Acta Universitatis Carolinae. Mathematica et Physica

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*Acta Universitatis Carolinae. Mathematica et Physica*, Vol. 52 (2011), No. 2, 89--95

Persistent URL: `http://dml.cz/dmlcz/143682`

# Dually Steady Rings

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*Received November 26, 2010*
*Revised April 29, 2011*

Two duals of steady rings are introduced and briefly studied.

## 1. Preliminaries

It is well known and easy to check that the following conditions are equivalent for a module $M$:

(A1) If $M_i$, $i < \omega$, is a countable family of submodules of $M$ such that $\Sigma M_i = M$, then $\Sigma_{i \le n} M_i = M$ for some $n < \omega$.

(A2) If $M_0 \subseteq M_1 \subseteq M_2 \subseteq \dots$ is a countable chain of submodules of $M$ such that $\bigcup M_i = M$ , then $M_n = M$ for some $n < \omega$.

(A3) If $\epsilon : \coprod_\omega A_i \to M$ is an epimorphism, then $\epsilon(\coprod_{i \le n} A_i) = M$ for some $n < \omega$.

(A4) If $\varphi : M \to \coprod_I A_i$ is a homomorphism, then $\mathrm{Im}(\varphi) \subseteq \coprod_J A_i$ for a finite subset $J$ of $I$.

(A5) The canonical mapping $\coprod_I \mathrm{Hom}_R(M, A_i) \to \mathrm{Hom}_R(M, \coprod_I A_i)$ is an isomorphism.

(A6) If $\varphi : M \to \coprod_\omega A_i$ is a homomorphism, then $\mathrm{Im}(\varphi) \subseteq \coprod_{i \le n} A_i$ for some $n < \omega$.

(A7) If $Q$ is a cogenerator for $R$-Mod and if $\varphi : M \to Q^{(\omega)}$ is a homomorphism, then $\mathrm{Im}(\varphi) \subseteq Q^{(n)}$ for some $n < \omega$.

Such a module M is called $\cup$-*compact* in this paper (other known names: $\Sigma$-compact, $\coprod$-slender, dually slender, small, e.t.c.) and one sees immediately that

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This work is part of the research project MSM 0021620839, financed by MŠMT

*2000 Mathematics Subject Classification.* 16D10

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every finitely generated module is $\cup$-compact. If the converse is true, then the ring $R$ is said to be *left steady*. Many such rings were studied in the literature (see e.g. [2, 11, 12, 16]), recall that left noetherian, left perfect, left semiartinian of countable socle length and countable commutative rings are known to be left steady [2, Theorem 1.8, Theorem 2.2], [16, Proposition 15].

However the conditions (A1)–(A7) are equivalent, dualization of them leads to three different notions; $\cap$-compact (dual to (A1)–(A3)), slim (dual to (A4)–(A5)), and slender modules (dual to (A6)–(A7)). These three duals of $\cup$-compact modules are extensively studied (see e.g. [1, 4, 5, 6, 7, 8, 9]), nevertheless there are no research of dualization of steady rings.

The main focus of this paper is to draw attention to possible duals of steady rings. First of all we sum up well known facts which allow us to define two variants of dually steady rings and which easily produce basic results about the notions. Thus this paper should serve as an introduction to the topic only and many results are straightforward consequences or generalization of older ones.

Throughout the paper, $R$ stands for a non-zero associative ring with unit and modules are unitary left $R$-modules. The terminology and notation used in what follows is quite standard (see e.g. [13] or [14]). The only exception is the (left $R$-)module $R^\omega/R^{(\omega)}$ which is denoted by $W$ in the sequel. Finally, a module will be called complete if it is complete in a non-discrete linear Hausdorff topology.

## 2. $\cap$-compact modules and 1-dually steady rings

The following conditions are equivalent for a module $M$:

(B1) If $M_i$, $i < \omega$, is a countable family of submodules of $M$ such that $\bigcap M_i = 0$, then $\bigcap_{i\leq n} M_i = 0$ for some $n < \omega$.

(B2) If $M_0 \supseteq M_1 \supseteq M_2 \supseteq \dots$ is a countable chain of submodules of $M$ such that $\bigcap M_i = 0$, then $M_n = 0$ for some $n < \omega$.

(B3) If $\iota : M \to \prod_\omega A_i$ is a monomorphism, then $\iota^{-1}(\prod_{i\geq n} A_i) = 0$ for some $n < \omega$.

Such a module will be called $\cap$-*compact* in the paper and one sees again immediately that every finitely cogenerated module is $\cap$-compact. If the converse is true, then we will say that the ring $R$ is *left 1-dually steady*. Note that any submodule of a $\cap$-compact module is $\cap$-compact and any factor of a left 1-dually steady ring is left 1-dually steady as well. For further basic properties of $\cap$-compact modules see [7].

**Lemma 2.1** *Every $\cap$-compact module has a finite uniform dimension.*

*Proof.* Any infinite direct sum of nonzero modules is not $\cap$-compact and $\cap$-compact modules are closed under submodules. □

**Proposition 2.2** *The following conditions are equivalent for a ring R.*

(1) *$R$ is left 1-dually steady,*

(2) *every $\cap$-compact cyclic module contains a simple submodule,*

(3) *every uniform* $\cap$*-compact cyclic module contains an essential simple submodule.*

*Proof.* As every nonzero finitely cogenerated module contains simple submodule, it is enough to prove (3)$\Rightarrow$(1). Assume that $R$ is not left 1-dually steady, hence there exist a $\cap$-compact module $M$ which is not finitely cogenerated. Then $M$ contains the finitely generated socle by 2.1, which is not essential submodule of $M$. Thus there exists a nonzero uniform cyclic $C$ submodule such that $C \cap \mathrm{Soc}(M) = 0$. $\square$

Some examples of left 1-dually steady rings:

**Proposition 2.3** *$R$ is left 1-dually steady in each of the following cases:*
(1) *$R$ is right noetherian and every left ideal is a two-sided ideal.*
(2) *$R$ is left semiartinian.*
(3) *$R$ is countable.*
(4) *$R$ is abelian regular.*

*Proof.* (1)–(3) See [7, Proposition 3].
(4) An immediate consequence of 2.2, since every non-artinian factor of abelian regular ring has an infinite uniform dimension. $\square$

Recall that a module $M$ is *strongly* $\cup$*-compact*, if for every countable sequence $m_0, m_1, \dots \in M$ there exists a finitely generated submodule $F$ of $M$ such that $m_i \in F$ for each $i < \omega$.

**Proposition 2.4** *Let $R$ be a left nonsingular ring with a left maximal ring of quotients $Q$. Then ${}_RR$ is $\cap$-compact if and only if $Q$ is semisimple and $Q_R$ is strongly $\cup$-compact.*

*Proof.* Let $R$ be $\cap$-compact. Since ${}_RQ$ is an injective envelope of ${}_RR$ and $Q$ is a von Neumann regular ring (see e.g. [13, Chapter XII]), $Q$ is semisimple by 2.1. Hence there exists a primitive set of orthogonal idempotents $\{e_i\}_{i \le k} \subseteq Q$. Let $m_1, m_2, \dots \in Q$ is a countable sequence. Put $A_n^i = \{r \in Re \mid re_i m_n \in R\}$. Note that that $\bigcap_n A_n^i \neq 0$ for each $i$ since $R$ is $\cap$-compact, hence there is a nonzero element $r_i e_i \in Re_i$ such that $re_i m_n \in R$ for all $n$. As $Qr_i e_i = Qe_i$, there exists $q_i \in Q$ for which $q_i r_i e_i = e_i$, thus $e_i m_n = q_i(r_i e_i m_n) \in q_i R$. As $r_i e_i m_n \in R$ and $m_n = \sum_i e_i m$ we get that $m_n \in \sum_{i \le k} q_i R$ for all $n < \omega$.

Suppose that $Q$ is semisimple and $Q_R$ is strongly $\cup$-compact and let $I_n$ be an increasing chain of left ideals such that $\bigcap_{j \le n} I_j \neq 0$ for each $n < \omega$. Note that we may suppose that $I_n = Rx_n$ and $Qx_n = Qe$, where $e^2 = e \in Q$, is a simple $Q$-module. Then there exist $s \in R$ and $y_n \in Q$ such that $se \in R \setminus \{0\}$ and $y_n x_n = e$. As $Q_R$ is strongly $\cup$-compact, there exist $u_1, \dots, u_k \in Q$ for which $y_n \in \sum_{i \le k} u_i R$. Moreover, there exists $v \in eQ$ and $a \in R$ satisfying $e = vsy_n x_n$ and $aevu_i \in R$ where $aevu_i \neq 0$ for at least one $i \le k$. Now, since $0 \neq ae = aevsy_n x_n \in R$ and $aevsy_n \in R$ for each $n$, we obtain that $\bigcap_n Rx_n \neq 0$. $\square$

**Corollary 2.5** *Let $R$ be a commutative domain with a quotient field $Q \neq R$. Then the following conditions are equivalent:*

(i) *$R$ is $\cap$-compact,*

(ii) *every countably generated submodule of ${}_RQ$ is a fractional ideal.*

*Moreover, if $R$ is uniserial, then the above conditions are equivalent to:*

(iii) *${}_RQ$ is $\cup$-compact,*

(iv) *${}_RQ$ is not countably generated.*

### 3. Slender and slim modules and 2-dually steady rings

Consider the following three conditions for a module $M$:

(B4) If $\psi : \prod_I B_i \to M$ is a homomorphism, then $\prod_J B_i \subseteq \mathrm{Ker}(\psi)$ for a cofinite subset $J$ of $I$.

(B5) The canonical mapping $\coprod_I \mathrm{Hom}_R(B_i, M) \to \mathrm{Hom}_R(\prod_I B_i, M)$ is an isomorphism.

(B6) If $\psi : \prod_\omega B_i \to M$ is a homomorphism, then $\prod_{i \geq n} B_i \subseteq \mathrm{Ker}(\psi)$ for some $n < \omega$.

(B7) If $\psi : R^\omega \to M$ is a homomorphism, then $R^{\omega \setminus n} \subseteq \mathrm{Ker}(\psi)$ for some $n < \omega$.

Clearly, the conditions (B6) and (B7) are equivalent and the corresponding modules are just the well known *slender* modules (see [3, Chapter III]). Similarly, the conditions (B4) and (B5) are equivalent and the corresponding modules are called *slim* in [4]. Every slim module is slender and, according to [8] or [9, Proposition 2.3] the converse is true if and only if there are no measurable cardinal numbers (and then the conditions (B4), (B5), (B6) and (B7) are equivalent). On the other hand, if $\kappa$ is a measurable cardinal and $|R| < \kappa$, then there exist no non-zero slim $R$-modules. Consequently, in case there are too many measurable cardinals, non-zero slim modules do not exist over any ring (the converse is also true – see [4, Theorem 8.2]).

**Proposition 3.1** ([1]) *A module $M$ is slender if and only if* $\mathrm{Hom}_R(W, M) = 0$, *$W = R^\omega / R^{(\omega)}$, and $M$ is not complete (i.e., not complete in any non-discrete linear Hausdorff topology).*

**Proposition 3.2** (i) *If a module $M$ is $\cap$-compact or if $|M| < 2^\omega$, then $M$ is slender if and only if* $\mathrm{Hom}_R(W, M) = 0$.

(ii) *If $S$ is a submodule of $R^\omega$ maximal with respect to $R^{(\omega)} \subseteq S$ and* $(1, 1, 1, \dots) \notin$ *$\notin S$ then the factor-module $T = R^\omega / S$ is cocyclic and not slender.*

(iii) *If $A$ is a nonzero slender module, then $A^{(\omega)}$ is a slender module that is not finitely cogenerated.*

*Proof.* (i) The assertion follows immediately from 3.1.

(ii) The socle of $T$ is an essential simple submodule, and hence $T$ is cocyclic. By 3.1, $T$ is not slender.

(iii) Slender modules are closed under direct sums. □

**Proposition 3.3** *Assume that every maximal left ideal of $R$ is a two-sided ideal. Then no non-zero finitely cogenerated module is slender.*

*Proof.* Due to 3.1, no simple module is slender and it is enough to take into account that slender modules are closed under submodules. □

**Proposition 3.4** *The following conditions are equivalent:*
(1) *All slender modules are finitely cogenerated,*
(2) *there exist no non-zero slender modules,*
(3) *there exist no non-zero slender cyclic modules,*
(4) $\mathrm{Hom}_R(W, R/I) \neq 0$ *for every proper left ideal $I$ such that the cyclic factor-module $R/I$ is not complete.*

*Moreover, if every left ideal of $R$ is a two-sided ideal, then the above conditions are equivalent to:*
(5) *Slender modules are closed under factor-modules.*

*Proof.* First, (1) implies (2) by 3.2 (3) and (2) implies (1), (3), and (5) trivially. Further, (3) is equivalent to (4) by 3.1 and (3) implies (1) due to the fact that slender modules are closed under submodules. Now, assume that all left ideals are two-sided and that the condition (5) is satisfied. We are going to show that then (3) is true. For, let $I$ be an ideal of $R$ such that the cyclic module $M = R_R/I$ is slender and let $\kappa$ be a cardinal number such that $\kappa \geq |R^\omega|$. Since slender modules are closed under direct sums, the module $N = M^{(\omega)}$ is slender. On the other hand, $N$ may be viewed as a free $R/I$-module and consequently there is an epimorphism $\varphi : N \to M^\omega$. According to (5), $M^\omega$ is a slender module and it follows immediately that $M = 0$. □

If the ring $R$ satisfies the equivalent conditions 3.4(1)–(4), then we will say that $R$ is *left 2-dually steady*.

**Lemma 3.5** *Let $I$ be an ideal, finitely generated as a right ideal, of $R$, and let $S = R/I$. If $M$ is a module such that $IM = 0$, then $M$ is a slender $R$-module if and only if $M$ is a slender $S$-module.*

*Proof.* There is a natural isomorphism $\theta : W/IW \to S^\omega/S^{(\omega)}$. □

**Corollary 3.6** *If $R$ is left 2-dually steady and $I$ is an ideal, finitely generated as a right ideal, of $R$, then the factor-ring $R/I$ is also left 2-dually steady.*

**Lemma 3.7** *No simple module is slender in each of the following cases:*
(1) *Every maximal left ideal is a two-sided ideal.*
(2) $I \neq I^2$ *for every maximal left ideal $I$.*
(3) *$R$ is a left V-ring.*
(4) *Every simple module is finite.*
(5) *$R/J(R)$ is a left 2-dually steady ring.*

*Proof.* Easy. □

**Proposition 3.8** *Assume that $R$ is left semiartinian. Then $R$ is left 2-dually steady in each of the following cases:*

(1) *Every maximal left ideal is a two-sided ideal,*
(2) *$R$ is a left V-ring,*
(3) *every simple module is finite,*
(4) *$R/J(R)$ is left 2-dually steady ring.*

*Proof.* Combine 3.7 and the fact that slender modules are closed under submodules. □

**Proposition 3.9** *$R$ is both left 1- and 2-dually steady in each of the following cases:*

(1) *$R$ is right perfect ring,*
(2) *$R$ is commutative semiartinian,*
(3) *$R$ is complete commutative principal ideal domain.*

*Proof.* (1), (2) An immediate consequence of 3.8 and 2.3.

(3) If $R$ is a principal ideal domain, then no simple module is slender and, up to isomorphism, the only Soc-torsionfree cyclic module is $R$ itself. Now, it is clear that $R$ satisfies the condition 3.4(4). Finally, $R$ is left 1-dually steady by 2.3(1) □

## 4. Small 2-dually steady rings

Throughout this section, a small ring is any ring $R$ with $|R| < 2^{\omega}$.

**Proposition 4.1** *If $R$ is small, then the following conditions are equivalent:*

(1) *$R$ is left 2-dually steady,*
(2) $\mathrm{Hom}_R(W, R/I) \neq 0$ *for every proper left ideal $I$,*
(3) $\mathrm{Hom}_R(W, M) \neq 0$ *for every non-zero module $M$.*

*Proof.* We have $|A| \geq 2^{\omega}$ for every complete module $A$ and the rest is clear from 3.1 and 3.4. □

**Proposition 4.2** *A small prime ring $R$ is (left, right) 2-dually steady if and only if $R$ is isomorphic to a full matrix ring over a division ring.*

*Proof.* The direct implication follows from [5, statement 4.1] and the converse one is clear. □

**Theorem 4.3** *The following conditions are equivalent for a small ring $R$:*

(1) *$R$ is right noetherian and left 2-dually steady.*
(2) *$R$ is right artinian.*

*If these conditions are satisfied, then $R$ is (left and right) steady, 1-dually steady and 2-dually steady.*

*Proof.* Assume (1) be true. Since $R$ satisfies maximal condition on ideals, the prime radical $P$ of $R$ is the intersection of a finite family of prime ideals, say

$P = P_1 \cap \cdots \cap P_n$. Further, by 3.6 and 4.2, all the factor-rings $R/P_i$ are completely reducible and consequently $P = J(R)$ and $R$ is semilocal. Since $R$ is right noetherian, $P$ is nilpotent and it follows easily that R is right artinian.

Conversely, if $R$ is right artinian, then $R$ is (left, right) steady, $R$ is 1-dually steady by 2.3(2) and $R$ is 2-dually steady by 3.9. □

**Proposition 4.4** *Let R be a commutative noetherian ring.*

(1) *R is 2-dually steady if and only if every non-zero* Soc*-torsionfree cyclic module is complete.*

(2) *If R is 2-dually steady, then R is semilocal and, moreover, if R is not artinian, then* $R/\mathrm{Soc}(R)$ *is complete.*

*Proof.* (1) Combine 3.4(3) and [6, statement 4.4].

(2) Use (1) and [6, statements 3.1 and 3.2]. □

## References


[1] Dimitrić, R.: *Slenderness in abelian categories*, in: Abelian Group Theory, Lecture Notes in Math. 1006, Springer, 1983, 375–383.

[2] Eklof, P. C., Goodearl, K. R., and Trlifaj, J.: *Dually slender modules and steady rings*, Forum Math. **9** (1997), 61–74.

[3] Eklof, P. C. and Mekler, A. H.: Almost Free Modules, North Holland, New York, 1990, revised ed. 2002.

[4] El Bashir, R. and Kepka, T.: *Modules commuting (via* Hom*) with some limits*. Fund. Math. **155** (1998), 271–292.

[5] El Bashir, R. and Kepka, T.: *On when small semiprime rings are slender*. Commun. Alg. **24** (1996), 1575–1580.

[6] El Bashir, R. and Kepka, T.: *On when commutative noetherian rings are slender*. Commun. Alg. **25** (1997), 2585–2591.

[7] Kepka, T.: *∩-compact modules*. Comment. Math. Univ. Carolinae **36** (1995), 423–426.

[8] Eda, K.: *Slender modules, endo-slender abelian groups and large cardinals*. Fund. Math. **135** (1990), 5–24.

[9] Göbel, R., Goldsmith, B., and Kolman, O.: *On modules which are self-slender*. Houston J. Math. **35** (2009), 725–736.

[10] Head, T.: *Preservation of coproducts by* $\mathrm{Hom}_R(M, -)$. Rocky Mount. J. Math. **2** (1972), 235–237.

[11] Rentschler, R.: *Sur les modules M tels que Hom*$(M, -)$ *commute avec les sommes directes*, C.R. Acad. Sci. Paris, **268**, 1969, 930–933.

[12] Růžička, P., Trlifaj, J., and Žemlička, J.: *Criteria of steadiness*, Abelian Groups, Module Theory, and Topology, New York 1998, Marcel Dekker, 359–372.

[13] Stenström, B.: Rings of quotients, Die Grundlehren der Mathematischen Wissenschaften, Band 217, Springer-Verlag, New York–Heidelberg, 1975.

[14] Wisbauer, R.: Grundlagen der Modul- und Ringtheorie, R. Fischerg, München 1988.

[15] Trlifaj, J.: *Steady rings may contain large sets of orthogonal idempotents. Abelian groups and modules* (Padova, 1994), Math. Appl., 343, Kluwer Acad. Publ., Dordrecht, 1995, 467–473.

[16] Žemlička, J. and Trlifaj, J.: *Steady ideals and rings*, Rend. Sem. Mat. Univ. Padova, **98** (1997), 161–172.