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WEAKER CONVERGENCE CONDITIONS FOR THE SECANT METHOD

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Abstract. We use tighter majorizing sequences than in earlier studies to provide a semilocal convergence analysis for the secant method. Our sufficient convergence conditions are also weaker. Numerical examples are provided where earlier conditions do not hold but for which the new conditions are satisfied.

Keywords: semilocal convergence; secant method; Banach space; majorizing sequence; Hölder condition; divided difference; Fréchet-derivative

MSC 2010: 65H10, 65B05, 65G99, 65N30, 49M15

1. Introduction

Let $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces and $\mathcal{D}$ be a convex subset in $\mathcal{X}$. Let $U(w,R)$ and $\overline{U}(w,R)$ stand for the open and closed ball in $\mathcal{X}$, respectively, with center w and radius $R > 0$. Denote by $\mathcal{L}(\mathcal{X},\mathcal{Y})$ the space of bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$. In the present paper we are concerned with the problem of approximating a locally unique solution $x^\star$ of the equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet-differentiable operator defined on $\mathcal{D}$ with values in $\mathcal{Y}$.

Many problems from computational sciences can be presented in the form of equation (1.1) using mathematical modelling [2], [5], [6]. The solution of these equations can rarely be found in closed form. That is why the solution methods for these equations are usually iterative. In particular, the practice of numerical analysis for finding such solutions is essentially connected to Newton-like methods (see [2] and the references therein). The study of convergence of iterative procedures is usually focused on two types: semilocal and local convergence analysis. The semilocal convergence is based on the information around an initial point, to give criteria ensuring the convergence of iterative procedures; while the local one is based on the information around a solution, to find estimates of the radii of convergence balls. A plethora of sufficient conditions for the local as well as the semilocal convergence of Newton-type methods as well as an error analysis for such methods can be found in [1]–[24].

We consider the secant method in the form

$$\begin{aligned}&x_{-1}, x_0 \text{ are initial points in } \mathcal{D}\\ &x_{n+1} = x_n - \delta F(x_{n-1}, x_n)^{-1} F(x_n) \quad \text{for each } n = 0, 1, \ldots,\end{aligned}$$

where $\delta F(x, y) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $x, y \in \mathcal{D}$, is a consistent approximation of the Fréchet-derivative of F [2], [14]. A popular choice for $\delta F(x, y)$ is given by

$$\delta F(x, y) = \int_0^1 F'(x + t(y - x))\,\mathrm{d}t.$$

Other popular choices for $\delta F(x, y)$ can be found in [2], [5], [7], [8], [9], [6] and the references therein (see also the examples at the end of this paper). The secant method (1.2) is an alternative of Newton's method

$$\begin{aligned}&x_0 \text{ is an initial point in } \mathcal{D}\\ &x_{n+1} = x_n - F'(x_n)^{-1} F(x_n) \quad \text{for each } n = 0, 1, \ldots\end{aligned}$$

Several studies were provided on the semilocal convergence analysis of the secant method using Lipschitz and Hölder-type conditions. Bosarge and Falb [10], Laasonen [15], Ortega and Rheinboldt [16], Dennis [11], Potra [17], [18], Potra and Pták [20], [19], Hernández et al. [13], Argyros [2] and others [12], [24] have provided sufficient convergence conditions for (1.2) based on Lipschitz-type conditions on δF (see also relevant works in [4], [1], [6], [15], [19], [21], [22], [23]). The conditions usually associated with the semilocal convergence of the secant method (1.2) in the above mentioned references are:

($\mathcal{H}_1$) F is a nonlinear operator defined on $\mathcal{D}$ with values in $\mathcal{Y}$;

($\mathcal{H}_2$) x_{-1} and x_0 are two points belonging to the interior $\mathcal{D}^0$ of $\mathcal{D}$ and satisfying

$$\|x_0 - x_{-1}\| \leqslant d;$$

($\mathcal{H}_3$) F is Fréchet-differentiable on $\mathcal{D}^0$ and there exists an operator $\delta F\colon \mathcal{D}^0 \times \mathcal{D}^0 \to \mathcal{L}(\mathcal{X}, \mathcal{Y})$ such that the linear operator $A = \delta F(x_{-1}, x_0)$ is invertible, its inverse A^{-1} is bounded and

$$\|A^{-1}F(x_0)\| \leqslant \eta;$$
$$\|A^{-1}(\delta F(x,y) - F'(z))\| \leqslant K(\|x-z\| + \|y-z\|), \quad \text{for all } x,y,z \in \mathcal{D};$$
$$\overline{U}(x_0, r) \subseteq \mathcal{D}^0, \text{ for some } r > 0 \text{ depending on } K,\ d,\ \eta, \text{ and}$$
$$Kd + 2\sqrt{K\eta} \leqslant 1. \tag{1.3}$$

The sufficient convergence condition (1.3) is easily violated. For example, let $K = 0.97689$, $\eta = 0.17895$ and $d = 0.17956$. Then, (1.3) is not satisfied, since

$$Kd + 2\sqrt{K\eta} = 1.011626765 > 1.$$

Moreover, our recently found corresponding conditions are also violated [7]. Hence, there is no guarantee that equation (1.1) has a solution that can be found using the secant method. In [8], using a combination of Lipschitz and center-Lipschitz conditions, we provided a semilocal convergence analysis for the secant method. Error bounds in [8] are tighter and our convergence conditions hold in cases where the corresponding hypotheses in earlier references [11], [13], [15], [16], [17], [18], [22], [24], [23] are not satisfied.

In the present paper, using more precise majorizing sequences we provide new semilocal convergence criteria and a tighter semilocal convergence analysis for the secant method (1.2) than in [7], [8], [11], [13], [15], [16], [17], [18], [22], [24], [23]. This way we expand the applicability of the secant method. Note that the global convergence of the secant method can be considerably improved by using the trust region strategy. Thus the iterative scheme (1.2) is practically never used.

The paper is organized as follows. In Section 2 we present our new semilocal convergence results for the secant method using majorizing sequences. Section 3 contains applications, special cases and numerical examples.

2. Semilocal convergence analysis

We need the following results on majorizing sequences for the secant method.

Lemma 2.1. *Let $K_0 > 0$, $K > 0$, $d \geqslant 0$ and $\eta > 0$ be constants. Set*

$$\alpha = \frac{2K}{K + \sqrt{K^2 + 4K_0K}} \tag{2.1}$$

and

$$\alpha_0 = \frac{K\Big((d+\eta)\dfrac{1-K_0d}{1-K_0(d+\eta)} - d\Big)}{1 - K_0\Big((d+\eta)\dfrac{1-K_0d}{1-K_0(d+\eta)} + \eta\Big)}. \tag{2.2}$$

Suppose that the following conditions hold

$$K_0(d+2\eta) < 1, \tag{2.3}$$

$$K_0\Big((d+\eta)\frac{1-K_0 d}{1-K_0(d+\eta)}+\eta\Big) < 1, \tag{2.4}$$

$$\alpha_0 \leqslant \alpha \tag{2.5}$$

and

$$(1-\alpha)(1-K_0(d+\eta))(K_0(d+2\eta)-1)+2K_0^2(d+\eta)\eta \leqslant 0. \tag{2.6}$$

Then, the scalar sequence $\{t_n\}$ ($n \geqslant -1$) given by

$$t_{-1}=0,\ t_0=d,\ t_1=d+\eta,\ t_2=d+\eta+\frac{K_0(d+\eta)\eta}{1-K_0(d+\eta)}, \tag{2.7}$$

$$t_{n+2}=t_{n+1}+\frac{K(t_{n+1}-t_{n-1})(t_{n+1}-t_n)}{1-K_0(t_{n+1}-t_0+t_n)} \quad \text{for each } n=1,2,\ldots$$

is non-decreasing, bounded from above by

$$t^{\star\star}=(d+\eta)\Big(1+\frac{1-K_0 d}{(1-\alpha)(1-K_0(d+\eta))}\Big) \tag{2.8}$$

and converges to its unique least upper bound $t^\star$, which satisfies $t^\star \in [0,t^{\star\star}]$. Moreover, the following estimates hold

$$0 < t_{n+2}-t_{n+1} \leqslant \frac{K_0(d+\eta)\eta}{1-K_0(d+\eta)}\alpha^{n+1} \quad \text{for each } n=1,2,\ldots \tag{2.9}$$

P r o o f. We know that $\alpha \in (0,1)$ by (2.1). The constant α_0 is non-negative by (2.3) and (2.4). Hypothesis (2.4) implies $K_0(t_2+t_1-t_0) < 1$, which together with (2.3) and $t_{-1} \leqslant t_0 \leqslant t_1$ imply $t_2 \leqslant t_3$. We use mathematical induction to prove that the following holds:

$$0 \leqslant \frac{K(t_{k+1}-t_{k-1})}{1-K_0(t_k+t_{k-1}-d)} \leqslant \alpha \quad \text{for each } k=1,2,\ldots \tag{2.10}$$

Estimate (2.10) is true for $k=1$ by (2.2). Then, we have by (2.7) that

$$\begin{aligned} 0 \leqslant t_3-t_2 \leqslant \alpha(t_2-t_1) &\Longrightarrow t_3 \leqslant t_2+\alpha(t_2-t_1)\\ &\Longrightarrow t_3 \leqslant t_2+(1+\alpha)(t_2-t_1)-(t_2-t_1)\\ &\Longrightarrow t_3 \leqslant t_1+\frac{1-\alpha^2}{1-\alpha}(t_2-t_1) < t^{\star\star}. \end{aligned} \tag{2.11}$$

Assume that (2.10) holds for all natural integers $n \leqslant k$. Then, we get by (2.7) and (2.10) that

$$0 \leqslant t_{k+2} - t_{k+1} \leqslant \alpha^k (t_2 - t_1) \tag{2.12}$$

and

$$t_{k+2} \leqslant t_1 + \frac{1-\alpha^{k+1}}{1-\alpha}(t_2 - t_1) < t^{\star\star}. \tag{2.13}$$

Evidently, estimate (2.10) is true if k is replaced by $k+1$, provided that

$$K(t_{k+2} - t_k) \leqslant \alpha(1 - K_0(t_{k+2} + t_{k+1} - t_0))$$

or

$$K(t_{k+2} - t_{k+1}) + K(t_{k+1} - t_k) + \alpha K_0(t_{k+2} + t_{k+1}) - \alpha(1 + K_0 t_0) \leqslant 0$$

or

$$\begin{aligned} & K\alpha^k(t_2 - t_1) + K\alpha^{k-1}(t_2 - t_1) \\ & \quad + \alpha K_0\Big(t_1 + \frac{1-\alpha^{k+1}}{1-\alpha}(t_2 - t_1) + t_1 + \frac{1-\alpha^k}{1-\alpha}(t_2 - t_1)\Big) - \alpha(1 + K_0 t_0) \leqslant 0. \end{aligned} \tag{2.14}$$

Estimate (2.14) motivates us to introduce recurrent functions defined on $[0,1)$ for each $k = 1, 2, \ldots$ by

$$\begin{aligned} f_k(s) = {} & K(s^k + s^{k-1})(t_2 - t_1) + sK_0(1 + s + \ldots + s^k)(t_2 - t_1) \\ & + sK_0(1 + s + \ldots + s^{k-1})(t_2 - t_1) + 2(K_0 t_1 - 1 - K_0 t_0)s \leqslant 0. \end{aligned} \tag{2.15}$$

We need the relationship between two consecutive functions f_k. We have that

$$\begin{aligned} f_{k+1}(s) &= f_k(s) - f_k(s) + K(s^{k+1} + s^k)(t_2 - t_1) \\ &\quad + sK_0(1 + s + \ldots + s^{k+1})(t_2 - t_1) \\ &\quad + sK_0(1 + s + \ldots + s^k)(t_2 - t_1) + 2(K_0 t_1 - 1 - K_0 t_0)s \\ &= f_k(s) + g(s)s^{k-1}(t_2 - t_1), \end{aligned} \tag{2.16}$$

where

$$g(s) = K_0 s^3 + (K_0 + K)s^2 - K. \tag{2.17}$$

Note that α given in (2.1) is the only positive root of polynomial g. In view of (2.14) and (2.15), we must prove that

$$f_k(\alpha) \leqslant 0 \quad \text{for each } k = 1, 2, \ldots \tag{2.18}$$

However, we have by (2.16) and (2.17) that

$$f_{k+1}(\alpha) = f_k(\alpha) \quad \text{for each } k = 1, 2, \ldots \tag{2.19}$$

We define the function f_∞ on $[0, 1)$ by

$$f_\infty(s) = \lim_{k \longrightarrow \infty} f_k(s). \tag{2.20}$$

Therefore, we must only show

$$f_\infty(\alpha) \leqslant 0. \tag{2.21}$$

But, we have by (2.14), (2.15) and (2.20) that

$$f_\infty(\alpha) = \alpha\Big(2K_0\Big(t_1 + \frac{t_2 - t_1}{1 - \alpha}\Big) - (1 + K_0 t_0)\Big). \tag{2.22}$$

Inequality (2.21) is then true by (2.6) and (2.22). The induction for (2.10) is complete. It then follows from (2.12) and (2.13) that the sequence $\{t_n\}$ is non-decreasing, bounded from above by $t^{\star\star}$ given in (2.8) and thus it converges to $t^\star \in [0, t^{\star\star}]$. The proof of Lemma 2.1 is complete. □

Lemma 2.2. *Under hypotheses* (2.3)–(2.5) *suppose further that*

$$(2K_0 + K)(t_2 - (d + \eta)) + K_0(d + 2\eta) < 1 \tag{2.23}$$

and

$$f_1(\alpha) = K_0(t_2 - t_1)\alpha^2 + ((2K_0 + K)(t_2 - t_1) + K_0(2t_1 - t_0) - 1)\alpha + K(t_2 - t_1) \leqslant 0 \tag{2.24}$$

hold, where t_2 *is given by* (2.7). *Then, the conclusions of Lemma* 2.1 *hold.*

P r o o f. Simply notice that (2.18) holds by (2.15) (for $k = 1$), (2.19), (2.23), and (2.24). The proof of Lemma 2.2 is complete. □

We also have the following useful generalizations of Lemmas 2.1 and 2.2.

Lemma 2.3. *Let $K_0 > 0$, $K > 0$, $d \geqslant 0$ and $\eta > 0$ be constants. Let also $N = 0, 1, \ldots$ be fixed. Set*

$$\alpha_N = \frac{K\Big(\dfrac{1-K_0t_N}{1-K_0t_{N+1}}t_{N+1}-t_N\Big)}{1-K_0\Big(\dfrac{1-K_0t_N}{1-K_0t_{N+1}}t_{N+1}+t_{N+1}-t_N\Big)}. \tag{2.25}$$

Suppose that the following conditions hold

$$K_0(2t_{N+1}-t_N) < 1, \tag{2.26}$$

$$K_0\Big(\frac{1-K_0t_N}{1-K_0t_{N+1}}t_{N+1}+t_{N+1}-t_N\Big) < 1, \tag{2.27}$$

$$\alpha_N \leqslant \alpha,$$

$$(1-\alpha)(1-K_0t_{N+1})(K_0(2t_{N+1}-t_N)-1)+2K_0^2(t_{N+1}-t_N)t_{N+1} \leqslant 0, \tag{2.28}$$

$$t_1 \leqslant t_2 < \ldots \leqslant t_{N+1}, \tag{2.29}$$

and

$$K_0(t_N+t_{N+1}) < 1+K_0d, \tag{2.30}$$

where α is defined by (2.1). *Then, the scalar sequence $\{t_n\}$ ($n \geqslant -1$) given by* (2.7) *is non-decreasing, bounded from above by*

$$t_N^{\star\star} = \Big(1+\frac{1-K_0t_N}{(1-\alpha)(1-K_0t_{N+1})}\Big)t_{N+1} \tag{2.31}$$

and converges to its unique least upper bound $t_N^\star$, which satisfies $t_N^\star \in [0, t_N^{\star\star}]$. Moreover, the following estimates hold

$$0 < t_{N+n}-t_{N+n-1} \leqslant (t_{N+1}-t_N)\alpha^{n-1} \quad \textit{for each } n = 1, 2, \ldots \tag{2.32}$$

P r o o f. Simply replace d, η, $d+\eta$, $d+2\eta$, α_0, $t^\star$, $t^{\star\star}$ by t_N, $t_{N+1}-t_N$, t_{N+1}, $2t_{N+1}-t_N$, α_N, $t_N^\star$, $t_N^{\star\star}$, respectively, in the proof of Lemma 2.1. The proof of Lemma 2.3 is complete. $\square$

Lemma 2.4. *Under hypotheses* (2.26), (2.27), (2.29), *and* (2.30) *suppose further that the following conditions hold:*

$$(2K_0+K)(t_{N+2}-t_{N+1})+K_0(2t_{N+1}-t_N) < 1 \tag{2.33}$$

and

$$f_1(\alpha) = K_0(t_{N+2} - t_{N+1})\alpha^2 + ((2K_0 + K)(t_{N+2} - t_{N+1}) + K_0(2t_{N+1} - t_N) - 1)\alpha + K(t_{N+2} - t_{N+1}) \leqslant 0. \tag{2.34}$$

Then, the conclusions of Lemma 2.3 *hold.*

Proof. Simply replace t_0, t_1, t_2, $d+\eta$, $d+2\eta$ by t_N, t_{N+1}, t_{N+2}, t_{N+1}, $2t_{N+1} - t_N$, respectively, in the proof of Lemma 2.2. The proof of Lemma 2.4 is complete. □

Remark 2.5. Hypotheses (2.3)–(2.6), (2.23), and (2.24) are used to determine the size of η and d. These are essentially linear or quadratic inequalities that can be solved for η. However, we decided to leave them uncluttered. Moreover, note that the verification of these inequalities requires only computations at the initial data d, η, K_0 and K.

Set

$$L = \frac{K}{1+K_0 d}, \quad L_1 = \frac{K_0}{1+K_0 d} \quad \text{and} \quad L_0 = 2L_1. \tag{2.35}$$

If the sequence $\{t_n\}$ is non-decreasing, then we have for each $n = 1, 2, \ldots$

$$\begin{aligned} t_{n+2} &= t_{n+1} + \frac{L(t_{n+1} - t_{n-1})(t_{n+1} - t_n)}{1 - L_1(t_{n+1} + t_n)} \\ &\leqslant t_{n+1} + \frac{L(t_{n+1} - t_{n-1})(t_{n+1} - t_n)}{1 - L_0 t_{n+1}}. \end{aligned} \tag{2.36}$$

Then, we have the following result for computing upper bounds on $t^\star$.

Lemma 2.6 ([9])**.** *Let a, b, c be real constants, $K_0 > 0$, L_0 as given in* (2.35) *and $\lambda \in [0, 1/L_0]$. Define quadratic polynomials p, q on $(-\infty, +\infty)$ and functions f, g on $[0, 1/L_0)$ by*

$$p(t) = at^2 + bt + c, \tag{2.37}$$

$$f(t) = \frac{p(t)}{1 - L_0 t},$$

$$g(t) = t + f(t), \tag{2.38}$$

and

$$q(t) = (1 - L_0 t)^2 g'(t) = L_0(L_0 - a)t^2 - 2(L_0 - a)t + 1 + b + L_0 c. \tag{2.39}$$

Suppose that the polynomial p *has a unique root* ϱ *in the interval* $[\lambda, 1/L_0]$,

$$p(\lambda) \geqslant 0, \tag{2.40}$$

$$p(1/L_0) \leqslant 0, \tag{2.41}$$

and

$$q(t) \geqslant 0 \quad \text{for each } t \in [\lambda, \varrho]. \tag{2.42}$$

Then, function g *is non-decreasing and bounded from above by* ϱ *for each* $t \in [\lambda, \varrho]$. *The root* ϱ *is given by*

$$\varrho = \frac{-b \pm (b^2 - 4ac)^{1/2}}{2a}. \tag{2.43}$$

Proof. The expression under the radical in (2.43) is non-negative, since p has a unique root ϱ in $[\lambda, 1/L_0]$. Polynomial p is obviously non-negative on the interval $[\lambda, \varrho)$, since ϱ is the only root of p in $[\lambda, \varrho]$. Moreover, the function f is non-negative on $[\lambda, \varrho]$ with the only exception of $\varrho = 1/L_0$; but the l'Hospital's theorem implies that f admits a continuous extension on the interval $[\lambda, \varrho]$. We must prove that function g is non-decreasing on the interval $[\lambda, \varrho]$. In fact, its derivative is given by

$$g'(t) = 1 + f'(t) = \frac{q(t)}{(1 - L_0 t)^2}. \tag{2.44}$$

It follows from (2.42) and (2.44) that $g'(t) \geqslant 0$ for each $t \in [\lambda, \varrho]$. Therefore, we have for each $t \in [\lambda, \varrho]$ that

$$g(t) = t + f(t) \leqslant \varrho + f(\varrho) = \varrho. \tag{2.45}$$

That completes the proof of Lemma 2.6. □

Lemma 2.7. *Let* $K > 0$, $K_0 > 0$ *be constants and* L, L_0 *as given in* (2.35). *Suppose that hypotheses of Lemma 2.6 hold. Define functions* φ *and* ψ *on* $\mathcal{I} = [\lambda, s] \times [r, t] \times [t, \varrho]$ *for each* $t \in [\lambda, \varrho]$ *by*

$$\varphi(r, s, t) = t + \frac{L(t - r)(t - s)}{1 - L_0 t} \tag{2.46}$$

and

$$\psi(r, s, t) = \begin{cases} (L - a)t^2 - (L(r + s) + b)t + Lrs - c & \text{if } t \neq \varrho, \\ 0 & \text{if } t = \varrho. \end{cases} \tag{2.47}$$

Moreover, suppose that

$$\psi(r,s,t) \leqslant 0 \quad \text{for each } (r,s,t) \in \mathcal{I}. \tag{2.48}$$

Then, the following assertion holds

$$\varphi(r,s,t) \leqslant g(t) \quad \text{for each } t \in [\lambda, \varrho]. \tag{2.49}$$

Proof. The inequality (2.49) follows immediately from the definition of functions g, φ, ψ and hypothesis (2.48). The proof of Lemma 2.7 is complete. □

Lemma 2.8. *Let $N = 0, 1, 2, \ldots$ be fixed. Under the hypotheses of Lemma 2.6 with $\lambda = t_N$, suppose further that*

$$t_1 \leqslant t_2 \leqslant \ldots \leqslant t_N \leqslant t_{N+1} \leqslant \varrho \tag{2.50}$$

and

$$p(t_{N+1}) \geqslant 0. \tag{2.51}$$

Then, the sequence $\{t_n\}$ generated by (2.7) is non-decreasing, bounded by ϱ and converges to its unique least upper bound $t^\star$, which satisfies $t^\star \in [t_N, \varrho]$.

Proof. We can write

$$t_{n+1} = \varphi(t_{n-1}, t_n, t_{n+1}). \tag{2.52}$$

Using (2.45), (2.49), and (2.52), we get that

$$t_{N+2} = \varphi(t_{N-1}, t_N, t_{N+1}) \leqslant g(t_{N+1}) \leqslant \varrho. \tag{2.53}$$

Moreover, we get by (2.50) and (2.51) that $t_{N+1} \leqslant t_{N+2}$. The proof of Lemma 2.8 is complete. □

Remark 2.9.

(a) Interval $\mathcal{I}$ can also be replaced by the more practical $\mathcal{J} = [\lambda_N, \varrho]^3$.

(b) Let us define function ψ_N on $[\lambda, \varrho]$ by

$$\psi_N(t) = (L-a)t^2 - (b + 2Lt_N)t + L\varrho^2 - c.$$

Then, since $-(b + L(r+s)) \leqslant -(b + 2Lt_N)$ for each $r, s \geqslant t_N$ and for $r \leqslant s \leqslant \varrho$, function ψ_N can replace ψ in hypothesis (2.48).

(c) Hypotheses of Lemma 2.6 are satisfied in many interesting cases. Let

$$a = \frac{K}{1+K_0 d}, \quad b = -\frac{1+Kd}{1+K_0 d}, \quad c = \frac{d+\eta}{1+K_0 d}, \quad \text{and} \quad \lambda = t_1. \tag{2.54}$$

Then, by (2.7) and (2.54) the sequence $\{t_n\}$ can be written as

$$\begin{aligned} &s_{-1} = 0,\ s_0 = d,\ s_1 = c+\eta,\ s_2 = d+\eta+\frac{K_0(d+\eta)\eta}{1-K_0(d+\eta)}, \\ &s_{n+2} = s_{n+1} + \frac{K(s_{n+1}-s_{n-1})(s_{n+1}-s_n)}{1-K_0(s_n+s_{n+1}-d)} \\ &\quad = s_{n+1} + \frac{Ks_{n+1}^2 - (1+Kd)s_{n+1} + d+\eta}{1-K_0(s_n+s_{n+1}-d)} \quad \text{for each } n = 1,2,\dots \end{aligned} \tag{2.55}$$

Moreover, if $K_0 = K$, the sequence $\{s_n\}$ reduces to $\{u_n\}$ given by

$$\begin{aligned} &u_{-1} = 0, \quad u_0 = d, \quad u_1 = d+\eta, \\ &u_{n+2} = u_{n+1} + \frac{K(u_{n+1}-u_{n-1})(u_{n+1}-u_n)}{1-K(u_n+u_{n+1}-d)} \\ &\quad = u_{n+1} + \frac{Ku_{n+1}^2 - (1+Kd)u_{n+1} + d+\eta}{1-K(u_n+u_{n+1}-d)} \quad \text{for each } n = 0,1,\dots \end{aligned} \tag{2.56}$$

The sequence $\{u_n\}$ has been used by many authors, e.g. [2], [14], [16], [22]. Hypotheses of Lemma 2.6 are then satisfied if (2.54) holds. Clearly, sequences $\{t_n\}$ and $\{s_n\}$ are tighter than $\{u_n\}$ if $K_0 < K$. In fact a simple inductive argument shows that for each $n = 2,3,\dots$

$$\begin{gathered} t_n < u_n, \quad s_n < u_n, \quad t_{n+1} - t_n < u_{n+1} - u_n, \\ s_{n+1} - s_n < u_{n+1} - u_n \quad \text{and} \quad t^\star \leqslant u^\star = \lim_{n\to\infty} u_n. \end{gathered}$$

(d) Polynomial p can even be a function, preferably in closed form, with a unique zero on $[0, 1/L_0]$ in Lemmas 2.6–2.8.

We shall study the secant method for triplets (F, x_{-1}, x_0) belonging to the class $\mathcal{C}(K, K_0, \eta, d)$ defined as follows:

Definition 2.10. Let K, K_0, η, d be non-negative constants satisfying the hypotheses of Lemma 2.1. A triplet (F, x_{-1}, x_0) belongs to the class $\mathcal{C}(K, K_0, \eta, d)$ if

($\mathcal{A}_1$) F is a nonlinear operator defined on $\mathcal{D}$ with values in $\mathcal{Y}$;

($\mathcal{A}_2$) x_{-1} and x_0 are two points belonging to the interior $\mathcal{D}^0$ of $\mathcal{D}$ and satisfying

$$\|x_0 - x_{-1}\| \leqslant d;$$

($\mathcal{A}_3$) F is Fréchet-differentiable on $\mathcal{D}^0$, there exists an operator $\delta F\colon \mathcal{D}^0 \times \mathcal{D}^0 \to \mathcal{L}(\mathcal{X},\mathcal{Y})$ such that $A^{-1} = \delta F(x_{-1},x_0)^{-1} \in \mathcal{L}(\mathcal{Y},\mathcal{X})$ and for all $x,y,z \in \mathcal{D}$, the following conditions hold:

$$\|A^{-1}F(x_0)\| \leqslant \eta,$$
$$\|A^{-1}(\delta F(x,y) - F'(z))\| \leqslant K(\|x-z\| + \|y-z\|),$$

and

$$\|A^{-1}(\delta F(x,y) - F'(x_0))\| \leqslant K_0(\|x-x_0\| + \|y-x_0\|);$$

($\mathcal{A}_4$)

$$\overline{U}(x_0,t^\star) \subseteq \mathcal{D}_c = \{x \in \mathcal{D}\colon\ F \text{ is continuous at } x\} \subseteq \mathcal{D},$$

where $t^\star$ is given in Lemma 2.1.

The semilocal convergence theorem for the secant method is as follows.

Theorem 2.11. *If $(F,x_{-1},x_0) \in \mathcal{C}(K,K_0,\eta,d)$, then the sequence $\{x_n\}$ ($n \geqslant -1$) generated by the secant method is well defined, remains in $\overline{U}(x_0,t^\star)$ for each $n = -1,0,\dots$ and converges to a unique solution $x^\star \in \overline{U}(x_0,t^\star)$ of (1.1). Moreover, the following estimates hold for each $n = 0,1,\dots$*

$$\|x_n - x_{n-1}\| \leqslant t_n - t_{n-1} \tag{2.57}$$

and

$$\|x_n - x^\star\| \leqslant t^\star - t_n, \tag{2.58}$$

where $\{t_n\}$ ($n \geqslant -1$) is given by (2.7). Furthermore, if there exists R such that

$$\overline{U}(x_0,R) \subseteq \mathcal{D}, \quad R \geqslant t^\star - d, \quad \text{and} \quad K_0(t^\star - d + \|A^{-1}(F'(x_0) - A)\|) < 1, \tag{2.59}$$

then the solution $x^\star$ is unique in $\overline{U}(x_0,R)$.

Proof. First, we show that $\mathcal{R} = \delta F(x_k,x_{k+1})$ is invertible for $x_k,x_{k+1} \in \overline{U}(x_0,t^\star)$. By ($\mathcal{A}_2$) and ($\mathcal{A}_3$), we have that

$$\begin{aligned}\|I - A^{-1}\mathcal{R}\| &= \|A^{-1}(\mathcal{R} - A)\| \\ &\leqslant \|A^{-1}(\mathcal{R} - F'(x_0))\| + \|A^{-1}(F'(x_0) - A)\| \\ &\leqslant K_0(\|x_k - x_0\| + \|x_{k+1} - x_0\| + \|x_0 - x_{-1}\|) \\ &\leqslant K_0(t_k - t_0 + t_{k+1} - t_0 + d) < 1 \quad \text{(by (2.11)).}\end{aligned} \tag{2.60}$$

Using the Banach lemma on invertible operators [2], [5], [14] and (2.60), $\mathcal{R}$ is invertible and

$$\|\mathcal{R}^{-1}A\| \leqslant (1 - K_0(t_{k+1} + t_k - t_0))^{-1}. \tag{2.61}$$

By $(\mathcal{A}_3)$, we get that

$$\|A^{-1}(F'(u) - F'(v))\| \leqslant 2K\|u - v\|, \quad u, v \in \mathcal{D}^0. \tag{2.62}$$

We can write the identity $F(x) - F(y) = \int_0^1 F'(y + t(x-y))\,\mathrm{d}t(x-y)$, then, for all $x, y, u, v \in \mathcal{D}^0$, we obtain that

$$\|A^{-1}(F(x) - F(y) - F'(u)(x-y))\| \leqslant K(\|x-u\| + \|y-u\|)\|x-y\| \tag{2.63}$$

and

$$\begin{aligned} \|A^{-1}(F(x) - F(y) - \delta F(u,v)(x-y))\| \\ \leqslant K(\|x-v\| + \|y-v\| + \|u-v\|)\|x-y\|. \end{aligned} \tag{2.64}$$

By a continuity argument, (2.62)–(2.64) remain valid if x and/or y belong to $\mathcal{D}_c$. Now we show (2.57). If (2.57) holds for all $n \leqslant k$ and if $\{x_n\}$ $(n \geqslant 0)$ is well defined for each $n = 0, 1, 2, \ldots, k$, then

$$\|x_n - x_0\| \leqslant t_n - t_0 < t^\star - t_0 \quad \text{for each } n \leqslant k. \tag{2.65}$$

That is, (1.2) is well defined for $n = k+1$. For $n = -1$ and $n = 0$, (2.57) reduces to $\|x_{-1} - x_0\| \leqslant d$ and $\|x_0 - x_1\| \leqslant \eta$. Suppose (2.57) holds for each $n = -1, 0, 1, \ldots, k$ $(k \geqslant 0)$. By (2.61), (2.64), and

$$F(x_{k+1}) = F(x_{k+1}) - F(x_k) - \delta F(x_{k-1}, x_k)(x_{k+1} - x_k), \tag{2.66}$$

we obtain the following estimates

$$\begin{aligned} \|A^{-1}F(x_{k+1})\| &\leqslant \overline{K}(\|x_{k+1} - x_k\| + \|x_k - x_{k-1}\|)\|x_{k+1} - x_k\| \\ &\leqslant K(t_{k+1} - t_k + t_k - t_{k-1})(t_{k+1} - t_k) \end{aligned}$$

and

$$\begin{aligned} \|x_{k+2} - x_{k+1}\| &= \|\delta F(x_k, x_{k+1})^{-1}F(x_{k+1})\| \\ &\leqslant \|\delta F(x_k, x_{k+1})^{-1}A\|\,\|A^{-1}F(x_{k+1})\| \\ &\leqslant \frac{\overline{K}(\|x_{k+1} - x_k\| + \|x_k - x_{k-1}\|)}{1 - K_0(\|x_{k+1} - x_0\| + \|x_k - x_0\| + d)}\|x_{k+1} - x_k\| \\ &\leqslant \frac{\overline{K}(t_{k+1} - t_{k-1})}{1 - K_0(t_{k+1} - t_0 + t_k - t_{-1})}(t_{k+1} - t_k) \\ &= t_{k+2} - t_{k+1}, \end{aligned} \tag{2.67}$$

where

$$\overline{K} = \begin{cases} K_0 & \text{if } k = 0, \\ K & \text{if } k \neq 0. \end{cases}$$

That is, the induction for (2.57) is completed. It follows from (2.57) and Lemma 2.1 that $\{x_n\}$ $(n \geqslant -1)$ is Cauchy in a Banach space $\mathcal{X}$ and therefore it converges to some $x^\star \in \overline{U}(x_0, t^\star)$ (since $\overline{U}(x_0, t^\star)$ is a closed set). By letting $k \to \infty$ in (2.67), we obtain $F(x^\star) = 0$. Estimate (2.58) follows from (2.57) by using standard majoration techniques [2], [5], [6], [14]. Finally, to show the uniqueness in $\overline{U}(x_0, t^\star)$, let $y^\star \in \overline{U}(x_0, t^\star)$ be a solution of (1.1). Set $\mathcal{M} = \int_0^1 F'(y^\star + t(y^\star - x^\star))\,\mathrm{d}t$. It then follows by $(\mathcal{A}_3)$ and (2.59) that

$$\begin{aligned} \|A^{-1}(A - \mathcal{M})\| &= K_0(\|y^\star - x_0\| + \|x^\star - x_0\|) + \|A^{-1}(F'(x_0) - A)\| \\ &\leqslant K_0((t^\star - t_0) + R) + \|A^{-1}(F'(x_0) - A)\| < 1. \end{aligned} \tag{2.68}$$

It follows from (2.68) and the Banach lemma on invertible operators that $\mathcal{M}^{-1}$ exists on $U(x_0, t^\star)$. Using the identity $F(x^\star) - F(y^\star) = \mathcal{M}(x^\star - y^\star)$, we deduce that $x^\star = y^\star$. The proof of Theorem 2.11 is complete. $\square$

3. Numerical examples

In this section, we present some numerical examples.

Example 3.1. (a) Let $K = 0.97689$, $K_0 = 0.845658$, $\eta = 0.17895$ and $d = 0.1677019$. Then, (1.3) is not satisfied, since

$$Kd + 2\sqrt{K\eta} = 1.000042706 > 1.$$

Using hypotheses of Lemma 2.1, we have that

$$\alpha = 0.6425703944 \quad \text{and} \quad \alpha_0 = 0.5018963491.$$

Hypotheses (2.3)–(2.6) hold since

$$\begin{gathered} K_0(d + 2\eta) = 0.4444794516 < 1, \\ K_0\Big((d+\eta)\frac{1 - K_0 d}{1 - K_0(d+\eta)} + \eta\Big) = 0.5072400253 < 1, \\ 0.5018963491 \leqslant 0.6425703944, \end{gathered}$$

and

$$(1-\alpha)(1 - K_0(d+\eta))(K_0(d+2\eta) - 1) + 2K_0^2(d+\eta)\eta = -0.5162722924\mathrm{e}-1 \leqslant 0.$$

Consequently, Lemma 2.1 is applicable but the condition (1.3) used in [11], [13], [15], [16], [17], [18], [22], [24], [23] is not satisfied.

(b) Let $K = 0.97689$, $K_0 = 0.845658$, $\eta = 0.17895$ and $d = 0.12956$. Then, (1.3) is satisfied, since

$$Kd + 2\sqrt{K\eta} = 0.9627822650 < 1.$$

Using hypotheses of Lemma 2.1, we have that

$$\alpha = 0.6425703944 \quad \text{and} \quad \alpha_0 = 0.4426273722.$$

Hypotheses (2.3)–(2.6) hold since

$$K_0(d + 2\eta) = 0.4122244487 < 1,$$
$$K_0\Big((d+\eta)\frac{1 - K_0 d}{1 - K_0(d+\eta)} + \eta\Big) = 0.4656419678 < 1,$$
$$0.4426273722 \leqslant 0.6425703944,$$

and

$$(1-\alpha)(1 - K_0(d+\eta))(K_0(d+2\eta) - 1) + 2K_0^2(d+\eta)\eta = -0.7631517220\mathrm{e} - 1 \leqslant 0.$$

Consequently, Lemma 2.1 is applicable. We can now compare our results of Lemma 2.1 (see also the sequence $\{t_n\}$ given by (2.7)) to the ones in [11], [13], [15], [16], [17], [18], [22], [24], [23] (see also the sequence $\{u_n\}$ given by (2.56)) Comparison Table 1 shows that our error bounds using sequence $\{t_n\}$ are tighter than those given in [11], [13], [15], [16], [17], [18], [22], [24], [23].

n	t_n	u_n	$t_{n+1} - t_n$	$u_{n+1} - u_n$
1	.30851	.0631668111	.30851	.1698509943
2	.3716768111	.0272575738	.4783609943	.0889850115
3	.3989343849	.0039292768	.5673460058	.0250726474
4	.4028636617	.0001884943	.5924186532	.0027762907
5	.4030521560	.0000011884	.5951949439	.0000731295
6	.4030533444	3e -10	.5952680734	1.966e-7
7	.4030533447	0	.5952682700	0
8	$\sim$	$\sim$		$\sim$

Comparison Table 1

Finally, we present examples where $K_0 < K$. The divided difference is defined by

$$\delta F(x, y) = \int_0^1 F'(y + t(x-y))\,\mathrm{d}t \quad \text{for each } x, y \in \mathcal{D}.$$

E x a m p l e 3.2. Define the scalar function F by $F(x) = c_0 x + c_1 + c_2 \sin \mathrm{e}^{c_3 x}$, $x_0 = 0$, where c_i, $i = 0, 1, 2, 3$ are given parameters. Then, it can easily be seen that for c_3 large and c_2 sufficiently small, K/K_0 can be arbitrarily large.

E x a m p l e 3.3 (Newton's method case). Let $\mathcal{X} = \mathcal{Y} = \mathcal{C}[0,1]$, equipped with the max-norm. Let $\theta \in [0,1]$ be a given parameter. Consider the "cubic" integral equation

$$u(s) = u^3(s) + \lambda u(s) \int_0^1 \mathcal{K}(s,t) u(t)\,\mathrm{d}t + y(s) - \theta. \tag{3.1}$$

Nonlinear integral equations of the form (3.1) are considered Chandrasekhar-type equations [2], [5], [6] and they arise in the theories of radiative transfer, neutron transport and in the kinetic theory of gasses. Here, the kernel $\mathcal{K}(s,t)$ is a continuous function of two variables $(s,t) \in [0,1] \times [0,1]$ satisfying

(i) $0 < \mathcal{K}(s,t) < 1$,
(ii) $\mathcal{K}(s,t) + \mathcal{K}(t,s) = 1$.

The parameter λ is a real number called the "albedo" for scattering; $y(s)$ is a given continuous function defined on $[0,1]$ and $x(s)$ is the unknown function sought in $\mathcal{C}[0,1]$. For simplicity, we choose

$$u_0(s) = y(s) = 1 \quad \text{and} \quad \mathcal{K}(s,t) = \frac{s}{s+t} \quad \text{for all } (s,t) \in [0,1] \times [0,1] \ (s+t \neq 0).$$

Let $\mathcal{D} = U(u_0, 1-\theta)$ and define the operator F on $\mathcal{D}$ by

$$F(x)(s) = x^3(s) - x(s) + \lambda x(s) \int_0^1 \mathcal{K}(s,t) x(t)\,\mathrm{d}t + y(s) - \theta \quad \text{for all } s \in [0,1]. \tag{3.2}$$

Then every zero of F satisfies equation (3.1). We obtain using, (3.2) and [2], [5], [6], that

$$\begin{aligned}[F'(x)v](s) = \lambda x(s) \int_0^1 \mathcal{K}(s,t) v(t)\,\mathrm{d}t + \lambda v(s) \int_0^1 \mathcal{K}(s,t) x(t)\,\mathrm{d}t \\ + 3x^2(s) v(s) - I(v(s)).\end{aligned}$$

Therefore, the operator F' satisfies the conditions of Theorem 2.11, with

$$\eta = \frac{|\lambda| \ln 2 + 1 - \theta}{2(1 + |\lambda| \ln 2)}, \quad K = \frac{|\lambda| \ln 2 + 3(2-\theta)}{1 + |\lambda| \ln 2}, \quad \text{and} \quad K_0 = \frac{2|\lambda| \ln 2 + 3(3-\theta)}{2(1 + |\lambda| \ln 2)}.$$

It follows from our main results that if our conditions holds, then problem (3.1) has a unique solution near u_0. This assumption is weaker than the one given before using the Newton-Kantorovich hypothesis. Note also that $K_0 < K$ for all $\theta \in [0,1]$ (see also Fig. 1).

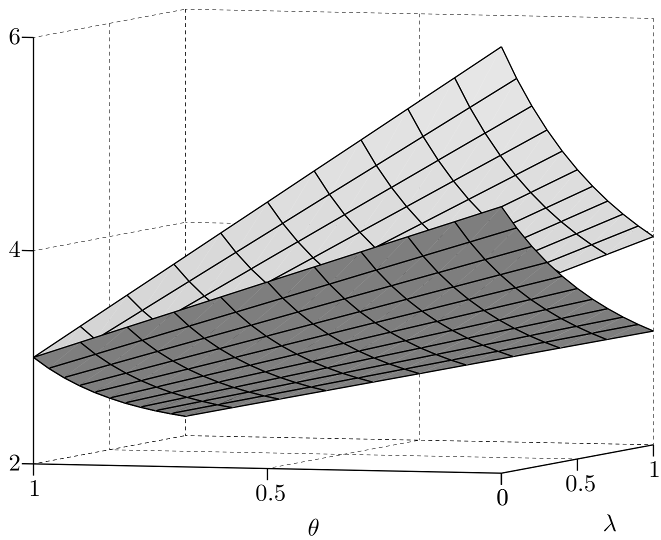

Figure 1. Functions K_0 and K in 3d with respect to (λ,θ) in $(0,1)\times(0,1)$, K is above K_0.

E x a m p l e 3.4 (Secant method case). Let $\mathcal{X}$ and $\mathcal{Y}$ be defined as in Example 3.3. Consider the following nonlinear boundary value problem [2], [5], [6]:

$$\begin{cases} u'' = -u^3 - \gamma u^2, \\ u(0) = 0, \quad u(1) = 1. \end{cases}$$

It is well known that this problem can be formulated as the integral equation

$$u(s) = s + \int_0^1 \mathcal{Q}(s,t)(u^3(t) + \gamma u^2(t))\,\mathrm{d}t, \tag{3.3}$$

where $\mathcal{Q}$ is the Green function given by

$$\mathcal{Q}(s,t) = \begin{cases} t(1-s) & \text{if } t \leqslant s, \\ s(1-t) & \text{if } s < t. \end{cases}$$

Then problem (3.3) is in the form (1.1), where $F\colon \mathcal{D} \longrightarrow \mathcal{Y}$ is defined as

$$[F(x)](s) = x(s) - s - \int_0^1 \mathcal{Q}(s,t)(x^3(t) + \gamma x^2(t))\,\mathrm{d}t.$$

Set $u_0(s) = s$ and $\mathcal{D} = U(u_0, R_0)$. The Fréchet derivative of F is given by (see [2], [5], [6])

$$[F'(x)v](s) = v(s) - \int_0^1 \mathcal{Q}(s,t)(3x^2(t) + 2\gamma x(t))v(t)\,\mathrm{d}t.$$

It is easy to verify that $U(u_0, R_0) \subset U(0, R_0 + 1)$ since $\|u_0\| = 1$. If $2\gamma < 5$, the conditions of Theorem 2.11 hold with

$$\eta = \frac{1+\gamma}{5-2\gamma}, \quad K = \frac{\gamma + 6R_0 + 3}{8(5-2\gamma)}, \quad \text{and} \quad K_0 = \frac{2\gamma + 3R_0 + 6}{16(5-2\gamma)}.$$

Note that $K_0 < K$ (see also Fig. 2).

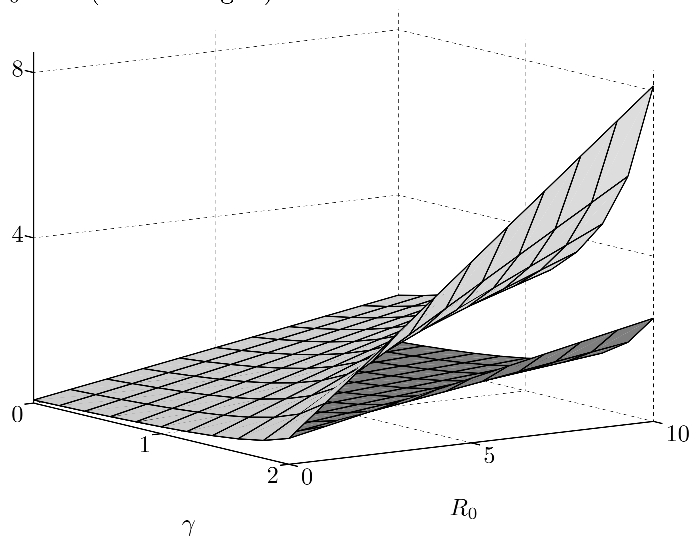

Figure 2. Functions $K_0/2$ and $K/2$ in 3d with respect to (γ, R_0) in $(0, 2.5) \times (0, 10)$, $K/2$ is also above $K_0/2$.

Conclusion

New sufficient convergence conditions for the secant method are provided. Using Lipschitz and center-Lipschitz conditions on the divided difference operator, we obtained the semilocal convergence analysis of the secant method. Our error bounds are more precise than in earlier studies such as [7], [8], [11], [13], [15], [16], [17], [18], [22], [24], [23]. Applications and numerical examples are also provided in this study.

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