 \)) is an integer. If \( f^n(z)(f(z)−1)f(z+c) \) and \( g^n(z)(g(z)−1)g(z+c) \) share \("(\alpha(z),2)"\) (or \("(\alpha(z),2)^*"\)), then \( f(z) \equiv g(z) \). Our results extend and generalize some well known previous results.

Keywords: entire function; difference polynomial; uniqueness

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1. Introduction, definitions and results

By a meromorphic function we shall always mean a meromorphic function in the complex plane. Let \( k \) be a positive integer or infinity and \( a \in \mathbb{C} \cup \{\infty\} \). Set \( E(a, f) = \{z: f(z) − a = 0\} \), where a zero point with multiplicity \( k \) is counted \( k \) times in the set. If these zeros points are only counted once, then we denote the set by \( \overline{E}(a, f) \). Let \( f \) and \( g \) be two nonconstant meromorphic functions. If \( E(a, f) = E(a, g) \), then we say that \( f \) and \( g \) share the value \( a \) CM; if \( \overline{E}(a, f) = \overline{E}(a, g) \), then we say that \( f \) and \( g \) share the value \( a \) IM. We denote by \( E_k(a, f) \) the set of all \( a \)-points of \( f \) with multiplicities not exceeding \( k \), where an \( a \)-point is counted according to its multiplicity. Also we denote by \( \overline{E}_k(a, f) \) the set of distinct \( a \)-points of \( f \) with multiplicities not greater than \( k \). It is assumed that the reader is familiar with the notations of Nevanlinna theory such as \( T(r, f) \), \( m(r, f) \), \( N(r, f) \), \( N(r, f) \), \( S(r, f) \) and so on, that can be found, for instance, in [5], [13]. We denote by \( N_k(r, 1/(f − a)) \) the counting function for zeros of \( f − a \) with multiplicity less or equal to \( k \), and by
\( N_k \left( r, \frac{1}{f-a} \right) \) the corresponding one for which multiplicity is not counted. Let \( N_{(k)}(r, 1/(f-a)) \) be the counting function for zeros of \( f-a \) with multiplicity at least \( k \) and \( \overline{N}_{(k)}(r, 1/(f-a)) \) the corresponding one for which multiplicity is not counted. Set

\[
N_k \left( r, \frac{1}{f-a} \right) = \overline{N} \left( r, \frac{1}{f-a} \right) + \overline{N} \left( r, \frac{1}{f-a} \right) + \ldots + \overline{N}(k) \left( r, \frac{1}{f-a} \right).
\]

Let \( N_E(r, a; f, g)(\overline{N}_E(r, a; f, g)) \) be the counting function (reduced counting function) of all common zeros of \( f-a \) and \( g-a \) with the same multiplicities and \( N_0(r, a; f, g)(\overline{N}_0(r, a; f, g)) \) the counting function (reduced counting function) of all common zeros of \( f-a \) and \( g-a \) ignoring multiplicities. If

\[
\overline{N}(r, \frac{1}{f-a}) + \overline{N}(r, \frac{1}{g-a}) - 2N_E(r, a; f, g) = S(r, f) + S(r, g),
\]

then we say that \( f \) and \( g \) share a “CM”. On the other hand, if

\[
\overline{N}(r, \frac{1}{f-a}) + \overline{N}(r, \frac{1}{g-a}) - 2N_0(r, a; f, g) = S(r, f) + S(r, g),
\]

then we say that \( f \) and \( g \) share a “IM”.

We now explain in the following definition the notion of weakly weighted sharing which was introduced by Lin and Lin [8].

**Definition 1** ([8]). Let \( f \) and \( g \) share a “IM” and \( k \) be a positive integer or \( \infty \). \( \overline{N}_k^E(r, a; f, g) \) denotes the reduced counting function of those \( a \)-points of \( f \) whose multiplicities are equal to the corresponding \( a \)-points of \( g \), and both of their multiplicities are not greater than \( k \). \( \overline{N}^O_{(k)}(r, a; f, g) \) denotes the reduced counting function of those \( a \)-points of \( f \) which are \( a \)-points of \( g \), and both of their multiplicities are not less than \( k \).

**Definition 2** ([8]). For \( a \in C \cup \{ \infty \} \), if \( k \) is a positive integer or \( \infty \) and

\[
\overline{N}_k \left( r, \frac{1}{f-a} \right) - \overline{N}_k^E \left( r, a; f, g \right) = S(r, f),
\]

\[
\overline{N}_k \left( r, \frac{1}{g-a} \right) - \overline{N}_k^E \left( r, a; f, g \right) = S(r, g),
\]

\[
\overline{N}_{(k+1)} \left( r, \frac{1}{f-a} \right) - \overline{N}^O_{(k+1)} \left( r, a; f, g \right) = S(r, f),
\]

\[
\overline{N}_{(k+1)} \left( r, \frac{1}{g-a} \right) - \overline{N}^O_{(k+1)} \left( r, a; f, g \right) = S(r, g),
\]

or if \( k = 0 \) and

\[
\overline{N} \left( r, \frac{1}{f-a} \right) - \overline{N}_0 \left( r, a; f, g \right) = S(r, f), \overline{N} \left( r, \frac{1}{g-a} \right) - \overline{N}_0 \left( r, a; f, g \right) = S(r, g),
\]

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then we say \( f \) and \( g \) weakly share \( a \) with weight \( k \). Here we write \( f, g \) share \( "(a, k)" \) to mean that \( f, g \) weakly share \( a \) with weight \( k \).

Now it is clear from Definition 2 that weakly weighted sharing is a scaling between IM and CM.

Recently, A. Banerjee and S. Mukherjee [1] introduced another sharing notion which is also a scaling between IM and CM but weaker than weakly weighted sharing.

**Definition 3 ([1]).** We denote by \( N(r, a; f| = p; g| = q) \) the reduced counting function of common \( a \)-points of \( f \) and \( g \) with multiplicities \( p \) and \( q \), respectively.

**Definition 4 ([1]).** Let \( f, g \) share a "IM". Also let \( k \) be a positive integer or \( \infty \) and \( a \in C \cup \{\infty\} \). If

\[
\sum_{p,q \leq k} N(r, a; f| = p; g| = q) = S(r),
\]

then we say \( f \) and \( g \) share \( a \) with weight \( k \) in a relaxed manner. Here we write \( f \) and \( g \) share \( (a, k)^* \) to mean that \( f \) and \( g \) share \( a \) with weight \( k \) in a relaxed manner.

W. K. Hayman proposed the following well-known conjecture in [6].

**Hayman’s conjecture.** If an entire function \( f \) satisfies \( f^n f' \neq 1 \) for all positive integers \( n \in N \), then \( f \) is a constant.

It has been verified by Hayman himself in [7] for the case \( n > 1 \) and Clunie in [3] for the case \( n \geq 1 \), respectively.

It is well-known that if \( f \) and \( g \) share four distinct values CM, then \( f \) is a Möbius transformation of \( g \). In 1997, corresponding to the famous conjecture of Hayman, Yang and Hua studied the unicity of differential monomials and obtained the following theorem.

**Theorem A ([12]).** Let \( f(z) \) and \( g(z) \) be two nonconstant entire functions, \( n \geq 6 \) a positive integer. If \( f^n f' \) and \( g^n g' \) share 1 CM, then either \( f(z) = c_1 e^{cz} \), \( g(z) = c_2 e^{-cz} \), where \( c_1, c_2, c \) are three constants satisfying \( (c_1 c_2)^{n+1} c^2 = -1 \), or \( f(z) \equiv tg(z) \) for a constant \( t \) such that \( t^{n+1} = 1 \).

In 2001, Fang and Hong studied the unicity of differential polynomials of the form \( f^n (f - 1)f' \) and proved the following uniqueness theorem.

**Theorem B ([4]).** Let \( f \) and \( g \) be two transcendental entire functions, \( n \geq 11 \) an integer. If \( f^n (f - 1)f' \) and \( g^n (g - 1)g' \) share the value 1 CM, then \( f \equiv g \).

In 2004, Lin and Yi extended the above theorem as to the fixed-point. They proved the following result.
Theorem C ([9]). Let \( f \) and \( g \) be two transcendental entire functions, \( n \geq 7 \) an integer. If \( f^n(f - 1)f' \) and \( g^n(g - 1)g' \) share \( z \) CM, then \( f \equiv g \).


Theorem D ([15]). Let \( f(z) \) and \( g(z) \) be two transcendental entire functions of finite order, and \( \alpha(z) \) be a small function with respect to both \( f(z) \) and \( g(z) \). Suppose that \( c \) is a non-zero complex constant and \( n \geq 7 \) is an integer. If \( f^n(z)(f(z) - 1) \times f(z + c) \) and \( g^n(z)(g(z) - 1)g(z + c) \) share \( \alpha(z) \) CM, then \( f(z) \equiv g(z) \).

Now one may ask the following question which is the motivation of the paper: Can the nature of small function \( \alpha(z) \) be relaxed in the above theorem? Considering this question, we prove the following results.

Theorem 1. Let \( f(z) \) and \( g(z) \) be two transcendental entire functions of finite order, and \( \alpha(z) \) be a small function with respect to both \( f(z) \) and \( g(z) \). Suppose that \( c \) is a non-zero complex constant and \( n \geq 7 \) is an integer. If \( f^n(z)(f(z) - 1) \times f(z + c) \) and \( g^n(z)(g(z) - 1)g(z + c) \) share \( \alpha(z) \) CM, then \( f(z) \equiv g(z) \).

Theorem 2. Let \( f(z) \) and \( g(z) \) be two transcendental entire functions of finite order, and \( \alpha(z) \) be a small function with respect to both \( f(z) \) and \( g(z) \). Suppose that \( c \) is a non-zero complex constant and \( n \geq 10 \) is an integer. If \( f^n(z)(f(z) - 1) \times f(z + c) \) and \( g^n(z)(g(z) - 1)g(z + c) \) share \( \alpha(z) \), then \( f(z) \equiv g(z) \).

Without the notions of weakly weighted sharing and relaxed weighted sharing we prove the following theorem which also improves Theorem D.

Theorem 3. Let \( f(z) \) and \( g(z) \) be two transcendental entire functions of finite order, and \( \alpha(z) \) a small function with respect to both \( f(z) \) and \( g(z) \). Suppose that \( c \) is a non-zero complex constant and \( n \geq 16 \) is an integer. If \( E_2)(\alpha(z), f^n(z) \times (f(z) - 1) \times f(z + c)) = E_2)(\alpha(z), g^n(z) \times (g(z) - 1) \times g(z + c)) \), then \( f(z) \equiv g(z) \).

2. Some lemmas

In this section, we present some lemmas which will be needed in the sequel. We will denote by \( H \) the following function:

\[
H = \left( \frac{F'''}{F'} - \frac{2F''}{F - 1} \right) - \left( \frac{G'''}{G'} - \frac{2G''}{G - 1} \right).
\]
Lemma 1 ([1]). Let \( H \) be defined as above. If \( F \) and \( G \) share \( "(1, 2)" \) and \( H \neq 0 \), then
\[
T(r, F) \leq N_2 \left( r, \frac{1}{F} \right) + N_2 \left( r, \frac{1}{G} \right) + N_2(r, F) + N_2(r, G) \]
\[
- \sum_{p=3}^{\infty} N(p, r, \frac{G'}{G}) + S(r, F) + S(r, G),
\]
and the same inequality holds for \( T(r, G) \).

Lemma 2 ([1]). Let \( H \) be defined as above. If \( F \) and \( G \) share \( (1, 2)^* \) and \( H \neq 0 \), then
\[
T(r, F) \leq N_2 \left( r, \frac{1}{F} \right) + N_2 \left( r, \frac{1}{G} \right) + N_2(r, F) + N_2(r, G) + N \left( r, \frac{1}{F} \right)
\]
\[
+ N(r, F) - m \left( r, \frac{1}{G-1} \right) + S(r, F) + S(r, G),
\]
and the same inequality holds for \( T(r, G) \).

Lemma 3 ([14]). Let \( H \) be defined as above. If \( H \equiv 0 \) and
\[
\limsup_{r \to \infty} \frac{N(r, F) + N(r, F) + N(r, \frac{1}{G}) + N(r, G)}{T(r)} < 1, \quad r \in I,
\]
where \( T(r) = \max\{T(r, F), T(r, G)\} \) and \( I \) is a set with infinite linear measure, then \( F \equiv G \) or \( FG \equiv 1 \).

Lemma 4 ([2]). Let \( f(z) \) be a meromorphic function in the complex plane of finite order \( \sigma(f) \), and let \( \eta \) be a fixed non-zero complex number. Then for each \( \varepsilon > 0 \), one has
\[
T(r, f(z + \eta)) = T(r, f(z)) + O(r^{\sigma(f) - 1 + \varepsilon}) + O(\log r)
\]

Lemma 5 ([11]). Let \( f(z) \) be an entire function of finite order \( \sigma(f) \), c a fixed non-zero complex number, and
\[
P(z) = a_n f^n(z) + a_{n-1} f^{n-1}(z) + \ldots + a_1 f(z) + a_0
\]
where \( a_j \) \( (j = 0, 1, \ldots, n) \) are constants. If \( F(z) = P(z) f(z + c) \), then
\[
T(r, F) = (n + 1)T(r, f) + O(r^{\sigma(f) - 1 + \varepsilon}) + O(\log r).
\]
Lemma 6 ([10]). Let $F$ and $G$ be two nonconstant entire functions, and $p \geq 2$ an integer. If $E_p(1, F) = E_p(1, G)$ and $H \neq 0$, then

$$T(r, F) \leq N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + 2\overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{G}\right) + S(r, F) + S(r, G).$$

3. Proof of Theorem 1

Let

$$F(z) = \frac{f^n(z)(f(z) - 1)f(z + c)}{\alpha(z)}, \quad G(z) = \frac{g^n(z)(g(z) - 1)g(z + c)}{\alpha(z)}.$$

Then $F(z)$ and $G(z)$ share “(1, 2)” except the zeros or poles of $\alpha(z)$. By Lemma 5, we have

$$T(r, F(z)) = (n + 2)T(r, f(z)) + O(r^{\sigma(f) - 1 + \varepsilon}) + S(r, f), \tag{3.1}$$

$$T(r, G(z)) = (n + 2)T(r, g(z)) + O(r^{\sigma(g) - 1 + \varepsilon}) + S(r, g). \tag{3.2}$$

Suppose $H \neq 0$, then by Lemma 1 and Lemma 4 we have

$$T(r, F) + T(r, G) \leq 2N_2\left(r, \frac{1}{F}\right) + 2N_2\left(r, \frac{1}{G}\right) + S(r, f) + S(r, g) \tag{3.3}$$

$$\leq 4\overline{N}\left(r, \frac{1}{f}\right) + 4\overline{N}\left(r, \frac{1}{g}\right) + 2\overline{N}\left(r, \frac{1}{f(z) - 1}\right) + 2\overline{N}\left(r, \frac{1}{g(z) - 1}\right)$$

$$+ 2\overline{N}\left(r, \frac{1}{f(z + c)}\right) + 2\overline{N}\left(r, \frac{1}{g(z + c)}\right) + S(r, f) + S(r, g)$$

$$\leq 8T(r, f) + 8T(r, g) + S(r, f) + S(r, g).$$

Substituting (3.1) and (3.2) into (3.3), we obtain

$$(n - 6)[T(r, f) + T(r, g)] \leq O(r^{\sigma(f) - 1 + \varepsilon}) + O(r^{\sigma(g) - 1 + \varepsilon}) + S(r, f) + S(r, g)$$

which contradicts with $n \geq 7$. Thus we have $H \equiv 0$. Note that

$$\overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{G}\right) \leq 3T(r, f) + 3T(r, g) + S(r, f) + S(r, g) \leq T(r)$$

where $T(r) = \max\{T(r, F), T(r, G)\}$. By Lemma 3, we deduce that either $F \equiv G$ or $FG \equiv 1$. Next we will consider the following two cases, respectively.
Case 1. $F \equiv G$, thus $f^n(z)(f(z) - 1)f(z + c) \equiv g^n(z)(g(z) - 1)g(z + c)$. Let $\varphi(z) = f(z)/g(z)$. If $\varphi^{n+1}(z)\varphi(z + c) \not\equiv 1$, we have

$$g(z) = \frac{\varphi^n(z)\varphi(z + c) - 1}{\varphi^{n+1}(z)\varphi(z + c) - 1}.$$  

Then $\varphi(z)$ is a transcendental meromorphic function of finite order since $g(z)$ is transcendental. By Lemma 4, we have

$$T(r, \varphi(z + c)) = T(r, \varphi(z)) + S(r, \varphi).$$  

If $\varphi^{n+1}(z)\varphi(z + c) = k(\neq 1)$, where $k$ is a constant, then Lemma 4 and (3.5) imply that

$$(n + 1)T(r, \varphi(z)) = T(r, \varphi(z + c)) + O(1) = T(r, \varphi(z)) + O(r^{\sigma(\varphi(z)) - 1 + \epsilon}) + O(\log r)$$

which contradicts with $n \geq 7$. Thus $\varphi^{n+1}(z)\varphi(z + c)$ is not a constant. Suppose that there exists a point $z_0$ such that $\varphi(z_0)^{n+1}\varphi(z_0 + c) = 1$. Then $\varphi(z_0)^n\varphi(z_0 + c) = 1$ since $g(z)$ is an entire function. Hence $\varphi(z_0) = 1$ and

$$N(r, \varphi^{n+1}(z)\varphi(z + c) - 1) \leq N(r, \varphi(z) - 1) \leq T(r, \varphi(z)) + O(1).$$

We apply the second Nevanlinna fundamental theorem to $\varphi(z)^{n+1}\varphi(z + c)$:

$$T(r, \varphi^{n+1}(z)\varphi(z + c)) \leq N(r, \varphi^{n+1}(z)\varphi(z + c)) + N(r, \varphi^{n+1}(z)\varphi(z + c) - 1) + S(r, \varphi) \leq 5T(r, \varphi(z)) + S(r, \varphi).$$

By Lemma 5 we deduce

$$n - 3T(r, \varphi(z)) \leq O(r^{\sigma(\varphi) - 1 + \epsilon}) + S(r, \varphi),$$

which contradicts with $n \geq 7$. So $\varphi^{n+1}(z)\varphi(z + c) \equiv 1$. Thus $\varphi(z) \equiv 1$, that is $f(z) \equiv g(z)$.

Case 2. $F(z)G(z) \equiv 1$, that is

$$f^n(z)(f(z) - 1)f(z + c)g^n(z)(g(z) - 1)g(z + c) \equiv \alpha^2(z).$$

Since $f$ and $g$ are transcendental entire functions, we can deduce from (3.7) that $N(r, 1/f) = S(r, f)$, $N(r, f) = S(r, f)$ and $N(r, 1/(f - 1)) = S(r, f)$. Then $\delta(0, f) + \delta(\infty, f) + \delta(1, f) = 3$, which contradicts the deficiency relation. This completes the proof of Theorem 1.

\[\Box\]
4. Proof of Theorem 2

Let

\[ F(z) = \frac{f^n(z)(f(z) - 1)f(z + c)}{\alpha(z)}, \quad G(z) = \frac{g^n(z)(g(z) - 1)g(z + c)}{\alpha(z)}. \]

Then \( F(z) \) and \( G(z) \) share \((1, 2)^*\) except the zeros or poles of \( \alpha(z) \). Obviously

\[
2N_2\left(r, \frac{1}{F}\right) + 2N_2\left(r, \frac{1}{G}\right) + \mathcal{N}\left(r, \frac{1}{F}\right) + \mathcal{N}\left(r, \frac{1}{G}\right) + S(r, F) + S(r, G) \\
\leq 11T(r, f) + 11T(r, g) + S(r, f) + S(r, g).
\]

According to (4.1) and Lemma 2, we can prove Theorem 2 in a similar way as in Section 3. \( \square \)

5. Proof of Theorem 3

Let

\[ F(z) = \frac{f^n(z)(f(z) - 1)f(z + c)}{\alpha(z)}, \quad G(z) = \frac{g^n(z)(g(z) - 1)g(z + c)}{\alpha(z)}. \]

Then \( \mathcal{E}_2(1, f^n(z)(f(z) - 1)f(z + c)) = \mathcal{E}_2(1, g^n(z)(g(z) - 1)g(z + c)) \) except the zeros or poles of \( \alpha(z) \). Obviously

\[
2N_2\left(r, \frac{1}{F}\right) + 2N_2\left(r, \frac{1}{G}\right) + 3\mathcal{N}\left(r, \frac{1}{F}\right) + 3\mathcal{N}\left(r, \frac{1}{G}\right) + S(r, F) + S(r, G) \\
\leq 17T(r, f) + 17T(r, g) + S(r, f) + S(r, g).
\]

Using (5.1) and Lemma 6, we can prove Theorem 3 in a similar way as in Section 3. \( \square \)

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