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(PRE)ORDER PRESERVING ADDITIVE HOMOMORPHISMS OF (PRE)ORDERED COMMUTATIVE SEMIGROUPS INTO REAL NUMBERS I

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Various necessary and/or sufficient conditions assuring the existence of various additive homomorphisms of commutative semigroups into real numbers are studied.

The aim of the present pseudo-expository note is to collect and order many scattered results concerning additive homomorphisms of commutative semigroups into real numbers. Similar topics were investigated e.g. in [1]–[20]. A kind reader should keep in mind that all the formulated results are fairly basic, and henceforth not attributed to any particular source.

1. Introduction

First, by a *preordering* (or *quasiordering*) we mean any reflexive and transitive relation defined on a set S. Thus $\mathrm{id}_S = \{(a,a) \mid a \in S\}$ is the smallest and $S \times S$ the largest preordering on S. An *equivalence* is a symmetric preordering and if ϱ is a preordering then the *symmetric core* (or *kernel*) $\ker(\varrho)$ of ϱ (we have $(a,b) \in \ker(\varrho)$ iff $(a,b) \in \varrho$ and $(b,a) \in \varrho$) is an equivalence. It is the largest equivalence contained in ϱ. If $\ker(\varrho) = \mathrm{id}_S$ then the preordering ϱ is antisymmetric and it is called *ordering*.

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Let ϱ be a preordering defined on a set S. A subset T of S is said to be *right* (*left*, resp.) *cofinal* in S if for every $a \in S$ there is at least one $v \in T$ such that $(a, v) \in \varrho$ ($(v, a) \in \varrho$, resp.).

1.1 Remark. Let ϱ be a preordering defined on a set S. Then $\sigma = (\varrho \setminus \ker(\varrho)) \cup \mathrm{id}_S$ is an ordering and $\sigma \subseteq \varrho$ (of course, $\sigma = \varrho$ iff ϱ is an ordering). Notice that $\sigma = \mathrm{id}_S$ iff ϱ is an equivalence.

In the remaining part of this section, let $A = A(+)$ be a commutative semigroup and ϱ be a preordering defined on A. Further, $0_A \in A$ means that the semigroup A has the neutral element 0_A.

1.2 Lemma. *The following conditions are equivalent:*
(i) $(a + c, b + c) \in \varrho$ *for all* $(a, b) \in \varrho$ *and* $c \in A$ (i.e., ϱ is *stable*).
(ii) $(a + c, b + d) \in \varrho$ *for all* $(a, b) \in \varrho$ *and* $(b, d) \in \varrho$ (i.e., ϱ is it compatible).

Proof. It is easy. □

The preordering ϱ is called *cancellative* if $(a, b) \in \varrho$ whenever $a, b, c \in A$ and $(a + c, b + c) \in \varrho$. Thus ϱ is both stable and cancellative if and only if $(a, b) \in \varrho \Leftrightarrow$ $\Leftrightarrow (a + c, b + c) \in \varrho$.

1.3 Lemma. (i) *If ϱ is stable then* $\ker(\varrho)$ *is a congruence of the semigroup A.*
(ii) *If ϱ is stable and cancellative then* $\ker(\varrho)$ *is a cancellative congruence of A.*

Proof. It is easy. □

1.4 Lemma. *Assume that ϱ is cancellative. If $a, b, c \in A$ are such that $a + c = b + c$ then* $(a, b) \in \ker(\varrho)$.

Proof. It is easy. □

1.5 Lemma. *If ϱ is a cancellative ordering then the semigroup A is cancellative.*

Proof. Use 1.4 □

1.6 Remark. Assume that ϱ is stable and cancellative. Then $\sigma = (\varrho \setminus \ker(\varrho)) \cup \mathrm{id}_A$ (see 1.1) is a stable ordering on the semigroup A. If A is cancellative (cf. 1.5) then σ is cancellative as well.

An element $a \in A$ will be called *almost* (ϱ)-*positive* (*negative*, resp.) if $(x, x+a) \in \varrho$ ($(x + a, x) \in \varrho$, resp.) for every $x \in A$.

1.7 Lemma. (i) *The set of almost positive (negative, resp.) elements is either empty or a subsemigroup of A.*
(ii) *If $0_A \in A$ then 0_A is both almost positive and almost negative.*
(iii) *If $a \in A$ is both almost positive and almost negative then* $(x + a, x) \in \ker(\varrho)$ *for every $x \in A$. If, moreover ϱ is an ordering then $a = 0_A$.*
(iv) *If ϱ is cancellative, $u \in A$ is almost negative and $v \in A$ is almost positive then* $(u, v) \in \varrho$.

Proof. It is easy. □

An element $a \in A$ will be called *right* (*left*, resp.) (ϱ-)*archimedean* if the one-generated (or cyclic) subsemigroup $\mathbb{N}a$ of A generated by the element a (here, $\mathbb{N}$ denotes the semiring of positive integers) is right (left, resp.) cofinal in A. This means that for every $b \in A$ there is $m \in \mathbb{N}$ such that $(b, ma) \in \varrho$ ($(ma, b) \in \varrho$, resp.).

1.8 Lemma. *If $a \in A$ and $m \in \mathbb{N}$ are such that ma is right (left, resp.) archimedean then a is such.*

Proof. It is obvious. □

1.9 Lemma. *Assume that ϱ is stable. Let $a \in A$ be right (left, resp.) archimedean and let $(a, b) \in \varrho$ ($(b, a) \in \varrho$, resp.). Then b is right (left, resp.) archimedean.*

Proof. It is easy. □

1.10 Lemma. *Assume that ϱ is stable and A contains at least one almost positive negative, resp.) element. If $a \in A$ is right (left, resp.) archimedean then ma is almost positive (negative, resp.) for at least one $m \in \mathbb{N}$.*

Proof. Let $v \in A$ be almost positive. Then $(x, x + v) \in \varrho$ for every $x \in A$ and there is $m \in \mathbb{N}$ such that $(v, ma) \in \varrho$. Now, $(x + v, x + ma) \in \varrho$, $(x, x + ma) \in \varrho$ and we see that ma is almost positive. □

1.11 Lemma. *Assume that ϱ is stable and cancellative. Let $a \in A$ be left (right, resp.) archimedean and almost positive (negative, resp.). Then:*
(i) *Every element from A is almost positive (negative, resp.).*
(ii) $(a, x) \in \varrho$ *($(x, a) \in \varrho$, resp.) for every $x \in A$.*
(iii) *If ϱ is an ordering then a is the smallest (largest, resp.) element in A.*

Proof. Given $x \in A$, we have $(x, x + a) \in \varrho$ and there is $m \in \mathbb{N}$ that is the smallest one with the property that $(ma, x) \in \varrho$. Now, $(ma, x + a) \in \varrho$ and, since ϱ is cancellative, we get $m = 1$. Thus $(a, x + a) \in \varrho$ for every $x \in A$. Consequently, $(y + a, y + x + a) \in \varrho$ and $(y, z + x) \in \varrho$ for every $y \in A$. The rest is clear. □

An element $a \in A$ will be called *right* (*left*, resp.) (ϱ-)*regular* if $m, n \in \mathbb{N}$ and $(ma, na) \in \varrho$ implies $m \leq n$ ($n \leq m$, resp).

1.12 Lemma. *If $a \in A$ and $\mathbb{N}a$ is finite then a is neither left nor right regular.*

Proof. It is easy. □

1.13 Lemma. *An element $a \in A$ is both right and left regular if an only if $\mathbb{N}a$ is infinite (equivalently, $\mathbb{N}a \cong \mathbb{N}$) and $\varrho|\mathbb{N}a = \mathrm{id}$.*

Proof. It is easy. □

1.14 Lemma. *Assume that every element from A is either right or left regular. Then the semigroup A is pretorsionfree (i.e., $\mathbb{N}a \cong \mathbb{N}$ is infinite for every $a \in A$).*

Proof. Use 1.12. □

1.15 Lemma. *Let $a \in A$ be right (left, resp.) regular. Then, for every $m \in \mathbb{N}$, the element ma is not almost negative (positive, resp.).*

Proof. If ma is almost negative then $((m+1)a, a) \in \varrho$ and $m+1 > 1$. Thus a is not right regular. □

1.16 Lemma. *Assume that ϱ is stable. Let $a \in A$ be right (left, resp.) archimedean and let $m \in \mathbb{N}$ be such that ma is almost negative (positive, resp.). Then no element from A is right (left, resp.) regular.*

Proof. Given $b \in A$, we have $(b, ma) \in \varrho$ for some $m \in \mathbb{N}$. Since ma is almost negative, we have $(ma + b, b) \in \varrho$. Now, $(mb, mna) \in \varrho$, $(mna + nb, nb) \in \varrho$, $((m + n)b, mna + nb) \in \varrho$, $((m + n)b, nb) \in \varrho$ and $m + n > n$. □

1.17 Lemma. *Assume that ϱ is stable and cancellative. If $a \in A$ is not right (left, resp.) regular then ma is almost negative (positive, resp.) for some $m \in \mathbb{N}$.*

Proof. We have $(ka, la) \in \varrho$, where $k > l$. Then $(ka + x, la + x) \in \varrho$ and $((k - l)a + x, x) \in \varrho$ for every $x \in A$ and it suffices to put $m = k - l$. □

1.18 Lemma. *Assume that ϱ is stable and cancellative. Let $a \in A$ be right (left, resp.) archimedean and not right (left, resp.) regular. Then no element from A is right (left, resp.) regular.*

Proof. Combine 1.17 and 1.16. □

1.19 Lemma. *Assume that ϱ is stable and cancellative. Let $a \in A$ be right (left, resp.) archimedean and not right (left, resp.) regular. Then there is $m \in \mathbb{N}$such that mx is almost negative (positive, resp.) for every $x \in A$.*

Proof. By 1.18, no element from A is right regular. By 1.17, for every $x \in A$ there is $m_x \in \mathbb{N}$ such that $m_x x$ is almost negative. Put $m = m_a$. Since a is right archimedean, we have $(x, n_x a) \in \varrho$ for some $n_x \in \mathbb{N}$. Now, $(mx, mn_x a) \in \varrho$, $(mx + mn_x a, mx) \in \varrho$, since $mn_x a$ is almost negative, and $(mx + mn_x a, mn_x a) \in \varrho$. Then $(mx + y = mn_x a, mn_x a + y) \in \varrho$ and $(mx + y, y) \in \varrho$ for every $y \in A$. Thus mx is almost negative. □

1.20 Lemma. *Assume that ϱ is stable and cancellative. Let $a \in A$ be neither left nor right regular. Then there is $m \in \mathbb{N}$ such that ma is both almost positive and almost negative (i.e., $(ma + x, x) \in \ker(\varrho)$ for every $x \in A$).*

Proof. The result follows easily from 1.17. □

1.21 Lemma. *Assume that ϱ is stable and cancellative and that no element from A is right or left regular. Then the factorsemigroup $A/\ker(\varrho)$ is a torsion group.*

Proof. By 1.20, for every $a \in A$ there is $m_a \in \mathbb{N}$ such that $(m_a a + x, x) \in \ker(\varrho)$ for every $x \in A$. It follows that $0_{\overline{A}} \in \overline{A}$ and $m_a \overline{a} = 0_{\overline{A}}$. Then, of course, $\overline{A}$ is a torsion group. □

1.22 Proposition. *Assume that ϱ is stable and cancellative and that the factorsemigroup $A/\ker(\varrho)$ is not a torsion group. Then every right (left, resp.) archimedean element from A is right (left, resp.) regular, provided that at least one of the following six conditions is satisfied:*

(1) *For every $m \in \mathbb{N}$ there is $v \in A$ such that mv is not almost negative (positive, resp.);*
(2) *At least one element from A is right (left, resp.) regular;*
(3) *At least one element from A is not left (right, resp.) regular;*
(4) *There are $k \in \mathbb{N}$ and $a \in A$ such that $k \geq 2$ and $(a, ka) \in \varrho$ ($(ka, a) \in \varrho$, resp.);*
(5) *At least one element from A is almost positive (negative, resp.);*
(6) *There are $l \in \mathbb{N}$ and $a \in A$ such that $l \geq 2$ and la is right (left, resp.) archimedean.*

Proof. If (1) is true then the result follows from 1.19. If (2) is true then 1.18 yields our result. f (3) is true then, by 1.21, at least one element from A is right regular and (2) is satisfied. The condition (4) is equivalent to (3) and (5) implies (4). Finally, (6) implies (4). □

1.23 Lemma. *Let $a \in A$ and $m \in \mathbb{N}$ be such that ma is right (left, resp.) ϱ-regular. Then a is right (left, resp.) ϱ-regular.*

Proof. It is easy. □

2. Extensions of homomorphisms – introduction

Throughout this section, let $A = A(+)$ be a commutative semigroup and let ϱ be a cancellative and stable preordering defined on A (i.e., for all $a, b, c \in A$ we have $(a, b) \in \varrho$ if and only if $(a + c, b + c) \in \varrho$). Furthermore, let B be a subsemigroup of A and let $h : B \to \mathbb{R}$ be an additive homomorphism such that $h(a) \leq h(b)$ whenever $(a, b) \in \varrho$.

For every $w \in A$ put

$$(2.1) \qquad (\underline{p}(w, A, B, h) =)\, \underline{p}(w) = \sup\{\tfrac{\mathrm{h(a)-h(b)}}{\mathrm{m}} \mid \mathrm{a, b \in B, m \in \mathbb{N}, (a, b + mw) \in \varrho}\}$$

and

$$(2.2) \qquad (\underline{q}(w, A, B, h) =)\, \underline{q}(w) = \inf\{\tfrac{\mathrm{h(c)-h(d)}}{\mathrm{n}} \mid \mathrm{c, d \in B, n \in \mathbb{N}, (d + nw, c) \in \varrho}\}.$$

2.1 Lemma. (i) $-\infty \leq \underline{p}(w) \leq \underline{q}(w) \leq +\infty$.
(ii) $\underline{p}(v) = h(v) = \underline{q}(v)$ *for every* $v \in B$.

Proof. (i) If either $\underline{p}(w) = -\infty$ or $\underline{q}(w) = +\infty$ then there is nothing to prove. On the other hand, if $(a, b + mw) \in \varrho$ and $(d + nw, c) \in \varrho$ for some $a, b, c, d \in B$ and $m, n \in \mathbb{N}$ then $(na, nb + nmw) \in \varrho$, $(md + mnw, mc) \in \varrho$, $(na + md + mnw, nb + mc + mnw) \in \varrho$ and, since ϱ is cancellative, we get $(na + md, nb + mc) \in \varrho$. Then $nh(a) + mh(d) \leq$ $\leq nh(b) + mh(c)$ and $\frac{h(a)-h(b)}{m} \leq \frac{h(c)-h(d)}{n}$. The rest is clear.
(ii) We have $(2v, v + 1v) = (2v, 2v) \in \varrho$ and $(v + 1v, 2v) \in \varrho$. Consequently, using (i), we get $h(v) = \frac{h(2v)-h(v)}{1} \leq \underline{p}(v) \leq \underline{q}(v) \leq h(v)$. Thus $h(v) = \underline{p}(v) = \underline{q}(v)$. □

2.2 Lemma. (i) *If B is right (left, resp.) ϱ-cofinal in A then $\underline{q}(w) < +\infty$ ($-\infty <$ $< \underline{q}(w)$, resp.) for every $w \in A$.*
(ii) *If at least one element from B is right (left, resp.) ϱ-archimedean in A then $\underline{q}(w) <$ $< +\infty$ ($-\infty < \underline{p}(w)$, resp.) for every $w \in A$.*

Proof. (i) For every $a \in B$ there is $b \in B$ with $(a + w, b) \in \varrho$ ($(b, a + w) \in \varrho$, resp.). Now, $\underline{q}(w) \leq h)b) - h(a)$ ($h(b) - h(a) \leq \underline{p}(w)$, resp.).
(ii) This follows immediately from (i). □

2.3 Lemma. *Assume that for all $u, v \in A$ such that $(u, v) \notin \varrho$ there are $a, b \in B$ with $(u + a, v + b) \in \varrho$. Then $-\infty < \underline{p}(w) \leq \underline{q}(w) < +\infty$ for every $w \in A$.*

Proof. Take any $c \in B$. Then there are $a_1, a_2, b_1, b_2 \in B$ such that $(c+a_1, w+b_1) \in \varrho$ and $(w + a_2, c + b_2) \in \varrho$. Now, we have $-\infty < h(c) + h(a_1) - h(b_1) \leq \underline{p}(w) \leq \underline{q}(w) \leq$ $\leq h(c) + h(b_2) - h(a_2) < +\infty$ (use 2.1(i)). □

2.4 Remark. Assume that B is both left and right ϱ-cofinal in A. Then, choosing $u, v \in A$, we can find $a, b \in B$ such that $(u, b) \in \varrho$ and $(a, v) \in \varrho$. Thus $(u+a, v+b) \in \varrho$ and 2.3 takes place (cf. 2.2(i)).

2.5 Lemma. *Let $w \in A$ be right (left, resp.) ϱ-archimedean. Then:*
(i) $-\infty < \underline{p}(w)$ *($\underline{q}(w) < +\infty$, resp.).*
(ii) *If $h(a) \geq 0$ ($h(a) \leq 0$, resp.) for at least one $a \in B$ then $\underline{p}(w) \geq 0$ ($\underline{q}(w) \leq 0$, resp.).*
(iii) *If $h(a) > 0$ ($h(a) < 0$, resp.) for at least one $a \in B$ then $\underline{p}(w) > 0$ ($\underline{q}(w) < 0$, resp.).*

Proof. For every $a \in A$ there is $m \in \mathbb{N}$ such that $(a, mw) \in \varrho$. Then $(2a, a+mw) \in \varrho$ and $\frac{h(a)}{m} \leq \underline{p}(w)$ due to (2.1). Thus $-\infty < \underline{p}(w)$ and, if $h(a) \geq 0$ or $h(a) > 0$ then $\underline{p}(w) \geq 0$ or $\underline{p}(w) > 0$. The other case is dual. □

2.6 Lemma. *Let $w \in A$ be such that kw is almost ϱ-positive (almost ϱ-negative, resp.) for some $k \in \mathbb{N}$. Then $\underline{p}(w) \geq 0$ ($\underline{q}(w) \leq 0$, resp.)*

Proof. We have $(a, a + kw) \in \varrho$ for every $a \in A$, and hence $0 = \frac{h(a)-h(a)}{k} \leq \underline{p}(w)$ by 2.1. The other case is dual. □

In the sequel, we put

(2.3) $\quad (\underline{W}(A, B, h) =) \, \underline{W} = \{ w \in A \mid -\infty < \underline{q}(w) \text{ and } \underline{p}(w) < +\infty \}$

and

(2.4) $\quad (\,\underline{V}(A,B,h) =)\, \underline{V} = \{\, w \in A \,|\, -\infty < \underline{p}(w) \text{ and } \underline{q}(w) < +\infty \,\}\,.$

2.7 Lemma. *$w \in \underline{W}$ if and only if $\underline{p}(w) \le r \le \underline{q}(w)$ for at least one $r \in \mathbb{R}$.*

Proof. We have $\underline{p}(w) \le \underline{q}(w)$ by 2.1(i) and our assertion follows from (2.3). □

2.8 Remark. The semigroup A is the disjoint union $A = \underline{W} \cup W_1 \cup W_2$, where $W_1 = \{\, w \in A \,|\, \underline{p}(w) = +\infty \,\}$ and $W_2 = \{\, w \in A \,|\, \underline{q}(w) = -\infty \,\}$. Of course, if $w \in W_1$ then $\underline{q}(w) = +\infty$ and $(d + nw, c) \notin \varrho$ for all $c, d \in B$ and $n \in \mathbb{N}$. Similarly, if $w \in W_2$ then $\underline{p}(w) = -\infty$ and $(a, b + mw) \notin \varrho$ for all $a, b \in B$ and $m \in \mathbb{N}$ (see (2.1) and (2.2)).

2.9 Lemma. *$\underline{V} = \{\, w \in W \,|\, \underline{p}(w) \in \mathbb{R}$ and $\underline{q}(w) \in \mathbb{R} \,\}$.*

Proof. The result follows by an easy combination of (2.4) and 2.1(i). □

2.10 Lemma. *$B \subseteq \underline{V} \subseteq \underline{W}$.*

Proof. First, $B \subseteq \underline{V}$ follows from 2.9 and 2.1(i). Next, $\underline{V} \subseteq \underline{W}$ follows from 2.7 and 2.1(i). □

2.11 Lemma. *Let C be a subsemigroup of A such that $B \subseteq C$ and h extends to an additive homomorphism $g : C \to \mathbb{R}$ such that $g(a) \le g(b)$ whenever $a, b \in C$ and $(a, b) \in \varrho$. Then $C \subseteq \underline{W}$ and $\underline{p}(c) \le g(c) \le \underline{q}(c)$ for every $c \in C$.*

Proof. If $a, b \in A$, $c \in C$ and $m \in \mathbb{N}$ are such that $(a, b + mc) \in \varrho$ then $h(a) = g(a) \le g(b) + mg(c) = h(b) + mg(c)$, and therefore $\frac{h(a)-h(b)}{m} \le g(c)$. Thus $\underline{p}(c) \le g(c)$ and, dually, $g(c) \le \underline{q}(c)$. By 2.7, $c \in \underline{W}$. □

2.12 Corollary. *Assume that h extends to an additive homomorphism $f : A \to \mathbb{R}$ such that $f(u) \le f(v)$ for all $(u, v) \in \varrho$. Then $\underline{W} = A$.* □

2.13 Lemma. *Assume that B is right (left, resp.) ϱ-cofinal in A* (see 2.2). *Then:*
(i) *$\underline{W} = \{\, w \in A \,|\, \underline{q}(w) > -\infty \,\}$ ($\underline{W} - \{\, w \in A \,|\, \underline{p}(w) < +\infty \,\}$, resp.).*
(ii) *$\underline{V} = \{\, w \in A \,|\, \underline{p}(w) > -\infty \,\}$ ($\underline{V} = \{\, w \in A \,|\, \underline{q}(w) < +\infty \,\}$, resp.).*
(iii) *If $w \in A$ is right (left, resp.) ϱ-archimedean then $w \in \underline{V}$*
(iv) *If $w \in A$ is such that kw is almost ϱ-positive (almost ϱ-negative, resp.) for some $k \in \mathbb{N}$ then $w \in \underline{V}$.*

Proof. (i) By 2.2(i), $\underline{q}(W) < +\infty$ for every $w \in A$. Since $\underline{p}(w) \le \underline{q}(w)$, we get $\underline{p}(w) < +\infty$ as well and the result follows from (2.3).
(ii) Again, $\underline{p}(w) \le \underline{q}(w) < +\infty$ and the result follows from (2.4).
(iii) Combine (ii) and 2.5.
(iv) Combine (ii) and 2.6. □

2.14 Lemma. *Let $w \in A$ be right (left, resp.) ϱ-archimedean. Then:*
(i) *$w \in \underline{W}$ if and only if $\underline{p}(w) < +\infty$ ($-\infty < \underline{q}(w)$, resp.).*
(ii) *$w \in \underline{V}$ if and only if $\underline{q}(w) < +\infty$ ($-\infty < \underline{p}(w)$, resp.).*

Proof. We have $-\infty < \underline{p}(w)$ ($\underline{q}(w) < +\infty$, resp.) by 2.5(i) and it remains to take into account (2.3) and (2.4). □

2.15 Lemma. *Let $w \in A$ be such that kw is almost ϱ-positive (almost ϱ-negative, resp.) for some $k \in \mathbb{N}$. Then:*
(i) *$w \in \underline{W}$ if and only if $\underline{p}(w) < +\infty$ ($-\infty < \underline{q}(w)$, resp.).*
(ii) *$w \in \underline{V}$ if and only if $\underline{p}(w) < +\infty$ ($-\infty < \underline{q}(w)$, resp.).*
(ii) *$w \in \underline{V}$ if and only if $\underline{q}(w) < +\infty$ ($-\infty < \underline{p}(w)$, resp.).*

Proof. We have $\underline{p}(w) \geq 0$ ($\underline{q}(w) \leq 0$, resp.) by 2.6 and it remains to take into account (2.3) and (2.4). □

2.16 Proposition. *$\underline{W} = A$ in each of the following five cases:*
(1) *B is right ϱ-cofinal in A and $\underline{q}(w) > -\infty$ for every $w \in A$;*
(2) *B is left ϱ-cofinal in A and $\underline{p}(w) < +\infty$ for every $w \in A$;*
(3) *If $w \in A \setminus B$ then $\underline{p}(w) < +\infty$ and either w is right ϱ-archimedean or kw is almost ϱ-positive for at least one $k \in \mathbb{N}$;*
(4) *If $w \in A \setminus B$ then $-\infty < \underline{q}(w)$ and either w is left ϱ-archimedean or kw is almost ϱ-negative for at least one $k \in \mathbb{N}$;*
(5) *If $w \in A \setminus B$ then at least one of the following four subcases takes place:*
  (5a) *$\underline{p}(w) < +\infty$ and w is right ϱ-archimedean;*
  (5b) *$-\infty < \underline{q}(w)$ and w is left ϱ-archimedean;*
  (5c) *$\underline{p}(w) < +\infty$ and kw is almost ϱ-positive for some $k \in \mathbb{N}$;*
  (5d) *$-\infty < \underline{q}(w)$ and kw is almost ϱ-negative for some $k \in \mathbb{N}$.*

Proof. Combine 2.13, 2.14 and 2.15. □

2.17 Proposition. *$\underline{V} = A$ in each of the following six cases:*
(1) *B is both left and right ϱ-cofinal in A;*
(2) *For all $u, v \in A$ such that $(u, v) \notin \varrho$ there are $a, b \in B$ with $(u + a, v + b) \in \varrho$;*
(3) *B is right ϱ-cofinal in A and for every $w \in A \setminus B$ at least one of the following three subcases takes place:*
  (3a) *$(a, b + mw) \in \varrho$ for some $a, b \in B$ and $m \in \mathbb{N}$;*
  (3b) *w is right ϱ-archimedean;*
  (3c) *kw is almost ϱ-positive for some $k \in \mathbb{N}$;*
(4) *B is left ϱ-cofinal in A and for every $w \in A \setminus B$ at least one of the following three subcases takes place:*
  (4a) *$(d + nw, c) \in \varrho$ for some $c, d \in B$ and $n \in \mathbb{N}$;*
  (4b) *w is left ϱ-archimedean;*
  (4c) *kw is almost ϱ-negative for some $k \in \mathbb{N}$;*
(5) *Every element from A is right ϱ-archimedean;*
(6) *Every element from A is left ϱ-archimedean.*

Proof. Combine 2.3, 2.13, 2.14 and 2.15. □

2.18 Remark. Let $w \in A$. If $\varrho|\mathbb{N} = \mathrm{id}$ then w is apparently both left and right ϱ-regular. Now, assume that $\varrho|\mathbb{N} \neq \mathrm{id}$. If w is not right ϱ-regular then $(nw, mw) \in \varrho$ for $n > m$, $((n-m)w + a, a) \in \varrho$ for every $a \in B$ and $\underline{q}(w) \leq 0$. Consequently, if $\underline{q}(w) > 0$ then w is right ϱ-regular. Similarly, if $\underline{p}(w) < 0$ then w is left ϱ-regular. Finally, if w is neither left nor right ϱ-regular then $\underline{p}(w) = 0 = \underline{q}(w)$.

2.19 Lemma. *Let $w \in A$ be an idempotent (i.e., $2w = w$). Then $\underline{p}(w) = 0 = \underline{q}(w)$.*

Proof. We have $(v + w, v + 2w) \in \varrho$ for every $v \in A$. Then $(v, v + w) \in \varrho$, since ϱ is cancellative. Similarly, $(v + w, v) \in \varrho$ and we have $9v + w, v) \in \ker(\varrho)$. The equalities $\underline{p}(w) = 0 = \underline{q}(w)$ are now clear from (2.1) and (2.2). □

2.20 Lemma. *Let $w \in A$ be such that $mw = w$ for some $m \in \mathbb{N}$, $m \geq 2$. Then $\underline{p}(w) = 0 = \underline{q}(w)$.*

Proof. We proceed similarly as in the proof of 2.19. □

2.21 Lemma. *Let $w \in A$ be such that $mw = nw$ for some $m, n \in \mathbb{N}$, $m > n$. Then $\underline{p}(w) = 0 = \underline{q}(w)$.*

Proof. Proceeding similarly as in the proof of 2.19, we show that $(v+(m-n)w, v) \in$ $\in \ker(\varrho)$ for every $v \in A$. The rest is clear from (2.1) and (2.2). □

2.22 Lemma. *Let $w_1, w_2 \in A$ be such that $-\infty < \underline{p}(w_1)$ and $-\infty < \underline{p}(w_2)$. Then $\underline{p}(w_1 + w_2) \geq \underline{p}(w_1) + \underline{p}(w_2)$.*

Proof. Let $(a_1, b_1 + m_1 w_1) \in \varrho$ and $(a_2, b_2 + m_2 w_2) \in \varrho$, where $a_1, a_2, b_1, b_2 \in B$ and $m_1, m_2 \in \mathbb{N}$. Then $(m_2 a_1, m_2 b_1 + m_1 m_2 w_1) \in \varrho)$, $m_1 a_2, m_1 b_2 + m_1 m_2 w_2) \in \varrho$ and $(m_2 a_1 + m_1 a_2, m_2 b_1 + m_1 b_2 + m_1 m_2 (w_1 + w_2)) \in \varrho$. Consequently, $\underline{p}(w_1 + w_2) \geq$ $\geq \frac{h(m_2 a_1 + m_1 a_2) - h(m_2 b_1 + m_1 b_2)}{m_1 m_2} = \frac{h(a_1) - h(b_1)}{m_1} + \frac{h(a_2) - h(b_2)}{m_2}$ and the rest is clear. □

2.23 Lemma. *Let $w_1, w_2 \in A$ be such $\underline{p}(w_1) < +\infty$ and $\underline{p}(w_2) < +\infty$. Then $\underline{p}(w_1 + w_2) \geq \underline{p}(w_1) + \underline{p}(w_2)$.*

Proof. The result follows from 2.22. If, say, $\underline{p}(w_1) = -\infty$ then $\underline{p}(w_1) + \underline{p}(w_2) = -\infty$ and there is noothing to prove. □

2.24 Lemma. *Let $w_1, w_2 \in A$ be such that $\underline{q}(w_1) < +\infty$ and $\underline{q}(w_2) < +\infty$. Then $\underline{q}(w_1 + w_2) \leq \underline{q}(w_1) + \underline{q}(w_2)$.*

Proof. This is dual to 2.22. □

2.25 Lemma. *Let $w_1, w_2 \in A$ be such that $-\infty < \underline{q}(w_1)$ and $-\infty < \underline{q}(w_2)$. Then $\underline{q}(w_1 + w_2) \leq \underline{q}(w_1) + \underline{q}(w_2)$.*

Proof. This is dual to 2.23. □

2.26 Proposition. *Let $w_1, w_2 \in \underline{W}$. Then $\underline{p}(w_1) + \underline{p}(w_2) \leq \underline{p}(w_1 + w_2) \leq \underline{q}(w_1 + w_2) \leq$ $\leq \underline{q}(w_1) + \underline{q}(w_2)$.*

Proof. By (2.3), we have $\underline{p}(w_1) < +\infty$, $\underline{p}(w_2) < +\infty$, $-\infty < \underline{q}(w_1)$, $-\infty < \underline{q}(w_2)$ and it remains to use 2.23 and 2.25. □

2.27 Proposition. *$\underline{V}$ is a subsemigroup of A.*

Proof. Combine 2.22 and 2.24. □

3. Extensions of homomorphisms – continued

This section immediately continues the preceding one. All the notation is fully kept.

3.1 Lemma. *Let $w \in A$, $a, b \in B$, $k \in \mathbb{N}$ and $r \in \mathbb{R}$ be such that $\underline{p}(w) \le r$ ($r \le \underline{q}(w)$, resp.) and $(b, a + kw) \in \varrho$ ($(b + kw, a) \in \varrho$, resp.). Then $h(b) \le h(a) + kr$ ($h(b) + kr \le h(a)$, resp.).*

Proof. Since $(b, a + kw) \in \varrho$, by (2.1) we have $\frac{h(b)-h(a)}{k} \le \underline{p}(w) \le r$. Thus $h(b) \le$ $\le h(a) + kr$. The other case is dual. □

3.2 Lemma. *Let $w \in A$, $a, b \in B$, $k, l \in \mathbb{N}$ and $r \in \mathbb{R}$ be such that $\underline{p}(w) \le r \le \underline{q}(w)$ and $(b + lw, a + kw) \in \varrho$. Then $lr + h(b) \le kr + h(a)$.*

Proof. First, if $l < k$ then $(v, a+(k-l)w) \in \varrho$, since the preordering ϱ is cancellative, and $lr + h(b) \le kr + h(a)$ by 3.1. Next, if $k < l$ then $(b + (l - k)w, a) \in \varrho$ and our result follows from 3.1 again. Finally, if $k = l$ then $(b, a) \in \varrho$ and $h(b) \le h(a)$. □

3.3 Lemma. *Let $w \in A$, $a \in B$, $k \in \mathbb{N}$ and $r \in \mathbb{R}$ be such that $\underline{p}(w) \le r$ ($r \le \underline{q}(w)$, resp.) and $(a, kw) \in \varrho$ ($(kw, a) \in \varrho$, resp.). Then $h(a) \le kr$ ($kr \le h(a)$. resp.).*

Proof. Since $(a, kw) \in \varrho$, we hace $(2a, a + kw) \in \varrho$ and $2h(a) = h(2a) \le h(a) + kr$ by 3.1. Thus $h(a) \le kr$. The other case is dual. □

3.4 Lemma. *Let $w \in A$, $a \in B$, $k\, l \in \mathbb{N}$ and $r \in \mathbb{R}$ be such that $\underline{p}(w) \le r \le \underline{q}(w)$ and $(lw, a + kw) \in \varrho$ ($(lw + a, kw) \in \varrho$, resp.). then $lr \le h(a) + kr$ ($lr + h(a) \le kr$, resp.).*

Proof. We have $(lw + a, 2a + kw) \in \varrho$ and 3.3 applies. The other case is dual. □

3.5 Lemma. *Let $w \in A$, $k, l \in \mathbb{N}$ and $r \in \mathbb{R}$ be such that $\underline{p}(w) \le r \le \underline{q}(w)$ and $(lw, kw) \in \varrho$. Then $lr \le kr$.*

Proof. Taking any $a \in B$, we get $(a + lw, a + kw) \in \varrho$ and the result follows from 3.2. □

3.6 Proposition. *Let $w \in A$ and let $B\langle w\rangle$ be the subsemigroup of A generated by $B \cup \{w\}$. The following conditions are equivalent:*

(i) $w \in \underline{W}$ (see(2.3)).

(ii) *There is at least one* $r \in \mathbb{R}$ *with* $\underline{p}(w) \le r \le \underline{q}(w)$ *and for any such* r *there exists (just one) additive homomorphism* $h_{w,r} : B\langle w\rangle \to \mathbb{R}$ *such that* $h_{w,r}$ *extends* h, $h_{w,r}(w) = r$ *and* $h_{w,r}(u) \le h_{w,r}(v)$ *whenever* $u, v \in B\langle w\rangle$ *and* $(u, v) \in \in \varrho$.

(iii) *There is at least one subsemigroup* C *of* A *such that* $B \subseteq C$, $w \in C$ *(then* $B\langle w\rangle \subseteq C$*) and* h *extends to an additive homomorphism* $g : C \to \mathbb{R}$ *such that* $g(u) \le g(v)$ *whenever* $u, v \in C$ *and* $(u, v) \in \varrho$.

Proof. (i) implies (ii). Let $r \in \mathbb{R}$ be such that $\underline{p}(w) \le r \le \underline{q}(w)$ (see 2.7). If $v \in \in B\langle w\rangle$ then either $v = a+kw$ for some $a \in B$ and $k \in \mathbb{N}$, and we put $h_{w,r}(v) = h(a)+kr$, or $v \in B$ and we put $h_{w,r}(v) = h(v)$, or, finally $v = kw$ for some $k \in \mathbb{N}$ and we put $h_{w,r}(v) = kr$. It follows from 3.1, 3.2, 3.3, 3.4 and 3.5 that the definition is correct and if $u, v \in B\langle w\rangle$ are such that $(u, v) \in \varrho$ then $h_{w,r}(u) \le h_{w,r}(v)$.
(ii) implies (iii). This implication is trivial.
(iii) implies (i). By 2.11, $C \subseteq \underline{W}$. Consequently, $w \in \underline{W}$. □

In what follows, let ($\underline{\mathscr{W}}(A, B, h) =$) $\underline{\mathscr{W}}$ denote the set of ordered pairs (C, g), where C is a subsemigroup of A with $B \subseteq C$ and $g : C \to \mathbb{R}$ is an additive homomorphism extending h such that $g(u) \le g(v)$ whenever $u, v \in C$ and $(u, v) \in \varrho$. The set $\underline{\mathscr{W}}$ is ordered by inclusion and we denote by $\underline{\mathscr{W}}_{\max}$ ($= \underline{\mathscr{W}}_{\max}(A, B, h)$) the set of maximal pairs from $\underline{\mathscr{W}}$.

3.7 Proposition. *Let* $(C, g) \in \underline{\mathscr{W}}_{\max}(A, B, h)$. *Then:*
(i) $B \subseteq \underline{V}(A, B, h) \subseteq C \subseteq \underline{W}(A, B, h)$.
(ii) $C = \underline{W}(A, C, g) = \underline{V}(A, C, g)$.
(iii) *If* $w \in A \setminus C$ *then either* $\underline{p}(w, A, C, g) = \underline{q}(w, A, C, g) = +\infty$ *or* $\underline{p}(w, A, C, g) = = \underline{q}(w, A, C, g) = -\infty$.

Proof. (i) By 2.10, $B \subseteq \underline{V}(A, B, h)$ and, by 2.11, $C \subseteq \underline{W}(A, B, h)$. On the other hand, if $w \in \underline{V}(A, B, h)$ then $-\infty < \underline{p}(w, A, B, h) \le \underline{p}(w, A, C, g) \le \underline{q}(w, A, C, g) \le \le \underline{q}(w, A, B, h) < +\infty$ (see (2.1), (2.2) and 2.1(i)). Consequently, $w \in \underline{V}(A, C, g)$. But $\underline{V}(A, C, g) = C$ by 3.6.
(ii) This assertion follows from 3.6 (where B is replaced by C).
(iii) This follows from the equality $C = \underline{W}(A, C, g)$. □

3.8 Proposition. *For every* $w \in \underline{W}(A, B, h)$ *there is at at least one pair* $(C, g) \in \in \underline{\mathscr{W}}_{\max}(A, B, h)$ *such that* $w \in C$.

Proof. The assertion follows from 3.6. □

3.9 Proposition. *Assume that* $\underline{V}(A, B, h) = A$. *Then* h *can be extended to an additive homomorphism* $f : A \to \mathbb{R}$ *such that* $f(u) \le f(v)$ *whenever* $(u, v) \in \varrho$. *Furthermore,* $(A, f) \in \underline{\mathscr{W}}_{\max}(A, B, h)$, *and if* $(C, g) \in \underline{\mathscr{W}}_{\max}(A, B, h)$ *then* $C = A$.

Proof. The result follows easily from 3.7. □

3.10 Remark. Various conditions that are sufficient for the equality $\underline{V}(A, B, h) = A$ are formulated in 2.17.

3.11 Proposition. *Assume that B is right (left, resp.) ϱ-cofinal in A and that for every $w \in A \setminus B$ there are $a, b \in B$ and $m \in \mathbb{N}$ such that $(a, b+mw) \in \varrho$ ($(b+mw, a) \in \varrho$, resp.). Then h extends to an additive homomorphism $f : A \to \mathbb{R}$ such that $f(u) \leq f(v)$ for all $(u, v) \in \varrho$.*

Proof. By 2.17, $\underline{V}(A, B, h) = A$ and 3.9 applies. □

3.12 Proposition. *Assume that every element from A is right (left, resp.) ϱ-archimedean and that $h(B) \neq 0$. Then $h(B) \subseteq \mathbb{R}^+$ ($h(B) \subseteq \mathbb{R}^-$, resp.) and h extends to an additive homomorphism $f : A \to \mathbb{R}^+$ ($f : A \to \mathbb{R}^-$, resp.) such that $f(u) \leq f(v)$ for all $(u, v) \in \varrho$.*

Proof. First, for every $a \in B$ there is $m \in \mathbb{N}$ with $(a, 2ma) \in \varrho$, hence $h(a) \leq$ $\leq 2mh(a)$, $(2m - 1)h(a) \geq 0$ and $h(a) \geq 0$. Thus $h(B) \subseteq \mathbb{R}_0^+$. Since $h(B) \neq 0$, we have $h(a_0$ for at least one $a_0 \in B$. Given $b \in B$, there is $n \in \mathbb{N}$ with $(a_0, nb) \in \varrho$. Then $0 < h(a_0) \leq nh(b)$ and $h(b) > 0$. Thus $h(B) \subseteq \mathbb{R}^+$. Furthermore, by 2.17, $\underline{V}(A, B, h) = A$ and, by 3.9, h extends to an additive homomorphism $f : A \to \mathbb{R}$. Proceeding similarly as above, we show that $f(A) \subseteq \mathbb{R}^+$. □

3.13 Proposition. *Assume that B is right (left, resp.) ϱ-cofinal in A and that for every $w \in B \setminus A$ (that is not right ϱ-archimedean) there is at least one $m_w \in \mathbb{N}$ such that $m_w w$ is almost ϱ-positive (almost ϱ-negative, resp.). Then h extends to an additive homomorphism $f : A \to \mathbb{R}$ such that $f(A \setminus B) \subseteq \mathbb{R}_0^+$ ($f(A \setminus B \subseteq \mathbb{R}_0^-$, resp.) and $f(u) \leq f(v)$ for all $(u, v) \in \varrho$. If $h(B) \subseteq \mathbb{R}_0^+$ ($h(B) \subseteq \mathbb{R}_0^-$, resp.) then $f(A) \subseteq \mathbb{R}_0^+$ ($f(A) \subseteq \mathbb{R}_0^-$, resp.).*

Proof. It follows easily from 3.11 that h extends to an additive homomorphism $f : A \to \mathbb{R}$ preserving the preordering. If $w \in A \setminus B$ and $a \in B$ then $(a, a + m_w w) \in \varrho$, so that $f(a) \leq f(a) + m_w f(w)$ and $0 \leq f(w)$. □

4. Extensions of homomorphisms of one-generated subsemigroups – introduction

Throughout this section, let A be a commutative semigroup, ϱ be a cancellative and stable preordering defined on A and $z \in A$ be right ϱ-regular. Then $B = \mathbb{N}z \cong \mathbb{N}$ and $(h_z =) h : B \to \mathbb{R}^+$, where $h(nz) = n$ for every $n \in \mathbb{N}$, is an injective additive homomorphism such that $h(z) = 1$ and $h(a) \leq h(b)$ whenever $a, b \in B$ and $(a, b) \in \varrho$.

4.1 Lemma. *Let $w \in A$. Then:*
(i) $\underline{p}(w) = \sup\{ \frac{\mathrm{k-l}}{\mathrm{m}} \mid \mathrm{k, l, m} \in \mathbb{N}, (\mathrm{kz}, \mathrm{lz} + \mathrm{mw}) \in \varrho \}$.
(ii) $\underline{q}(w) = \inf\{ \frac{\mathrm{k-l}}{\mathrm{n}} \mid \mathrm{k.l.n} \in \mathbb{N}, (\mathrm{nw} + \mathrm{lz}, \mathrm{kz}) \in \varrho \}$.
(iii) $-\infty \leq \underline{p}(w) \leq \underline{q}(w) \leq +\infty$.
(iv) $\underline{p}(mz) = \underline{q}(mz) = m$ *for every $m \in \mathbb{N}$.*

Proof. We have $B = \mathbb{N}z$ and the reast is clear from (2.1), (2.2) and 2.1. □

4.2 Lemma. *Assume that at least one of the following three conditions is satisfied for $w \in A$:*

(1) *w is right ϱ-archimedean in A;*

(2) $(k_0 z, l_0 z + m_0 w) \in \varrho$ *for some* $k_0, l_0, m_0 \in \mathbb{N}$, $k_0 > l_0$*;*

(3) $\underline{p}(w) > 0$.

Then $\underline{p}(w) = \sup\{\, \frac{\mathrm{k}}{\mathrm{m}} \mid \mathrm{k}, \mathrm{m} \in \mathbb{N}, (\mathrm{kz}, \mathrm{mw}) \in \varrho \,\} > 0$.

Proof. Clearly, (1) implies (2) and (2) is equivalent to (3). Now, if (2) is true then $\underline{p}(w) = \sup\{\, \frac{\mathrm{k-l}}{\mathrm{m}} \mid \mathrm{k}, \mathrm{l}, \mathrm{m} \in \mathbb{N}, \mathrm{k} > \mathrm{l}, (\mathrm{kz}, \mathrm{lz} + \mathrm{mw}) \in \varrho \,\}$ and our assertion follows from the fact that ϱ is cancellative. □

4.3 Lemma. *Assume that* $\underline{p}(w) \geq 0$ *(e.g. if $m_0 w$ is almost ϱ-positive for some $m_0 \in \mathbb{N}$). Then* $\underline{p}(w) = \sup(\{0\} \cup \{\, \frac{\mathrm{k}}{\mathrm{m}} \mid \mathrm{k}, \mathrm{m} \in \mathbb{N}, (\mathrm{kz}, \mathrm{mw}) \in \varrho \,\}) \geq 0$.

Proof. Clearly, $\underline{p}(w) = \sup\{\, \frac{\mathrm{k-l}}{\mathrm{m}} \mid \mathrm{k}, \mathrm{l}, \mathrm{m} \in \mathbb{N}, \mathrm{k} \geq \mathrm{l}, (\mathrm{kz}, \mathrm{lz} + \mathrm{mw}) \in \varrho \,\}$ and the rest is clear. □

4.4 Lemma. *Assume that* $\underline{q}(w) > 0$*. Then* $\underline{q}(w) = \inf\{\, \frac{\mathrm{l}}{\mathrm{n}} \mid \mathrm{l}, \mathrm{n} \in \mathbb{N}, (\mathrm{nw}, \mathrm{lz}) \in \varrho \,\}$.

Proof. Since $\underline{q}(w) > 0$, we have $k > l$ whenever $k, l, n \in \mathbb{N}$ are such that $(nw + {} + lz, kz) \in \varrho$. Then $(nw, (k - l)z) \in \varrho$ and our result follows. □

4.5 Proposition. *If* $\underline{p}(w) > 0$ *then* $\underline{p}(w) = \sup\{\, \frac{\mathrm{k}}{\mathrm{m}} \mid \mathrm{k}, \mathrm{m} \in \mathbb{N}, (\mathrm{kz}, \mathrm{mw}) \in \varrho \,\}$ *and* $\underline{q}(w) = \inf\{\, \frac{\mathrm{l}}{\mathrm{n}} \mid \mathrm{l}, \mathrm{n} \in \mathbb{N}, (\mathrm{nw}, \mathrm{lz}) \in \varrho \,\}$.

Proof. We have $\underline{q}(w) \geq \underline{p}(w)$ and it suffices to use 4.2 and 4.4. □

4.6 Proposition. *Assume that* $\underline{p}(w) = 0$*. Then:*

(i)*$k \geq l$ whenever $k, l, m \in \mathbb{N}$ are such that $(lz, lz + mw) \in \varrho$.*

(ii) *There are $k_0, l_0, m_0 \in \mathbb{N}$ such that $k_0 \leq l_0$ and $(k_0 z, l_0 z + m_0 w) \in \varrho$. If $k_0 = l_0$ then $m_0 w$ is almost ϱ-positive.*

(iii) *Suppose that $m_1 w$ is not almost ϱ-positive for any $m_1 \in \mathbb{N}$. Then* $0 = \underline{p}(w) = {} = \sup\{\, \frac{\mathrm{1-t}}{\mathrm{m}} \mid \mathrm{t}, \mathrm{m} \in \mathbb{N}, \mathrm{t} \geq 2, (\mathrm{z}, \mathrm{tz} + \mathrm{mw}) \in \varrho \,\}$ *and $(t - 1)z + mw$ is almost ϱ-positive.*

Proof. (i) This follows from 4.1(i).

(ii) The existence of the numbers k_0, l_0, m_0 follows from 4.1(1) and the fact that $\underline{p}(w) = 0$. Furthermore, if $k_0 = l_0$ then $(v + k_0 z, v + l_0 z + m_0 w) \in \varrho$ for every $v \in A$. Since ϱ is cancellative, we get $(v, v + m_0 w) \in \varrho$ and this means that $m_0 w$ is almost ϱ-positive.

(iii) If $k, l, m \in \mathbb{N}$ are such that $(kz, lz + mw) \in \varrho$ then from (i) and (ii) follows that $k < l$ and we get $(z, (l - k + 1)z + mw) \in \varrho$, $t = l - k + 1 \geq 2$. The rest is clear from 4.1(i). □

4.7 Proposition. (cf. 4.5 and 4.6) *Assume that* $\underline{q}(w) = 0$*. Then at least one of the following two cases holds:*

(1) $\underline{q}(w) = \inf\{\, \frac{\mathrm{l}}{\mathrm{n}} \mid \mathrm{l}, \mathrm{n} \in \mathbb{N}, (\mathrm{nw}.\mathrm{lz}) \in \varrho \,\}$*;*

(2) *$k = l$ whenever $k, l, n \in \mathbb{N}$ are such that $(mw + lz, kz) \in \varrho$, and there are $n_0, k_0 \in \mathbb{N}$ such that $(n_0w + k_0z, k_0z) \in \varrho$ and n_0w is almost ϱ-positive (then $(n_0w + z, z) \in \varrho$).*

Proof. Assume that (1) is not true. We have $\underline{q}(w) = 0$ and it follows that $k \geq l$ whenever $k, l, n \in \mathbb{N}$ are such that $(nw + lz, kz) \in \varrho$. If $k > l$ then $(nw, (k-l)z) \in \varrho$. Now, since (1) is not true, there are $n_0, k_0 \in \mathbb{N}$ with $(n_0w + k_0z, k_0z) \in \varrho$. Then n_0w is almost ϱ-negative and $(n_0w + z, z) \in \varrho$. Put $\alpha = \inf\{\,\frac{\mathrm{k-l}}{\mathrm{n}} \mid \mathrm{k, l, n} \in \mathbb{N}, \mathrm{k} > \mathrm{l}, (\mathrm{nw} + \mathrm{lz}, \mathrm{kz}) \in \varrho\,\} \subseteq \mathbb{R}_0^+ \cup \{+\infty\}$. Since (1) is not true, we have $\alpha > 0$. If $\alpha = +\infty$ then (2) is true. Consequently, assume finally that $\alpha < +\infty$. Since $\alpha > 0$, there is $t \in \mathbb{N}$ such that $tk \geq n + tl$ whenever $k, l, n \in \mathbb{N}$ are such that $k > l$ and $(nw+lz, kz) \in \varrho$. Furthermore, since $\alpha < +\infty$, $(n_1w+l_z, k_1z) \in \varrho$ for some $k_1, l_1, n_1 \in \mathbb{N}$, $k_1 > l_1$. We have $p = tk_1 - tl_1 - n_1 \geq 0$ and there is $q \in \mathbb{N}$ with $qn_0 > p$. However, qn_0w is almost ϱ-negative, and hence $((n_1 + qn_0)w + (l_1 + 1)z, (k_1 + 1)z) \in \varrho$. Now, $t(k_1 + 1) \geq n_1 + qn_0 + t(l+1+1)$ and then $p = tk_1 - tl_1 - n_1 \geq qn_0$, a contradiction. □

4.8 Proposition. *Assume that $\underline{q}(w) > 0$ and that mw is not almost ϱ-negative for any $m \in \mathbb{N}$. Then $\underline{q}(w) = \inf\{\,\frac{\mathrm{l}}{\mathrm{n}} \mid \mathrm{l, n} \in \mathbb{N}, (\mathrm{nw}, \mathrm{lz}) \in \varrho\,\}$.*

Proof. Combine 4.4 and 4.7. □

4.9 Remark. Assume that $\overline{p}(w) = 0$ (see 4.6) and 4.7(2) is true. Then n_0w is almost ϱ-negative for some $n_0 \in \mathbb{N}$. Furthermore, $(k_0z, l_0z + m_0w) \in \varrho$ for some $k_0, l_0, m_0 \in \mathbb{N}$, $k_0 \leq l_0$. If $k_0 = l_0$ then m_0w is almost ϱ-positive. In such a case, the element tw, where $t = n_0m_0$, is both almost ϱ-positive and almost ϱ-negative. Consequently, $(v, v + tw) \in \ker(\varrho)$ for every $v \in A$ (if ϱ is an ordering then $tw = 0_A \in A$).

4.10 Proposition. (i) *If z is right ϱ-archimedean then $\underline{q}(w) < +\infty$ for every $w \in A$.*
(ii) *If z is left ϱ-archimedean then $1 \leq \overline{p}(w)$ for every $w \in A$.*
(iii) *If $w \in A$ is right ϱ-archimedean then $\overline{p}(w) > 0$.*
(iv) *If $w \in A$ is left ϱ-archimedean then $\underline{q}(w) \leq 1$.*
(v) *If mw is almost ϱ-positive for some $m \in \mathbb{N}$ then $\overline{p}(w) \geq 0$.*
(vi) *If nw is almost ϱ-negative for some $n \in \mathbb{N}$ then $\underline{q}(w) \leq 0$.*

Proof. (i) There is $m \in \mathbb{N}$ with $(w, mz) \in \varrho$. Then $(w+z, (m+1)z) \in \varrho$ and $\underline{q}(w) \leq m$ by 4.1(ii).
(ii) There is $n \in \mathbb{N}$ with $(nz, w) \in \varrho$. Then $((n+1)z, z + w) \in \varrho$ and $\overline{p}(w_{\geq}n$ by 4.1(i).
(iii) There is $m \in \mathbb{N}$ with $(z, mw) \in \varrho$. Then $(2z, z + mw) \in \varrho$ and $\overline{p}(w) \geq \frac{1}{m} > 0$ by 4.1(i).
(iv) There is $n \in \mathbb{N}$ with $(nw, z) \in \varrho$. Then $(nw + z, 2z) \in \varrho$ and $\underline{q}(w) \leq \frac{1}{n} \leq 1$ by 4.1(ii).
(v) We have $(z, z + mw) \in \varrho$, and hence $\overline{p}(w) \geq 0$ by 4.1(i).
(vi) We have $(nz + z, z) \in \varrho$, and gence $\underline{q}(w) \leq 0$ by 4.1(ii). □

4.11 Proposition. (i) *If $w \in A$ is right ϱ-archimedean then* $0 < \sup\{\frac{k}{m} \mid k, m \in \mathbb{N}, (kz, mw) \in \varrho\} = \underline{p}(w) \le \overline{q}(w) = \inf\{\frac{j}{n} \mid l, n \in \mathbb{N}, (nw, lz) \in \varrho\}$. *If, moreover, z is right ϱ-archimedean then $\overline{q}(w) < +\infty$. If z is left ϱ-archimedean then $1 \le \underline{p}(w)$.*
(ii) *If both z and w are left ϱ-archimedean then* $\sup\{\frac{k}{m} \mid k, m \in \mathbb{N}, (kz, mw) \in \varrho\} = \underline{p}(w) = 1 = \overline{q}(w) - \inf\{\frac{l}{n} \mid l, n \in \mathbb{N}, (nw, lz) \in \varrho\}$.

Proof. (i) By 4.10(iii), we have $\underline{p}(w) > 0$ and the rest follows from 4.2, 4.4, 4.10(i) and 4.10 (iv).
(ii) We have $1 \le \underline{p}(w) \le \overline{q}(w) \le 1$ by 4.10(ii),(iv). Thus $\underline{p}(w) = 1 = \overline{q}(w)$ and the rest follows from 4.2 and 4.4. □

4.12 Proposition. *Let $w \in A$ be such that $m_0 w$ is almost ϱ-positive for some $m_0 \in \mathbb{N}$. Then at least one of the following four cases holds:*

(1) $0 < \sup\{\frac{k}{m} \mid k, m \in \mathbb{N}(kz, mw) \in \varrho\} = \underline{p}(w) \le \overline{q}(w) = \inf\{\frac{l}{n} \mid l, n \in \mathbb{N}, (nw, lz) \in \varrho\}$.
(2) $0 = \underline{p}(w) < \overline{q}(w) = \inf\{\frac{l}{n} \mid l, n \in \mathbb{N}, (nw, lz) \in \varrho\}$.
(3) $0 = \underline{p}(w) = \overline{q}(w) = \inf\{\frac{l}{n} \mid l, n \in \mathbb{N}, (nw, lz) \in \varrho\}$.
(4) $\underline{p}(w) = 0 = \overline{q}(w)$ *and there is $t \in \mathbb{N}$ such that tw is both almost ϱ-positive and almost ϱ-negative* (i.e., $(tw + v, v) \in \ker(\varrho)$ *for every $v \in A$).*

Proof. We have $\underline{p}(w) \ge 0$ by 4.10(v). The rest follows from 4.2, 4.4 and 4.8. □

4.13 Remark. Let $z_1 \in A$ be right ϱ-regular. Put $p_1 = \underline{p}(z_1, A, \mathbb{N}z, h)$, $q_1 = \overline{q}(z_1, A, \mathbb{N}z, h)$, $p_2 = \underline{p}(z, A, \mathbb{N}z_1, h_{z_1})$, $q_2 = \overline{q}(z, A, \mathbb{N}z_1, h_{z_1})$ (see (2.1) and (2.2)).
(i) Now, assume that $0 < p_1$ and $0 < q_2$. Then $p_1 = \sup\{\frac{k}{m} \mid k, m \in \mathbb{N}, (kz, mz_1) \in \varrho\}$ and $q_2 = \inf\{\frac{m}{k} \mid k, m \in \mathbb{N}, (kz, mz_1) \in \varrho\}$. Since $p_1 > 0$, we have $q_2 < +\infty$ and, since $q_2 > 0$, we have $p_1 < +\infty$. Using this, we calculate easily that $p_1 q_2 = 1$. Similarly, if $0 < p_2$ and $0 < q_1$ then $p_2 q_1 = 1$. (Notice that $0 < p_1$ implies $0 < q_1$ and $0 < p_2$ implies $0 < q_2$. Thus $0 < p_1$ and $0 < p_2$ implies $q_2 = \frac{1}{p_1}$ and $q_1 = \frac{1}{p_2}$.)
(ii) If $p_1 = 1$ and $0 < q_2$ then $q_2 = 1$. If $q_2 = 1$ and $0 < p_1$ then $p_1 = 1$. If $p_2 = 1$ and $0 < q_1$ then $q_1 = 1$. If $q_1 = 1$ and $0 < p_2$ then $p_2 = 1$. If $p_1 = 1 = q_1$ and $0 < p_2$ then $p_2 = 1 = q_2$. If $p_2 = 1 = q_2$ and $0 < p_1$ then $p_1 = 1 = q_1$.

4.14 Remark. Let $w \in A$ be such that $\underline{p}(w) = 1 = \overline{q}(w)$. Then $\sup\{\frac{k}{m} \mid k, m \in \mathbb{N}, (kz, mw) \in \varrho\} = 1 = \inf\{\frac{l}{n} \mid l, n \in \mathbb{N}, (nw, lz) \in \varrho\}$. Furthermore, suppose that $f : A \to \mathbb{R}$ is an additive homomorphism such that $f(u) \le f(v)$ for all $(u, v) \in \varrho$. Then $\frac{kf(z)}{m} \le f(w) \le \frac{lf(z)}{n}$ and we conclude that $f(z) = f(w)$.

4.15 Proposition. *Let $w \in A$ Then:*
(i) *If $\overline{q}(w) > 0$ then w is right ϱ-regular.*
(ii) *If $\underline{p}(w) < 0$ then w is left ϱ-regular.*
(iii) *If w is neither right nor left ϱ-regular then $\varrho|\mathbb{N}w \ne \mathrm{id}$ and $\underline{p}(w) = 0 = \overline{q}(w)$.*

Proof. See 2.18. □

5. Local summary

As usual in this paper, let ϱ be a stable and cancellative preordering defined on a commutative semigroup A.

5.1 Theorem. *Let $z \in A$ be right ϱ-archimedean and right ϱ-regular* (cf. 1.22, 5.4). *Suppose that for every $w \in A$ there are positive integers m, n such that $nz + mw$ or mw is almost ϱ-positive. Then there is an additive homomorphism $f : A \to \mathbb{R}$ such that $f(z) = 1$ and $f(u) \le f(v)$ for all $(u, v) \in \varrho$.*

Proof. Put $B = \mathbb{N}z$ and $h(kz) = k$ for every $k \in \mathbb{N}$. Since z is right ϱ-regular, $B \cong \mathbb{N}$ and h is an injective additive homomorphism of B into $\mathbb{R}$ such that $h(z) = 1$ and $h(a) \le h(b)$ for all $a, b \in B$ such that $(a, b) \in \varrho$. If $nz + mw$ is almost ϱ-positive then $(z, (n + 1)z + mw) \in \varrho$ and $-\frac{n}{m} \le \underline{p}(w)$ by 4.1(i). If mw is almost ϱ-positive then $(z, z + mw) \in \varrho$ and $0 \le \underline{p}(w)$ by 4.1(i). Since z is right ϱ-archimedean, we have $\overline{q}(w) < +\infty$ by 4.10(i). Thus $-\infty < \underline{p}(w) \le \overline{q}(w) < +\infty$ for every $w \in A \setminus B$ and it follows from 2.9 and 2.10 that $A = \underline{V}(A, B, h)$. Now it remains to use 3.9. □

5.2 Theorem. *Let $z \in A$ be right ϱ-archimedean and right ϱ-regular* (cf. 1.22, 5.4). *Suppose that for every $w \in B \setminus A$ (such that w is not right ϱ-archimedean) there is a positive integer m such that mw is almost ϱ-positive. Then there is an additive homomorphism $f : A \to \mathbb{R}_0^+$ such that $f(z) = 1$ and $f(u) \le f(v)$ for all $(u, v) \in \varrho$.*

Proof. Put $B = \mathbb{N}z$ and $h(kz) = k$ for every $k \in \mathbb{N}$. Since z is right ϱ-regular, $B \cong \mathbb{N}$ and h is an injective additive homomorphism such that $h(z) = 1$ and $h(a) \le h(b)$ for all $a, b \in B$, $(a, b) \in \varrho$. If mw is almost ϱ-positive then $(z, z + mw) \in \varrho$ and $\underline{p}(w) \ge 0$ by 4.1(i). If w is right ϱ-archimedean then $\underline{p}(w) > 0$ by 4.10(iii). Furthermore, since z is right ϱ-archimedean, we have $\overline{q}(w) < +\infty$ by 4.10(i). Thus $0 \le \underline{p}(w) \le \overline{q}(w) < +\infty$ for every $w \in A \setminus B$ and it remains to use 3.13. □

5.3 Theorem. (cf. 5.4) *Assume that every element from A is right ϱ-archimedean. Then, for every right ϱ-regular element $z \in A$, there is an additive homomorphism $f : A \to \mathbb{R}^+$ such that $f(z) = 1$ and $f(u) \le f(v)$ for all $(u, v) \in \varrho$.*

Proof. Again, put $B = \mathbb{N}z$, $h(nz) = z$ and use 3.12. □

5.4 Remark. Assume that $A/\ker(\varrho)$ is not a torsion group. Let $z \in A$ be such that z is right ϱ-archimedean, but not right ϱ-regular. By 1.22, every element from A is neither right ϱ-regular nor almost ϱ-positive. Besides, for all $a \in A$ and $m \in \mathbb{N}$, $m \ge 2$, the element ma is not right ϱ-archimedean.

5.5 Theorem. *Let $z \in A$ be right ϱ-regular. Suppose that for all $u_1, v_1 \in A$ such that $(u_1, v_1) \notin \varrho$ there is a positive integer m such that either $(u_1 + mz, v_1) \in \varrho$ or $(u_1, v_1 + mz) \in \varrho$. Then there is an additive homomorphism $f : A \to \mathbb{R}$ such that $f(z) = 1$ and $f(u) \le f(v)$ for all $(u, v) \in \varrho$.*

Proof. Put $B = \mathbb{N}z$ and $h(nz) = n$ for every $n \in \mathbb{N}$. If $u_1, v_1 \in A$ are such that $(u_1, v_1) \notin \varrho$ then $(u_1 + mz, v_1) \in \varrho$ ($(u_1, v_1 +, z) \in \varrho$, resp.) for some $m \in \mathbb{N}$ and we get $(u_1 + (m+1)z, v_1 + z) \in \varrho$ ($(u_1 + z, v_1 + (m+1)z) \in \varrho$, resp.). Consequently, the condition 2.17(2) is satisfied and it remains to use 3.9. □

5.6 Proposition. *Let $z \in A$ be right ϱ-regular, $B = \mathbb{N}z$ and $h(mz) = m$ for every $m \in \mathbb{N}$. Then $\underline{V}(A, B, h) = A$ if and only if every element $w \in A$ ($w \in A \setminus B$) satisfies at least one of the following four conditions:*

(1) *$(n_1 z, m_1 w) \in \varrho$ and $(m_2 w, n_2 z) \in \varrho$ for some $n_1, n_2, m_1, m_2 \in \mathbb{N}$ (then $(n_1 m_2 z, m_1 m_2 w) \in \varrho$, $(m_1 m_2 w, n_2 m_1 z) \in \varrho$, $(n_1 m_2 z, n_2 m_2 z) \in \varrho$, $n_1 m_2 \leq n_2 m_1$, $(n_1 m_2 w, n_1 n_2 z) \in \varrho$, $(n_1 n_2 z, n_2 m_1 w) \in \varrho$, $(n_1 m_2 w, n_2 m_1 w) \in \varrho$ and $0 < \underline{p}(w) \leq \leq \underline{q}(w) < +\infty$);*

(2) *There are $n_1, n_2, m_1, m_2 \in \mathbb{N}$ such that $m_1 w + n_1 z$ is almost ϱ-positive and $(m_2 w, n_2 z) \in \varrho$ (then $(m_1 m_2 w, n_2 m_1 z) \in \varrho$, $m_1 m_2 w + n_1 m_2 z$ is almost ϱ-positive, $(m_1 m_2 w + n_1 m_2 z, (n_1 m_2 + n_2 m_1) z) \in \varrho$ and $(n_1 m_2 + n_2 m_1) z$ is almost ϱ-positive);*

(3) *mw is both almost ϱ-positive and almost ϱ-negative for some $m \in \mathbb{N}$ (then $(x, x + mv) \in \ker(\rho)$ for every $x \in A$ and $\underline{p}(w) = 0 = \underline{q}(w)$);*

(4) *$mw + nz$ is both almost ϱ-positive and almost ϱ-negative for some $n, m \in \mathbb{N}$ (then $\underline{p}(w) = -\frac{n}{m} = \underline{q}(w) < 0$).*

Proof. (i) Let $w \in \underline{V}(A, B, h)$. Then we have $-\infty < \underline{p}(w) \leq \underline{q}(w) < +\infty$ and, according to 4.1(i),(ii), there are $k_1, k_2, l_1, l_2, m_1, m_2 \in \mathbb{N}$ such that $(k_1 z, l_z + m_1 w) \in \varrho$ and $(m_2 w + l_2 z, k_2 z) \in \varrho$. Now, we have to distinguish the following eight cases:

(i1) Let $k_1 > l_1$ and $k_2 > l_2$. Since ϱ is cancellative, we get $(n_1 z, m_1 w) \in \varrho$ and $(m_2 w, n_2 z) \in \varrho$, where $n_1 = k_1 - l_1 \in \mathbb{N}$ and $n_2 = k_2 - l_2 \in \mathbb{N}$. Thus (1) is true.

(i2) Let $k_1 > l_2$ and $k_2 \leq l_2$. Then $(n_1 z, m_1 w) \in \varrho$ and $(m_2 w + n_2 z, z) \in \varrho$, where $n_1 = k_1 - l_1 \in \mathbb{N}$ and $n_2 = l_2 - k_2 + 1 \in \mathbb{N}$. Consequently, $(n_1 m_2 z, m_1 m_2 w) \in \varrho$, $(m_1 m_2 w + m_1 n_2 z, m_1 z) \in \varrho$, $((n_1 m_2 + m_1 n_2) z, m_1 m_2 w + m_1 n_2 z) \in \varrho$, $((n_1 m_2 + m_1 n_2) z, m_1 z) \in \varrho$ and $n_1 m_2 + m_1 n_2 \leq m_1$, since z is right ϱ-regular. But his is a contradiction.

(i3) Let $k_1 = l_1$ and $k_2 > l_2$. Then $(z, z + m_1 w) \in \varrho$ and $(m_2 w, n)2z) \in \varrho$, where $n_2 = k_2 - l + 2 \in \mathbb{N}$. Now, $m_1 w$ is almost ϱ-positive, $m_1 m_2 w$ is almost ϱ-positive, $n_2 m_1 z$ is almost ϱ-positive and, finally, $m_1 w + n_2 m_1 z$ is almost ϱ-positive. Thus (2) is true.

(i4) Let $k_1 = l_1$ and $k_2 = l + 2$. Then $(z, z + m_1 w) \in \varrho$, $(m)2w + z, z) \in \varrho$, $m_1 w$ is almost ϱ-positive and $m_2 w$ is almost ϱ-negative. Now, $m_1 m_2 w$ is both almost ϱ-positive and almost ϱ-negative and (4) is true.

(i5) Let $k_1 = l_1$ and $k_2 < l_2$. Then $(z, z + m_1 w) \in \varrho$ and $(m_2 w + n_2 z, z) \in \varrho$, where $n_2 = l_2 - k_2 + 1 \in \mathbb{N}$, $n_2 \geq 2$. Now, $m_1 w$ is almost ϱ-positive, $m_1 m_2 w$ is almost ϱ-positive, $(m_1 z, m_1 z + m_1 m_2 w) \in \varrho$, $(m_1 m_2 w + m_1 n_2 z, m_1 z) \in \varrho$, $(m_1 m_2 w + w + m + +1 n_2 z, m_1 m_2 w + m_1 z) \in \varrho$, $(m_1 n_2 z, m_1 z) \in \varrho$, $m_1 n_2 \leq m_1$ and and $n_2 \leq 1$ since z is right ϱ-regular, but this is a contradiction.

(i6) Let $k_1 < l_1$ and $k_2 > l_2$. Then $(z, k_1 z + m_1 w) \in \varrho$ and $(m_2 w, n_2 z) \in \varrho$, where $n_2 = k_2 - l_2 \in \mathbb{N}$. Put $k_3 = l_1 - k_1 + 1 \in \mathbb{N}$. Then $k_3 \geq 2$ and $n_1 z + m_1 w$ is almost ϱ-positive, where $n_1 = k_3 - 1 \in \mathbb{N}$. Thus (2) is true.

(i7) Let $k_1 < l_1$ and $k_2 = l_2$. Then $(z, k_3z+m_1w) \in \varrho$, where $k_3 = l_1-k_1+1 \in \mathbb{N}$, $k_3 \geq 2$, and $(z + m_2w, z) \in \varrho$. Now, $n_1z + m_1w$ is almost ϱ-positive, where $n_1 = k_3 - 1 \in \mathbb{N}$, and m_2w is almost ϱ-negative. Consequently, $n_1m_2z+m_1m_2w$ is almost ϱ-positive and m_1m_2w is almost ϱ-negative. It follows easily that n_1m_2z is almost ϱ-positive. Now, $(m_1m_2w + z, z) \in \varrho$, $(z, (n_1m_2+)z) \in \varrho$, and hence $(m_1m_2w, m_1m_2z) \in \varrho$. Thus (2) is true.

(i8) Let $k_1 < l_1$ and $k_2 < l_2$. Then $(z, k_3z + m_1w) \in \varrho$, where $k_3 = l_1 - k_1 + 1 \in \mathbb{N}$, $k_3 \geq 2$ and $(m_2w + k_4z, z) \in \varrho$, where $k_4 = l_2 - k_2 + 1 \in \mathbb{N}$, $k_4 \geq 2$. Now, $n_1z + m_1w$ is almost ϱ-positive and $m_2w + n_2z$ is almost ϱ-negative, where $n_1 = k_3 - 1 \in \mathbb{N}$ and $n_2 = k_4 - 1 \in \mathbb{N}$. Consequently, $n_1m_2z + m_1m_2w$ is almost ϱ-positive, $n_2m_1z + m_1m_2w$ is almost ϱ-negative, $(z, (n_1m_2 + 1)z + m_1m_2w) \in \varrho$, $((n_2m_1 + 1)z + m_1m_2w, z) \in \varrho$, $((n_2m_1 + 1)z, (n_1m_2 + 1)z) \in \varrho$, $(n_2m_1z, n_1m_2z) \in \varrho$ and $n_2m_1 \leq n_1m_2$ since z is right ϱ-regular.

If $n_2m_1 < n_1m_2$ then $(n_1m_2 - n_2m_1)z$ is almost ϱ-positive. On the other hand, $(n_1m_2 - n_2m_1)n_2m_1z + (n_1m_2 - n_2m_1)m_1m_2w$ is almost ϱ-negative and $(n_1m_2 - n_2m_1)$ n_2m_1z is almost ϱ-positive. Now, it follows easily that the element $(n_1m_2 - n_2m_1)$ m_1m_2w is almost ϱ-negative. Thus (3) is true.

Finally, if $n_2m_1 = n_1m_2$ then $mw + nz$ is both almost ϱ-positive and almost ϱ-negative, where $m = m_1m_2$ and $n = n_1m_2 = n_2m_1$. Thus (4) is true.

(ii) Let $w \in A$ satisfy at least one of the four conditions (1),...,(4). One checks easily that $-\infty < \underline{p}(w)$ and $\underline{q}(w) < +\infty$. □

5.7 Proposition. *Assume that no element from A is both almost ϱ-positive and almost ϱ-negative (equivalently, $0 \notin A/\ker(\varrho)$). Let $z \in A$ be right ϱ-regular, $B = \mathbb{N}z$ and $h(mz) = m$ for every $m \in \mathbb{N}$. Then $\underline{V}(A, B, h) = A$ if and only if every element $w \in A$ ($w \in A \setminus B$) satisfies at least one of the following two conditions:*

1. *$(n_1z, m_1w) \in \varrho$ and $(m_2w, n_2z) \in \varrho$ for some $n_1, n_2, m_1, m_2 \in \mathbb{N}$;*
2. *There are $n_1, n_2, m_1, m_2 \in \mathbb{N}$ such that $m_1w + n_1z$ is almost ϱ-positive and $(m_2w, n_2z) \in \varrho$.*

Proof. Use 5.6. □

5.8 Proposition. *Let $z \in A$ be right ϱ-regular and $B = \mathbb{N}z$. Assume that every element from $A \setminus B$ is almost ϱ-positive. Then $\underline{V}(A, B, h) = A$ if and only if every element $w \in A$ ($w \in A \setminus B$) satisfies at least one of the following two conditions:*

1. *$(mw, nz) \in \varrho$ for some $m, n \in \mathbb{N}$ (then nz is almost ϱ-positive);*
2. *mw is both almost ϱ-positive and almost ϱ-negative for some $m \in \mathbb{N}$ (then $mw \notin B$ and $(x, x + mw) \in \ker(\varrho)$ for every $x \in A$).*

Proof. Since every element from $A \setminus B$ is almost ϱ-positive, we have $\underline{p}(w) \geq 0$ for every $w \in A$. Now, $w \in \underline{V}$ if and only if $\underline{q}(w) < +\infty$, i.e., $(mw + lz, kz) \in \varrho$ for some $k, l, m \in \mathbb{N}$. Suppose that this is true. If $w \in B$ then (1) is true. If $w \notin B$ then w is almost ϱ-positive, and hence $(lz, mw + lz) \in \varrho$. Then $(lz, kz) \in \varrho$ and $l \leq k$, since z is right ϱ-regular. If $l < k$ then $(mw, nz) \in \varrho$, where $n + k - l \in \mathbb{N}$. If $k = l$ then mw is both almost ϱ-positive and almost ϱ-negative. The converse is obvious. □

5.9 Proposition. *Let $z \in A$ be right ϱ-regular and $B = \mathbb{N}z$. Assume that every element from $A \setminus B$ is almost ϱ-positive but not almost ϱ-negative. Then $\underline{V}(A, B, h) = A$ if and only if z is right ϱ-archimedean.*

Proof. If $\underline{V}(A, B, h) = A$ and $w \in A$ then $(mw, nz) \in \varrho$ for some $m, n \in \mathbb{N}$ by 5.8. If $w \notin B$ then w is almost ϱ-positive, $(w, mw) \in \varrho$ amd $(w, nz) \in \varrho$. If $w \in B$ then $w = kz$ for some $k \in \mathbb{N}$. The rest is obvious. □

6. The cancellative cover

Let ϱ be a stable preordering defined on a commutative semigroup A. Define a relation $\sigma = \underline{\mathrm{cn}}(\varrho)$ on A by $(a, b) \in \sigma$ if and only if $(a + c, b + c) \in \varrho$ for at least one $c \in A$.

6.1 Proposition. *σ is a stable and cancellative preordering. It is the smallest cancellative relation containing ϱ (the cancellative cover or envelope of ϱ).*

Proof. Since $(2a, 2a) \in \varrho$, we have $(a, a) \in \sigma$ and σ is reflexive. If $(a, b) \in \sigma$ and $(b, c) \in \sigma$ and $(a + c_1, b + c_1) \in \varrho$, $(b + c_2, c + c_2) \in \varrho$ for suitable $c_1, c_2 \in A$ and we get $(a + c_1 + c_2, b + c_1 + c_2) \in \varrho$, $(b + c_1 + c + 2, c + c + 1 + c + 2) \in \varrho$ and $(a + c_1 + c_2, c + c_1 + c_2) \in \varrho$. Thus $(a, c) \in \sigma$ and we see that $\sigma$is transitive. It means that σ is a preordering.

If $(a, c) \in \sigma$, $(a + c, b + c) \in \varrho$ and $d \in A$ then $(a + d + c, b + d + c) \in \varrho$ and $(a + d, b + d) \in \sigma$ and $(a + d, b + d) \in \sigma$. It follows that σ is stable.

If $(a + d, b + d) \in \sigma$ then $(a + d + c, b + d + c) \in \varrho$ for some $c \in A$, and hence $(a, b) \in \sigma$. It follows that σ is cancellative.

If $(a, b) \in \varrho$ then $(a + c, b + c) \in \varrho$ for every $c \in A$ and we have $(a.b) \in \sigma$. Thus $\varrho \subseteq \sigma$.

Finally, if λ is a cancellative relation defined on A such that $\varrho \subseteq \lambda$ and if $(a + c, b + c) \in \varrho$ then $(a, b) \in \lambda$. Consequently, $\sigma \subseteq \lambda$ and σ is just the smallest cancellative relation containing ϱ. □

6.2 Corollary. *$\varrho = \sigma$ if and only if ϱ is cancellative.* □

6.3 Lemma. $\ker(\sigma) = \underline{\mathrm{cn}}(\ker(\varrho))$ *is a cancellative congruence of the semigroup A.*

Proof. If $(a, b) \in$ ker(sigma) then $(a + c, b + c) \in \varrho$ and $(b + d, c + d) \in \varrho$ for some $c, d \in A$. Then $(a + c + d, b + c + d) \in \ker(\varrho)$ and $(a, b) \in \underline{\mathrm{cn}}(\ker(\varrho))$. The rest is clear. □

6.4 Proposition. $\underline{\mathrm{cn}}(\mathrm{id}_A)$ *is the smallest cancellative congruence of the semigroup A.*

Proof. It is obvious. □

6.5 Lemma. *σ is an ordering if and only if ϱ is and ordering and the semigroup A is cancellative.*

Proof. If σ is an ordering then $\underline{\mathrm{cn}}(\ker(\varrho)) = \mathrm{id_A}$ by 6.3. Then $\ker(\varrho = \mathrm{id_A}$, ϱ is an ordering, $\underline{\mathrm{cn}}(\mathrm{id_A}) = \mathrm{id_A}$ and A is cancellative. The converse implication is similar. □

6.6 Lemma. (i) *Every almost ϱ-positive (almost ϱ-negative, resp.) element is almost σ-positive (almost σ-negative, resp.).*
(ii) *Every right (left, resp.) ϱ-archimedean element is right (left, resp.) σ-archimedean.*
(iii) *Every right (left, resp.) σ-regular element is right (left, resp.) ϱ-regular.*

Proof. It is obvious. □

6.7 Remark. Notice that $A/\ker(\sigma)$ is not a torsion groups if and only if the following condition is satisfied:
(6.1) There is at least one element $w \in A$ such that for every $m \in \mathbb{N}$ there is $v_m \in A$ such that for every $u \in A$ we have either $(mw + v_m + u, v_m + u) \notin \varrho$ or $(v_m + u, mw + v_m + u) \notin \varrho$.

If ϱ is an ordering (i.e., $\ker(\varrho) = \mathrm{id_A}$) then (6.1) is equivalent to
(6.2) There is at least one element $w \in A$ such that for every $m \in \mathbb{N}$ there is $v_m \in A$ such that for every $u \in A$ we have $mw + v_m + u \neq v_m + u$.

6.8 Lemma. *An element $a \in A$ is right (left, resp.) σ-regular if and only if $m \leq n$ whenever $(m, n \in \mathbb{N}$ and $b \in A$ are such that $(ma+b, na+b) \in \varrho$ $((na+b, ma+b) \in \varrho$, resp.).*

Proof. It is obvious. □

6.9 Remark. Let B be a subsemigroup of A and $h : B \to \mathbb{R}$ be an additive homomorphism such that $h(a) \leq h(b)$ whenever $a, b \in B$, $v \in A$ and $(a + v, b + v) \in \varrho$. This means that $h(a) \leq h(b)$ whenever $a, b \in B$ and $(a, b) \in \sigma$. Now, we can make use of all the results from the foregoing four sections. In particular, when $B = \mathbb{N}z$, $z \in A$ being right σ-regular.

6.10 Theorem. *Let $z \in A$ be right $\underline{\mathrm{cn}}(\varrho)$-regular (*i.e. *$l \leq k$ whenever $k, l \in \mathbb{N}$ and $u \in A$ are such that $(lz+u, kz+u) \in \varrho$). Assume that every element $w \in A$ ($w \in A \setminus \mathbb{N}z$) satisfies at least one of the following three conditions:*

1. *There are $n_1, n_2, m_1, m_2 \in \mathbb{N}$ and $u, v \in A$ such that $(n_1z + u, m_1w + u) \in \varrho$ and $(m_2w + v, n_2z + v) \in \varrho$;*
2. *There are $n_1, n_2, m_1, m_2 \in \mathbb{N}$ and $u, v \in A$ such that $(z + u, m_1w + n_1z + u) \in \varrho$ and $(m_2w + v, n_2z + v) \in \varrho$;*
3. *$(z + u, mw + nz + u) \in \varrho$ and $(mw + nz + u, z + u) \in \varrho$ for some $n, m \in \mathbb{N}$ and $u \in A$.*

Then there is an additive homomorphism $f : A \to \mathbb{R}$ such that $f)z) = 1$ and $f(x) \leq f(y)$ for all $(x, y) \in \varrho$.

Proof. As we know, $\sigma = \underline{\mathrm{cn}}(\varrho)$ is a cancellative stable preordering and, since z is right σ-regular, we have $B = \mathbb{N}z \cong \mathbb{N}$ and $h : B \to \mathbb{R}$, where $h(mz) = m$ for every $m \in \mathbb{N}$, is an additive homomorphism such that $h(a) \le h(b)$ for all $a, b \in B$, $(a, b) \in \sigma$. In view of 3.9, we have to check that $\underline{V}(A, B, h) = A$ (where ϱ is replaced by σ). Of course, $B \subseteq \underline{V}$. Let $w \in A \setminus B$. If (1) is true then $((n_1 + 1)z, m_1 w) \in \sigma$, $\frac{n_1}{m_1} \le \underline{p}(w)$, $(m_2 w + z, (n_2 + 1)z) \in \sigma$, $\underline{q}(w) \le \frac{n_2}{m_2}$. If (2) is true then $(z, m_1 w + n_1 z) \in \sigma$, $\frac{1-n_1}{m_1} \le \underline{p}(w)$, $(m_2 w + z, (n_2 + 1)z) \in \sigma$, $\underline{q}(w) \le \frac{n_2}{m_2}$. If (3) is true then $(z, mw + nz) \in \sigma$, $\frac{1-n}{m} \le \underline{p}(w)$, $(mu + nz, z) \in \sigma$, $\underline{q}(w) \le \frac{1-n}{m}$. □

6.11 Theorem. *Let $z \in A$ be $\underline{\mathrm{cn}}(\varrho)$-regular (*i.e.*, $l \le k$ whenever $k, l \in \mathbb{N}$ and $u \in A$ are such that $(lz + u, kz + u) \in \varrho$). Assume that every element $w \in A$ ($w \in A \setminus \mathbb{N}z$) satisfies the following two conditions:*

(1) *$(mw + u, nz + u) \in \varrho$ for some $n, m \in \mathbb{N}$ and $u \in A$;*

(2) *For every $k \in \mathbb{N}$ there are $n_k, m_k \in \mathbb{N}$ and $u_k \in A$ such that $(z + u_k, m_k w + {} + n_k z + u_k) \in \varrho$ and $m_k \ge k(n_k - 1)$.*

There there is an additive homomorphism $f : A \to \mathbb{R}_0^+$ such that $f(z) = 1$ and $f(x) \le f(y)$ for all $(x, y) \in \varrho$.

Proof. By 6.10, there is an additive homomorphism$f : A \to \mathbb{R}$ such that $f(z) = 1$ and $f(x) \le f(y)$ for all $(x, y) \in \varrho$. Of course, $f(\mathbb{N}z) \subseteq \mathbb{R}^+$. On the other hand, if $(z + u, m_k w + n_k z + u_k) \in \varrho$ then $1 \le m_k f(w) + n_k$, and hence $-f(w) \le \frac{n_k - 1}{m_k} \le \frac{1}{k}$. Thus $-f(w) \le 0$ and $0 \le f(w)$. □

6.12 Theorem. *Let $z \in A$ be $\underline{\mathrm{cn}}(\varrho)$-regular (*i.e.*, $l \le k$ whenever $k, l \in \mathbb{N}$ and $u \in A$ are such that $(lz + u, kz + u) \in \varrho$). Assume that for every $w \in A$ ($w \in A \setminus \mathbb{N}z$) there are $n_1, n_2, m_1, m_2 \in \mathbb{N}$ and $uv, \in A$ such that $(n_1 z_u, m_1 w + u) \in \varrho$ and $(m_2 w + v, n_2 z + v) \in \varrho$. Then there is an additive homomorphism $f : A \to \mathbb{R}^+$ such that $f(z) = 1$ and $f(x) \le {} \le f(y)$ for all $(x, y) \in \varrho$.*

Proof. By 6.10, there is an additive homomorphism$f : A \to \mathbb{R}$ such that $f(z) = 1$ and $f(x) \le f(y)$ for all $(x, y) \in \varrho$. Of course, $f(\mathbb{N}z) \subseteq \mathbb{R}^+$. On the other hand, if $(n_1 z + u, m_1 w + u) \in \varrho$ then $n_1 \le m_1 f(w)$ and $0 < \frac{n_1}{m_1} \le f(w)$. Thus $f(A) \subseteq \mathbb{R}^+$. □

7. The cancellative factor

In this section, let ϱ be a stable and cancellative preordering defined on a commutative semigroup A. As we know, $\underline{\alpha}_A = \underline{\mathrm{cn}}(\mathrm{id_A})$ is just the smallest cancellative congruence of A; we have $\underline{\alpha}_A \subseteq \ker(\varrho)$ and $(a, b) \in \underline{\alpha}_A$ if and only if $a + c = b + c$ for at least one $c \in A$. Now, let $\varphi : A \to \overline{A} = A/\underline{\alpha}_A$ denote the natural projection. Then $\overline{A}$ is a cancellative semigroup and, for every $a \in A$, we put $\overline{a} = \varphi(a)$.

7.1 Lemma. *Let $a, b, c, d \in A$ be such that $(a, b) \in \varrho$, $\overline{a} = \overline{c}$ and $\overline{b} = \overline{d}$. Then $(c, d) \in \varrho$.*

Proof. We have $a+u=c+u$ and $b+v=d+v$ for some $u,v\in A$. Now, $a+w=c+w$ and $b+w=d+w$, where $w=u+v$, and $(c+w,d+w)=(a+w,b+w)\in\varrho$. Since ϱ is cancellative, we get $(c,d)\in\varrho$. □

In view of the preceding lemma, we see that ϱ induces a relation $\overline{\varrho}=\varphi(\varrho)=\varrho/\underline{\alpha}_A$ defined on $\overline{A}$ such that $(\overline{a},\overline{b})=(\varphi(a),\varphi(b))\in\overline{\varrho}$ for all $(a,b)\in\varrho$ (in fact, $(\overline{a},\overline{b})\in\overline{\varrho}$ if and only if $(a,b)\in\varrho$.

7.2 Lemma. *$\overline{\varrho}$ is a stable and cancellative preordering defined on the cancellative semigroup $\overline{A}$.*

Proof. It is easy. □

7.4 Lemma. *$\overline{\varrho}$ is an ordering if and only if* $\ker(\varrho)=\underline{\alpha}_A$ (i.e., *for every* $(a,b)\in$ $\in\ker(\varrho)$ *there is $c\in A$ with $a+c,b+c$).*

Proof. It is easy. □

7.5 Remark. Of course, if ϱ is an ordering then $\underline{\alpha}_A=\mathrm{id}_A$ and A is cancellative.

7.6 Lemma. (i) *If $a\in A$ is almost ϱ-positive (almost ϱ-negative, resp.) then $\overline{a}\in\overline{A}$ is almost $\overline{\varrho}$-positive (almost $\overline{\varrho}$-negative, resp.).*
(ii) *If $a\in A$ is right (left, resp.) ϱ-archimedean then $\overline{a}$ is right (left, resp.) $\overline{\varrho}$-archimedean.*
(iii) *If $a\in A$ is right (left, resp.) ϱ-regular then $\overline{a}\in\overline{A}$ is right (left, resp.) $\overline{\varrho}$-regular.*

Proof. It is obvious. □

7.7 Remark. Let B be a subsemigroup of A and $h:B\to\mathbb{R}$ be an additive homomorphism such that $h(a)\le h(b)$ for all $a,b\in B$, $(a,b)\in\varrho$. Assume, furthermore, that $h(a_1)=h(b_1)$ whenever $a_1,b_1\in B$ and $u\in A$ are such that $a_1+u=b_1+u$ (i.e., $(a_1,b_1)\in\underline{\alpha}_A$). Then h induces an additive homomorphism $\overline{h}:\overline{A}\to\mathbb{R}$ such that $\overline{h}(\overline{a})=h(a)$ for every $a\in A$ and $\overline{h}(\overline{(a_2)}\le\overline{h}(\overline{b_2}$ for all $\overline{a_2},\overline{b_2}\in\overline{B}$, $(\overline{a_2},\overline{b_2})\in\overline{\varrho}$.

8. The antisymmetric factor

Let ϱ be a stable preordering defined on a commutative semigroup A. Then $\ker(\varrho)$ is a congruence of A and we put $\widetilde{A}=A/\ker(\varrho)$. Let $\psi:A\to\widetilde{A}$ be the natural projection. Now, ϱ induces a relation $\tau=\widetilde{\varrho}=\psi(\varrho)=\varrho/\ker(\varrho)$ on $\widetilde{A}$, where $(\widetilde{a},\widetilde{b})\in\widetilde{\varrho}$ if and only if $(a,b)\in\varrho$.

8.1 Proposition. *τ is a stable ordering defined on the factorsemigroup $\widetilde{A}$.*

Proof. It is easy. □

8.2 Lemma. *τ is cancellative if and only if ϱ is such (then* $\ker(\varrho)$ *is cancellative and $\widetilde{A}$ is a cancellative semigroup).*

Proof. It is easy. □

8.3 Lemma. (i) *If $a \in A$ is almost ϱ-positive (almost ϱ-negative, resp.) then $\widetilde{a} \in \widetilde{A}$ is almost τ-positive (almost τ-negative, resp.)*
(ii) *If $a \in A$ is right (left, resp.) ϱ-archimedean then $\widetilde{a} \in \widetilde{A}$ is right (left, resp.) τ-archimedean.*
(iii) *If $a \in A$ is right (left, resp.) ϱ-regular then $\widetilde{a} \in \widetilde{A}$ is right (left, resp.) τ-regular.*

Proof. It is obvious. □

8.5 Remark. Let B be a subsemigroup of A and let $h : B \to \mathbb{R}$ be an additive homomorphism such that $h(a) \leq h(b)$ for all $a, b \in B$ with $(a, b) \in \varrho$. If $a_1, b_1 \in B$ are such that $(a_1, b_1) \in \ker(\varrho)$ then $h(a_1) = h(b_1)$, and so $\ker(\varrho)|\mathrm{B} \subseteq \ker(\mathrm{h})$. Then, of course, h induces an additive homomorphism $\widetilde{h} : \widetilde{B} \to \mathbb{R}$ such that $\widetilde{h(\widetilde{a})} \leq \widetilde{(b)}$ for all $\widetilde{a}, \widetilde{b} \in \widetilde{B}$ with $(\widetilde{a}, \widetilde{b}) \in \tau$. We have $h = \widetilde{h}\psi$.

8.6 Assume that ϱ is cancellative and put $\sigma = (\varrho \setminus \ker(\varrho)) \cup \mathrm{id}_{\mathrm{A}}$ (see 1.1). Then σ is an ordering and $\sigma \subseteq \varrho$. If $(a, b) \in \sigma$ and $a \neq b$ then $(a, b) \in \varrho$ and $(b, a) \notin \varrho$.Now, $(a+c, b+c) \in \varrho$ and $(b+c, a+c) \notin \varrho$ for every $c \in A$, since ϱ is stable and cancellative. It means that σ is a stable ordering. Similarly, if $(a + c, b + c) \in \varrho$ and $a + c \neq b + c$ then $(b + c, a + c) \notin \varrho$, $(b, a \notin \varrho$ and $(a, b) \in \sigma$. Thus σ is cancellative, provided that the semigroup A is cancellative.

Let $a \in A$ be almost ϱ-positive. If a is not almost σ-positive then $(u, a + u) \notin \sigma$ for some $u \in A$ and we have $a + u \neq u$, $(a + u, u) \in \varrho$ and $(u, a + u) \in \ker(\varrho)$. Since ϱ is cancellative, we see that a is almost ϱ-negative as well. Thus $a/\ker(\varrho) = 0_{\mathrm{A}/\ker(\varrho)}$.

Let $a \in A$ be right ϱ-archimedean. If a is not right σ-archimedean then there is $u \in A$ such that $(u, ma) \notin \sigma$ for every $m \in \mathbb{N}$. It means that $u \neq ma$ and either $(u, ma) \notin \varrho$ or $(u, ma) \in \ker(\varrho)$. Since a is right ϱ-archimedean, there is $n \in \mathbb{N}$ such that $(u, na) \in \varrho$. Consequently, $u \neq na)$ and $(u, na) \in \ker(\varrho)$. Now, assume that a is almost ϱ-positive. Then $(na, 2na) \in \varrho$, $(u, 2na) \in \varrho$, $(u, 2na) \in \ker(\varrho)$, $(na, 2na) \in \ker(\varrho)$ and $na/\ker(\varrho) = 0_{\mathrm{A}/\ker(\varrho)}$. If $a/\ker(\varrho) = 0_{\mathrm{A}/\ker(\varrho)}$ then a is almost ϱ-negative.

9. The unperforated cover

As always, let ϱ be a stable preordering defined on a commutative semigroup A. The preordering ϱ is called *unperforated* if $(a, b) \in \varrho$ whenever $a, b \in A$ and $m \in \mathbb{N}$ are such that $(ma, mb) \in \varrho$.

9.1 Lemma. *If ϱ is unperforated then the factor-semigroup $A/\ker(\varrho)$ is torsionfree and $\ker(\varrho)$ is unperforated.*

Proof. If $ma/\ker(\varrho) = \mathrm{mb}/\ker(\varrho)$ for some $a, b \in A$ and $m \in \mathbb{N}$ then $(ma, mb) \in$ $\in \ker(\varrho)$. Since ϱ is unperforated, we have $(a, b) \in \ker(\varrho)$ and $a/\ker(\varrho) = \mathrm{b}/\ker(\varrho)$. □

9.2 Lemma. (cf. 1.8 and 1.23) *Assume that ϱ is unperforated. If $a \in A$ and $m \in \mathbb{N}$ are such that ma is almost ϱ-positive (almost ϱ-negative, resp.) then a is almost ϱ-positive (almost ϱ-negative, resp.).*

Proof. We have $(mx, mx + ma) \in \varrho$ for every $x \in A$. Since ϱ is unperforated, it follows that $x, x + a) \in \varrho$. Thus a is almost ϱ-positive. □

Now, define a relation $\tau = \underline{\mathrm{up}}(\varrho)$ on A by $(a, b) \in \tau$ if and only if $(ma, mb) \in \varrho$ for some $m \in \mathbb{N}$.

9.3 Lemma. (i) *τ is a stable preordering.*
(ii) *$\varrho \subseteq \tau$ and τ is unperforated.*
(iii) *τ is just the smallest unperforated relation containing ϱ* (the *unperforated cover* of ϱ).

Proof. It is easy. □

9.4 Lemma. (i) $\ker(\tau) = \underline{\mathrm{up}}(\ker(\varrho)$.
(ii) *τ is an ordering if and only if ϱ is and ordering and the semigroup A is torsionfree.*

Proof. It is easy. □

9.5 Lemma. *If ϱ is cancellative then τ is cancellative.*

Proof. It is easy. □

9.6 Lemma. (i) $\lambda = \underline{\mathrm{cn}}(\underline{\mathrm{up}}(\varrho)) = \underline{\mathrm{up}}(\underline{\mathrm{cn}}(\varrho))$ *is a stable cancellative unperforated preordering.*
(ii) *λ is just the smallest cancellative unperforated relation containing ϱ.*

Proof. First, let $(a, b) \in \underline{\mathrm{cn}}(\underline{\mathrm{up}}(\varrho))$. Then $(a + c, b + c) \in \underline{\mathrm{up}}(\rho)$ for some $c \in A$ and there is $m \in \mathbb{N}$ with $(ma + mc, mb + mc) \in \varrho$. Consequently, $(ma, mb) \in \underline{\mathrm{cn}}(\varrho)$ and $(a, b) \in \underline{\mathrm{up}}(\underline{\mathrm{cn}}(\varrho))$. Thus $\underline{\mathrm{cn}}(\underline{\mathrm{up}}(\varrho)) \subseteq \underline{\mathrm{up}}(\underline{\mathrm{cn}}(\varrho))$.

Conversely, let $(a, b) \in \underline{\mathrm{up}}(\underline{\mathrm{cn}}(\varrho))$. Then there is $n \in \mathbb{N}$ with $(na, nb) \in \underline{\mathrm{cn}}(\varrho)$ and $(na + d, nb + d) \in \varrho$ for some $d \in A$. Consequently, $(na + nd, nb + nd) \in \varrho$, $(a + d, b + d) \in \underline{\mathrm{up}}(\varrho)$ and $(a, b) \in \underline{\mathrm{cn}}(\underline{\mathrm{up}}(\varrho))$. Thus $\underline{\mathrm{up}}(\underline{\mathrm{cn}}(\varrho)) \subseteq \underline{\mathrm{cn}}(\underline{\mathrm{up}}(\varrho))$. □

9.7 Lemma. (i) $\ker(\lambda) = \underline{\mathrm{cn}}(\underline{\mathrm{up}}(\ker(\varrho))) = \underline{\mathrm{up}}(\underline{\mathrm{cn}}(\ker(\varrho)))$.
(ii) *λ is an ordering if and only if ϱ is an ordering and A is a cancellative torsionfree semigroup.*

Proof. Use 9.6(i). □

Put $\underline{\beta}_A = \underline{\mathrm{up}}(\mathrm{id}_A)$. As we know, $\underline{\beta}_A$ is the smallest congruence of A such that the corresponding factor-semigroup is torsionfree; we have $(a, b) \in \underline{\beta}_A$ if and only if $ma = mb$ for some $m \in \mathbb{N}$. Clearly, $\underline{\beta}_A = A \times A$ if and only if A is torsion.

Put $\underline{\gamma}_A = \underline{\mathrm{cn}}(\underline{\mathrm{up}}(\mathrm{id}_A))$ $(= \underline{\mathrm{up}}(\underline{\mathrm{cn}}(\mathrm{id}_A)))$. As we know, $\underline{\gamma}_A$ is the smallest congruence of A such that the corresponding factor-semigroup is cancellatine and torsionfree; we have $(a, b) \in \underline{\gamma}_A$ if and only if $ma + c = mb + c$ for some $m \in \mathbb{N}$ and $c \in A$.

9.8 Remark. Let B be a subsemigroup of A and let $h : B \to \mathbb{R}$ be an additive homomorphism such that $h(a) \le h(b)$ for all $a, b \in B$ with $(a, b) \in \varrho$. If $a_1, b_1 \in B$ are such that $(a_1, b_1) \in \underline{\text{up}}(\varrho))$ then $(ma_1, mb_1) \in \rho$ for some $m \in \mathbb{N}$, $mh(a_1) \le mh(b_1)$ and $h(a_1) \le h(b_1)$.

Now, assume that $h(a_2) \le h(b_2)$ for all $a_2, b_2 \in B$ such that $(a_2, b_2) \in \underline{\text{cn}}(\varrho)$ (cf. 6.9). Then $h(a_3) \le h(b_3)$ for all $a_3, b_3 \in B$ with $(a_3, b_3) \in \underline{\text{up}}(\underline{\text{cn}}(\varrho))$.

9.9 Remark. Assume that ϱ is unperforated (unperforated and cancellative, resp.) Then ϱ induces and unperforated preordering $\varrho/\underline{\beta}_A$ ($\varrho/\underline{\gamma}_A$, resp.) on the torsionfree (torsionfree and cancellative, resp.) semigroup $A/\underline{\beta}_A$ ($A/\underline{\gamma}_A$, resp.).

9.10 Remark. Assume that ϱ is unperforated (unperforated and cancellative, resp.) (see 9.9). Let $h : B \to \mathbb{R}$ be an additive homomorphism such that $h(a_1) = h(b_1)$ whenever $a_1, b_1 \in B$ are such that $(a_1, b_1) \in \underline{\alpha}_A$. Then h induces an additive homomorphism $h/\underline{\beta}_A : B/\underline{\beta}_A \to \mathbb{R}$ ($h/\underline{\gamma}_A : B/\underline{\gamma}_A \to \mathbb{R}$, resp.) and this induced homomorphism preserves the induced preordering (see 9.9). In this situation, notice that $\underline{\beta}_B = \underline{\beta}_A | B \times B$.

10. Homomorphisms into $\mathbb{R}$

In 10.1 – 10.7, let ϱ be a stable preordering defined on a commutative semigroup A and let $f : A \to \mathbb{R}$ be an additive homomorphism such that $f(a) \le f(b)$ for all $(a, b) \in \varrho$.

10.1 Lemma. $\ker(\varrho) \cup \underline{\alpha}_A \cup \underline{\beta}_A \subseteq \ker(\varrho) \cup \underline{\gamma}_A \subseteq \ker(\underline{\text{cn}}(\underline{\text{un}}(\varrho))) \subseteq \ker(\varrho)$ *and* $A/\ker(\varrho) \cong \mathrm{f(A)}$ *is a cancellative torsionfree semigroup.*

Proof. If $(a, b) \in \ker(\underline{\text{cn}}(\underline{\text{un}}(\varrho)))$ then $ma + c = mb + c$ for some $m \in \mathbb{N}$ and $c \in A$. It follows immediately that $f(a) = f(b)$. The rest is clear. □

10.2 Lemma. *If* $(a, b) \in \underline{\text{cn}}(\underline{\text{un}}(\varrho))$ *then* $f(a) \le f(b)$.

Proof. It is easy. □

10.3 Lemma. *If* $a \in A$ *is almost* ϱ*-positive (almost* ϱ*-negative, resp.) then* $f(a) \ge 0$ ($f(a) \le 0$*, resp.).*

Proof. We have $(a, 2a) \in \varrho$, and so $0 \le f(a)$. □

10.4 Lemma. *Let* $a \in A$ *be right (left, resp.)* ϱ*-archimedean.*
(i) *If* $f(u) > 0$ ($f(u) < 0$*, resp.) for at least one* $u \in A$ *then* $f(a) > 0$ ($f(a) < 0$*, resp.).*
(ii) *If* $f(v) \ge 0$ ($f(v) \le 0$*, resp.) for at least one* $v \in A$ *then* $f(a) \ge 0$ ($f(a) \le 0$*, resp.).*
(iii) *If* $f(a) \in \mathbb{R}^-$ ($f(a) \in \mathbb{R}^+$*, resp.) then* $f(a)$ *is the greatest (the smallest, resp.) number in* $f(A)$.

Proof. For every $w \in A$ there is $m \in \mathbb{N}$ with $\frac{f(w)}{m} \le f(a)$. The rest is clear. □

10.5 Lemma. *Let* $a \in A$ *be such that* $f(a) > 0$ ($f(a) < 0$*, resp.). Then* a *is right (left, resp.)* ϱ*-regular.*

Proof. It is easy. □

10.6 Define a relation μ on A by $(a,b) \in \mu$ if and only if $f(a) \le f(b)$. Then $\varrho \subseteq \underline{\text{cn}}(\underline{\text{un}}(\varrho)) \subseteq \mu$ and μ is a stable, cancellative and unperforated preordering defined on the semigroup A. Clearly, $\ker(\mu) = \ker(\text{f})$, and hence μ is an ordering if and only if the homomorphism f is injective.

An element $a \in A$ is almost μ-positive (almost μ-negative, resp.) if and only if $f(a) \ge 0$ ($f(a) \le 0$, resp.).

If $f(u) > 0$ ($f(u) < 0$, resp.) for at least one $u \in A$ then an element $a \in A$ is right (left, resp.) μ-archimedean if and only if $f(a) > 0$ ($f(a) < 0$, resp.).

If $f(A) \le 0$ ($0 \le f(A)$, resp.) and $f(v) = 0$ for at least one $v \in A$ then an element $a \in A$ is right (left, resp.) μ-archimedean if and only if $f(a) = 0$.

If $f(A) < 0$ ($0 < f(A)$, resp.) then an element $a \in A$ is right (left, resp.) μ-archimedean if and only if $f(a)$ is the greatest (the smallest, resp.) number in $F(A)$.

If $f(a) > 0$ ($f(a) < 0$, resp.) then a is right (left, resp.) μ-regular. In fact, we have $(ma, na) \in \mu$ for all $m, n \in \mathbb{N}$ such that $m \le n$ ($n \le m$, resp.). If $f(a) = 0$ then a is neither right nor left μ-regular.

Finally, notice that $\mu = \text{id}_\text{A}$ if and only if $|A| = 1$ and that $\mu = A \times A$ if and only if $f = 0$.

10.7 Define a relation ν on A by $(a,b) \in \nu$ if and only if either $(a,b) \in \ker(\varrho)$ or $f(a) < f(b)$. Then ν is a stable preordering on A and $\nu \subseteq \mu$ (see 10.6). Clearly, $\ker(\nu) = \ker(\varrho)$, and hence ν is an ordering if and only if ϱ is so. If $\ker(\varrho)$ is cancellative then ν is cancellative. If $\ker(\varrho)$ is unperforated then ν is unperforated. If $(a,b) \in \nu$ then $f(a) \le f(b)$.

If $a \in A$ is such that $f(a) > 0$ ($f(a) < 0$, resp.) then a is almost ν-positive (almost ν-negative, resp.), right (left, resp.) ν-archimedean and right (left, resp.) ν-regular.

Finally, notice that $\nu = \text{id}_\text{A}$ if and only if ϱ is and ordering and $f = 0$, and that $\nu = A \times A$ if and only if $\varrho = A \times A$ (and then $f = 0$).

10.8 Let $f : A \to \mathbb{R}$ be a non-zero additive homomorphism. If $z \in A$ is such that $r = f(z) \ne 0$ then the mapping $g = r^{-1} f$ is again an additive homomorphism from A to $\mathbb{R}$. Of course, we have $g(z) = 1$.

Define a relation ν on A by $(a,b) \in \nu$ if and only if $f(a) < f(b)$ or $a = b$ (see 10.7). Then ν is a stable ordering on the semigroup A. If A is cancellative then ν is so (in fact, $(a+c, b+c) \in \nu \setminus \text{id}_\text{A}$ always implies $(a,b) \in \nu \setminus \text{id}_\text{A}$). If A is torsionfree then ν is unperforated (in fact, $(ma, mb) \in \nu \setminus \text{id}_\text{A}$ always implies $(a,b) \in \nu \setminus \text{id}_\text{A}$).

Put $\nu_1 = \underline{\text{cn}}(\nu)$, $\nu_2 = \underline{\text{un}}(\nu)$ and $\nu_3 = \underline{\text{cn}}(\underline{\text{un}}(\nu))$. Now, $(a,b) \in \nu_1$ iff either $(a,b) \in \nu$ or $a + c = b + c$ for some $c \in A$. Thus $\nu_1 = \nu \cup \underline{\alpha}_A$. Similarly, $\nu_2 = \nu \cup \underline{\beta}_A$ and $\nu_3 = \nu \cup \underline{\gamma}_A$.

Now, choose $z \in A$ with $f(z) > 0$. Then z is almost ν-positive, right ν-archimedean and right ν-regular (in fact, $(mz, nz) \in \nu$ iff $m \le n$). Moreover, z is right ν_i-regular for $i = 1, 2, 3$ and for every $w \in A$ there are $n_1, n_2 \in \mathbb{N}$ such that $w + n_1 z$ is almost ν-positive and $(w, n_2 z) \in \nu$ (see 5.6).

10.9 Theorem. *The following conditions are equivalent for a commutative semigroup A:*

(i) *There is at least one non-zero additive homomorphism $f : A \to \mathbb{R}$.*

(ii) *There is at least one additive homomorphism $f : A \to \mathbb{R}$ such that $1 \in f(A)$.*

(iii) *There is a stable ordering $\leq$ on A such that the following conditions are true:*

(iii1) *If $a, b, c \in A$ are such that $a + c \leq b + c$ then either $a \leq b$ or $a + c = b + c$;*

(iii2) *If $a, b \in A$ and $m \in \mathbb{N}$ are such that $ma \leq mb$ then either $a \leq b$ or $ma = mb$;*

(iii3) *There is at least one right $\leq$-archimedean and almost $\leq$-positive element $z \in A$ such that $m \leq n$ whenever $mz + u \leq nz + u$, $m, n \in \mathbb{N}$, $u \in A$, and for every $w \in A$ there is at least one $k \in \mathbb{N}$ with $w = kz$ being almost $\leq$-positive (we can also assume that $m_1 z \leq n_1 z$ for all $m_1, n_1 \in \mathbb{N}$, $m_1 \leq n_1$).*

(iv) *There is a stable preordering ϱ on A such that at least one element $z \in A$ satisfies the following conditions:*

(iv1) *$l \leq k$ whenever $k, l \in \mathbb{N}$ and $u \in A$ are such that $(lz + u, kz + u) \in \varrho$;*

(iv2) *For every $w \in A \setminus \mathbb{N}z$ there are $m_1, m_2, n_1, n_2 \in \mathbb{N}$ and $u \in A$ such that either $(n_1 z + u, m_1 w + u) \in \varrho$ and $(m_w + u, n_z + u) \in \varrho$, or $(z + u, m_1 w + n_1 z + u) \in \varrho$ and $(n_2 w + u, n_2 z + u) \in \varrho$, or $(z + u, m_1 w + n_1 z + u) \in \varrho$ and $(m_1 w + n_1 z + u, z + u) \in \varrho$.*

Proof. (i) implies (ii). See 10.2.
(ii) implies (iii). See 10.3.
(iii) implies (iv). This is clear.
(iv) implis (i). See 6.10. □

10.10 Remark. (i) Let A be a non-trivial cancellative and torsionfree commutative semigroup. The group $G = A - A$ of differences is torsionfree, and hence for every $0 \neq u \in G$ there is an additive homomorphism $g : G \to \mathbb{Q}$ such that $g(u) = 1$. In particular, for every $a \in A$, $a \neq 0_A$, there is an additive homomorphism $f : A \to \mathbb{Q}$ with $f(a) = 1$.
(ii) Let A be a commutative semigroup. If $\underline{\gamma}_A = A \times A$ (i.e., no non-trivial homomorphic image of A is a cancellative and torsionfree semigroup) then there is no non-zero additive homomorphism of A into $\mathbb{R}$. On the other hand, if $\underline{\gamma}_A \neq A \times A$ then $\overline{A} = A/\underline{\gamma}_A$ is a non-trivial cancellative and toesionfree semigroup and it follows from (i) that there are non-zero additive homomorphisms of A into $\mathbb{R}$. In fact, if $a \in A$ is such that $(a, 2a) \notin \underline{\gamma}_A$ (i.e., $ma + u \neq 2ma + u$ for all $m \in \mathbb{N}$ and $u \in A$) then there is an additive homomorphism $f : A \to \mathbb{Q}$ with $f(a) = 1$.
(iii) Let A be a commutative semigroup and $f : A \to \mathbb{Q}$ be an additive homomorphism such that $f(A) \cap \mathbb{Q}^- \neq \emptyset \neq f(A) \cap \mathbb{Q}^+$. Then $A/\mathrm{ker(f)} \cong \mathrm{f(A)}$ is a non-zero torsionfree group.
(iv) Let A be a commutative semigroup such that $\underline{\gamma}_A \neq A \times A$ and no non-trivial homomorphic image of A is a torsionfree group. Then there is at least one non-zero additive

homomorphism $f : A \to \mathbb{Q}_0^+$. Of course, $\underline{\gamma}_A \subseteq \ker(f) \neq A \times A$ and $A/\ker(f) \cong f(A)$ is a cancellative torsionfree semigroup.

(v) Let A be an additive subsemigroup of $\mathbb{Q}$ and let r be a cancellative congruence of A, $r \neq \mathrm{id}_A$. We claim that A/r is a torsion group.

If $A = \{0\}$ then $r = A \times A = \mathrm{id}_A$, a contradiction. If $A \subseteq \mathbb{Q}_0^-$ then $-A \subseteq \mathbb{Q}_0^+$ and $-A$ is an isomorphic copy of A. Thus we can assume that $A \cap \mathbb{Q}^+ \neq \emptyset$. Since $r \neq \mathrm{id}_A$ and $A \cap \mathbb{Q}^+ \neq \emptyset$, there are $p, q \in A \cap \mathbb{Q}^+$ such that $(p, q) \in r$ and $p < q$. We have $p = \frac{m}{n}$, $q = \frac{k}{l}$, $m, n, k, l \in \mathbb{N}$, $ml < nk$, $t = nk - ml \in \mathbb{N}$ and $nkp/r = mlp/r$ in A/r. Since A/r is a cancellative semigroup, we get $tp/r = tq/r = 0_{A/r}$. Now, given $s \in A$, there is $m_1 \in \mathbb{N}$ with $0 < m_1 p + s$. Of course, $(m_1 p + s, m_1 q + s) \in r$, $m_1 p + s < m_1 q + s$ and there is $t_1 \in \mathbb{N}$ such that $t_1(m_1 p + s)/r = 0_{A/r}$. Thus $0_{A/r} = tt_1(m_1 p + s)/r = tt_1 s/r$ and we see that A/r is a torsion group.

(vi) Let A be an additive subsemigroup of $\mathbb{Z}$ and let r be a congruence of A, $r \neq \mathrm{id}_A$. We claim that A/r is a finite semigroup.

We can assume that $A \subseteq \mathbb{N}$. The semigroup A is finitely generated, and so the same is true for the factor-semigroup A/r. Now, it is enough to prove that every one-generated subsemigroup of A/r is finite. For, let $m \in A$ and $B = \mathbb{N}m$. Since $r \neq \mathrm{id}_A$, we get $s = R|B \times B \neq \mathrm{id}_B$. But $B \cong \mathbb{N}$ and the rest is clear.

(vii) Let A be an additive subsemigroup of $\mathbb{Q}$ and let r be a congruence of A, $r \neq \mathrm{id}_A$. We claim that the factor-semigroup A/r is locally finite (i.e., every finitely generated subsemigroup of A/r is finite).

First, if $A \cap \mathbb{Q}^- \neq \emptyset \neq A \cap \mathbb{Q}^+$ then A is a subgroup of $\mathbb{Q}$ and A/r is a torsion group (see (v)). If $A \subseteq \mathbb{Q}_0^-$ then $-A \subseteq \mathbb{Q}_0^+$ and $-A \cong A$. Consequently, we can assume that $A \subseteq \mathbb{Q}_0^+$. We have $A \neq \{0\}$ and we put $B = A \cap \mathbb{Q}^+$ and $s = r|B \times B$. Clearly, $s \neq \mathrm{id}_B$. Let C be a finitely generated subsemigroup of B. We can assume that $t = s|C \times C \neq \mathrm{id}_C$ (if $(p, q) \in s$, $p \neq q$ then $C + \mathbb{N}_0 p + \mathbb{N}_0 q$ is again finitely generated). Since C is finitely generated, $mC \subseteq \mathbb{N}$ for some $m \in \mathbb{N}$. Now, $C \cong mC$ and we use (vi) to show that C/t is finite.

(viii) Let A be an additive subsemigroup of $\mathbb{Q}$. Let r be a congruence of A. If $A \cap \mathbb{Q}^+ \neq$ $\neq \emptyset \neq A \cap \mathbb{Q}^-$ then A is a subgroup of $\mathbb{Q}$, and hence the factor-semigroup A/r has just one idempotent element, namely the zero element. If $A \subseteq \mathbb{Q}_0^-$ then for all $a, b \in A$ there are $m, n \in \mathbb{N}$ with $ma = nb$, and hence the factor-semigroup A/r has at most one idempotent element (just one if $r \neq \mathrm{id}_A$). Assume, finally, that $A \subseteq \mathbb{Q}_0^+$ and $0 \in A$. If $A = \{0\}$ or if $r = \mathrm{id}_A$ then A/r has just one idempotent element, namely $0_{A/r}$. If $(0, a) \in r$ for some $a \in A$, $a > 0$ then A/r is a torsion group. If $A \neq \{0\}$, $r \neq \mathrm{id}_A$ and $(0, b) \notin r$ for every $b \in B = A \setminus \{0\}$ then $r|B \times B \neq \mathrm{id}_B$ and the factor-semigroup A/r has just two idempotent elements.

10.11 Proposition. *The following conditions are equivalent for a commutative semigroup A:*

(i) *There is at least one non-zero additive homomorphism $f : A \to \mathbb{Q}$.*
(ii) *There is at least one non-zero additive homomorphism $g : A \to \mathbb{R}$.*

(iii) *There is at least one element* $w \in A$ *such that* $mw + a \neq 2mw + a$ *for all* $m \in \mathbb{N}$ *and* $a \in A$ *(then* f *from (i) can be chosen such that* $f(w) = 1$*).*

Proof. (i) implies (ii). This implication is trivial.
(ii) implies (iii). Just choose any $w \in A$ with $g(w) \neq 0$.
(iii) implies (i). See 10.10(ii). □

10.12 Remark. Consider the situation from 10.11. If A is cancellative then 10.11(iii) means that $mw \neq 0_A$ for every $m \in \mathbb{N}$. Thus a cancellative semigroup A satisfies the equivalent conditions of 10.11 if and only if A is not a torsion group. A (possibly non-cancellative) semigroup A satisfies the conditions of 10.11 if and only if $A/\underline{\alpha}_A$ is not a torsion group. Notice that if $A/\underline{\alpha}_A$ is finite then it is a torsion group. On the other hand, if A is finitely generated and $A/\underline{\alpha}_A$ is a torsion group then $A/\underline{\alpha}_A$ is finite. Consequently, a finitely generated commutative semigroup A satisfies the equivalent conditions of 10.11 if and only if the factor-semigroup $A/\underline{\alpha}_A$ is not finite.

10.13 Proposition. *The following conditions are equivalent for a commutative semigroup A:*

(i) *A is isomorphic to an additive subsemigroup of* $\mathbb{Q}^+$.

(ii) *A is cancellative, torsionfree, uniform (i.e., for all* $a, b \in A$ *there are* $m, n \in \mathbb{N}$ *with* $ma = nb$*; it means that the intersection of any two or finitely many subsemigroups of A is non-empty) and* $0_A \notin A$ *(equivalently, A has no idempotent element).*

(iii) *A is cancellative, torsionfree,* $0_A \notin A$ *and if r is a congruence of A such that* $r \neq \mathrm{id_A}$ *then* A/r *is locally finite.*

(iv) *A is cancellative, torsionfree,* $0_A \notin A$ *and if r is a cancellative congruence of A such* $\mathrm{id_A} \neq \mathrm{r} \neq \mathrm{A} \times \mathrm{A}$ *then* A/r *is not torsionfree (*A/r *is a torsion group).*

Proof. (i) implies (ii). This is easy.
(ii) implies (i). The group $G = A - A$ of differences is a non-trivial torsionfree group. If $a_1, a_2 \in A$ are such that $a_1 \neq a_2$ and $b \in A$ is arbitrary then $ma_1 = n_1 b$ and $ma_2 = n_2 b$ for some $m, n_1, n_2 \in \mathbb{N}$. Now, $m(a_1 - a_2) = (n_1 - n_2)b$ and $n_1 - n_2 \neq 0$, since $a_1 \neq a_2$. It follows that every non-zero subgroup H of G contains a subsemigroup $B_H \subseteq H \cap A$. Since A is uniform and $0_A \notin A$, we conclude that G is a torsionfree group of rank 1, and G is isomorphic to an additive subgroup of $\mathbb{Q}$. The rest is clear,
(i) implies (iii). See 10.10(vii).
(iii) implies (i). By 10.10(i), there is at least one non-zero additive homomorphism $f : A \to \mathbb{Q}$. Clearly, $\ker(\mathrm{f}) = \mathrm{id_A}$, and hence A is isomorphic to a subsemigroup of $\mathbb{Q}$. Since $0_A \notin A$, A is isomorphic to a subsemigroup of $\mathbb{Q}^+$.
(i) implies (iv). See 10.10(v).
(iv) implies (i). Use 10.10(i). □

10.14 Remark. Using 10.13, we can formulate various characterizations of additive subsemigroups of $\mathbb{Q}^+$ and of $\mathbb{Q}$. Furthermore, taking into account that subsemigroups of $\mathbb{Z}$ are finitely generated, we can obtain characterizations of additive subsemigroups of $\mathbb{Z}$, $\mathbb{N}_0$ and $\mathbb{N}$.

The additive group of real numbers is divisible of rank 2^ω. Consequently, a commutative semigroup A is isomorphic to a subsemigroup of $\mathbb{R}$ if and only if A is cancellative, torsionfree and $|A| \le 2^\omega$.

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