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Commutative Semigroups with Almost Transitive Endomorphism Semiring

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In the paper, commutative semigroups with almost transitive endomorphism semirings are investigated.

In many classical situations, endomorphisms and/or automorphisms operate transitively on some algebraic structures. Such considerations appeared e.g. in our investigation of commutative semigroups that are simple over their endomorphism semirings (see [1]). In this note, we present a slight generalization of the transitive action.

Throughout the paper, let $A = A(+)$ be a commutative semigroup and $E = \mathrm{End}(A(+))$ be the full endomorphism semiring of A (clearly, E is a unitary semiring and A is a left E-semimodule). Further, $\mathrm{Aut}(A)$ is the group of automorphisms of $A(+)$, $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0$ is the set of non-negative integers. As usual, $0 = 0_A$ ($o = o_A$, resp.) will denote the neutral (absorbing, resp.) element of A and $0_A \in A$ ($o \in A$, resp.) means that A has the neutral (absorbing, resp.) element. An element $a \in A$ is *idempotent* if $a = a + a$ and $\mathrm{Id}(A)$ denotes the set of all idempotent elements. A is a *semilattice* if $A = \mathrm{Id}(A)$. A subset I of A is an *ideal* if $I \neq \emptyset$ and $A + I \subseteq I$. A subsemigroup B of A is fully invariant if $f(B) \subseteq B$ for every $f \in E$. We shall say that A is *ems-simple* if $|A| \geq 2$ and $|B| = 1$ whenever B is a fully invariant subsemigroup with $B \neq A$ (then $B = \{a\}$ for some $a \in \mathrm{Id}(A)$).

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Obviously, for each $a \in A$, $E(a) = \{ f(a) \mid f \in E \}$ is a fully invariant subsemigroup of A and $a \in E(a)$. In particular, if $E(a) = \{a\}$ then $a \in \mathrm{Id(A)}$. We shall say that E operates on A

- *transitively* if for all $a, b \in A$ there is $f \in E$ such that $f(a) = b$ (i.e., $E(a) = A$ for every $a \in A$);
- *almost transitively* if there is $w \in A$ such that for all $a, b \in B_w = A \setminus \{w\}$ there is $f \in E$ such that $f(a) = b$ (i.e., $B_w \subseteq E(a)$ for every $a \in B_w$.

Clearly, if E operates on A transitively then it operates almost transitively and for w can be chosen any element. Further, if $|A| = 2$ then E operates almost transitively on A (indeed, if $w \in A$ then $B_w = \{v\}$ and $v \in E(v)$).

In the rest of the paper, we shall always assume that E operates almost transitively on A (i.e., $w \in A$ is such that $B_w = A \setminus \{w\} \subseteq E(a)$ for every $a \in B_w$) and $|A| \geq 2$.

1. Basic properties

1.1 Lemma. *If* $w \in \mathrm{Id(A)}$ *then* $E(a) = A$ *for every* $a \in B_w$.

Proof. The mapping f defined by $f(x) = w$ for each $x \in A$ is an endomorphism, and hence $w = f(a) \in E(a)$ for every $a \in B_w$. □

1.2 Lemma. *If* $a \in \mathrm{Id(A)}$ *then* $E(a) \subseteq \mathrm{Id(A)}$.

Proof. Obvious. □

1.3 Lemma. *Just one of the following two cases takes place:*

- $E(w) = \{w\}$ *(and then* $w \in \mathrm{Id(A)}$*).*
- $E(w) = A$. □

Proof. If $E(w) \neq \{w\}$ then there is $f \in E$ such that $a = f(w) \neq w$. Then $B_w \subseteq$ $\subseteq E(a) = E(f(w)) \subseteq E(w)$ and, of course, $w \in E(w)$. □

1.4 Lemma. *If* $w \in \mathrm{Id(A)}$ *and either* $\mathrm{Id(A)} \neq \{\mathrm{w}\}$ *or* $E(w) \neq \{w\}$ *then* A *is a semilattice and E operates transitively on A.*

Proof. Combine 1.1, 1.2 and 1.3. □

1.5 Lemma. *Assume that* $w \notin E(a_0)$ *for at least one* $a_0 \in B_w$. *Then:*
(i) B_w *is a fully invariant subsemigroup of A and* $w \notin E(a) = B_w$ *for every* $a \in B_w$.
(ii) End(B) *operates transitively on B.*

Proof. (i) $B_w = E(a_0)$ is a fully invariant subsemigroup of A. If $a \in B$ and $f \in E$ are such that $w \in E(a)$ then $a = g(a_0)$ for some $g \in E$ and $w = fg(a_0) \in E(a_0)$, a contradiction.
(ii) For every $f \in E$, the restriction $f|B_w$ is an endomorphism of B_w by (i). □

1.6 Corollary. *Just one of the following two cases takes place:*

- *$E(a) = A$ for every $a \in B_w$.*
- *$w \notin E(a)$ for every $a \in B_w$.* □

1.7 Remark. Let $T = \{(u, v) \in A \times A \mid u \notin E(v)\}$. According to 1.5, either $u \neq w$ for all $(u, v) \in T$ or $(w, a) \in T$ for every $a \in B_w$. Similarly, using 1.3, either $v \neq w$ for all $(u, v) \in T$ or $(a, w) \in T$ for every $a \in B_w$.

1.8 Proposition. *If E does not operate transitively on A and $|A| \geq 3$ then w is uniquely determined.*

Proof. Suppose that there are $v, w \in A$ such that $B_w \subseteq E(x)$ for all $x \in B_w$, $B_v \subseteq$ $\subseteq E(y)$ for all $y \in B_v$ and $v \neq w$. As $|A| > 2$, there is $c \in A$ with $v \neq c \neq w$. With respect to 1.6, if $E(a) \neq A$ for some $a \neq w$ then $w \notin E(c)$ and $w \notin E(v)$, hence $E(v) = \{v\}$ by 1.3, $v \in \mathrm{Id(A)}$ and $E(c) = A$ by 1.1, a contradiction. Thus $E(a) = A$ for all $a \neq w$. Symmetrically, $E(a) = A$ for all $a \neq v$, hence $E(w) = A$ and E operates transitively on A. □

2. Classification with respect to idempotents

2.1 Assume now that $w \notin \mathrm{Id(A)}$ and $\mathrm{Id(A)} \cap \mathrm{B_w} \neq \emptyset$. By 1.2, B is a semilattice. Of course, $E(w) = A$ by 1.3, $w \neq v = 2w = 4w = 2v$, $B = \mathrm{Id(A)}$ is a fully invariant subsemigroup of A and $A = B \cup \{w\}$. Since $w \notin \mathrm{Id(A)}$, $f(w) = w$ and $f(v) = v$ for each $f \in \mathrm{Aut(A)}$. Thus automorphisms do not operate almost transitively on A whenever $|A| \geq 3$. If $|A| \leq 3$ then A is isomorphic to one of the following semigroups A_1, A_2, A_3, A_4:

A_1	w	v
w	v	v
v	v	v

A_2	w	v	u
w	v	v	v
v	v	v	v
u	v	v	u

A_3	w	v	u
w	v	v	u
v	v	v	u
u	u	u	u

A_4	w	v	u
w	v	v	w
v	v	v	v
u	w	v	u

2.2 Now, suppose that $w \in \mathrm{Id(A)} = \{\mathrm{w}\}$. Then $E(a) = A$ for every $a \in B_w$ by 1.1 and $E(w) = \{w\}$. Of course, A is ems-simple and E does not operate transitively on A. Further, $f(w) = w$ and $f(B) = B$ for every $f \in \mathrm{Aut(A)}$. Nevertheless, it may happen that Aut(A) operates transitively on B (i.e., for all $a, b \in B$ there is $f \in \mathrm{Aut(A)}$ such that $f(a) = b$).

2.3 Now, let us suppose that $w \in \text{Id(A)}$ and $B \cap \text{Id(A)} \neq \emptyset$. Then A is a semilattice, $A = B \cup \{w\}$ and E operates transitively on A.

2.4 Finally, suppose that $\text{Id(A)} = \emptyset$. Then A is infinite. Moreover, $E(w) = A$ and $B \subseteq E(a)$ for every $a \in B$. If $E(a) = A$ (i.e., $w \in E(a)$) for at least one $a \in B$ then E operates transitively on A. On the other hand, if $E(a) = B$ for every $a \in B$ then B is a fully invariant subsemigroup of A and End(B) operates transitively on B.

2.5 Suppose that A is not ems-simple. Then just one of the following two cases takes place:

- Id(A) = B, $A = B \cup \{w\}$, $2w \neq w$ and B is a fully invariant subsemigroup of A (and a semilattice).
- $\text{Id(A)} = \emptyset$, $A = B \cup \{w\}$, $2w \neq w$, B is a fully invariant subsemigroup of A and End(B) operates transitively on B.

References

[1] Ježek, J., Kepka, T., and Němec, P.: *Commutative semigroups that are simple over their endomorphism semirings*, Acta. Univ. Carolinae Math. Phys. **52/2** (2011), 37–50.