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Semicontinuous integrands as jointly measurable maps

ORIOLE CARBONELL-NICOLAU

Abstract. Suppose that (X, \mathcal{A}) is a measurable space and Y is a metrizable, Souslin space. Let \mathcal{A}^u denote the universal completion of \mathcal{A} . For $x \in X$, let $\underline{f}(x, \cdot)$ be the lower semicontinuous hull of $f(x, \cdot)$. If $f : X \times Y \rightarrow \overline{\mathbb{R}}$ is $(\mathcal{A}^u \otimes \mathcal{B}(Y), \mathcal{B}(\overline{\mathbb{R}}))$ -measurable, then \underline{f} is $(\mathcal{A}^u \otimes \mathcal{B}(Y), \mathcal{B}(\overline{\mathbb{R}}))$ -measurable.

Keywords: lower semicontinuous hull; jointly measurable function; measurable projection theorem; normal integrand

Classification: 54C30, 28A20

Let (X, \mathcal{A}) be a measurable space. For every bounded measure μ on (X, \mathcal{A}) , let \mathcal{A}^μ denote the completion of \mathcal{A} with respect to μ . Let

$$\mathcal{A}^u := \bigcap \{ \mathcal{A}^\mu : \mu \text{ is a bounded measure on } (X, \mathcal{A}) \}.$$

The σ -algebra \mathcal{A}^u is called the *universal completion* of \mathcal{A} .

Let Y be a topological space, and let $\mathcal{B}(Y)$ represent the σ -algebra of Borel subsets of Y . The space Y is said to be *Souslin* if it is Hausdorff and there exist a Polish space P and a continuous surjection from P to Y .

Given $f : X \times Y \rightarrow \overline{\mathbb{R}}$, define the map $\underline{f} : X \times Y \rightarrow \overline{\mathbb{R}}$ by

$$\underline{f}(x, y) := \sup_{V_y} \inf_{z \in V_y} f(x, z),$$

where V_y ranges over all neighborhoods of y . For each $x \in X$, $\underline{f}(x, \cdot)$ is the *lower semicontinuous hull* of $f(x, \cdot)$. If Y is metrizable, \underline{f} can be expressed as

$$\underline{f}(x, y) = \sup_{n \in \mathbb{N}} \inf_{z \in N_{\frac{1}{n}}(y)} f(x, z),$$

where $N_{\frac{1}{n}}(y)$ represents the open $\frac{1}{n}$ -neighborhood of y .

Theorem. *Suppose that (X, \mathcal{A}) is a measurable space and Y is a metrizable, Souslin space. Suppose further that the map $f : X \times Y \rightarrow \overline{\mathbb{R}}$ is $(\mathcal{A}^u \otimes \mathcal{B}(Y), \mathcal{B}(\overline{\mathbb{R}}))$ -measurable. Then \underline{f} is $(\mathcal{A}^u \otimes \mathcal{B}(Y), \mathcal{B}(\overline{\mathbb{R}}))$ -measurable.*

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PROOF: Define $g^n : X \times Y \rightarrow \overline{\mathbb{R}}$ by

$$g^n(x, y) := \inf_{z \in N_{\frac{1}{n}}(y)} f(x, z).$$

We first show that g^n is $(\mathcal{A}^u \otimes \mathcal{B}(Y), \mathcal{B}(\overline{\mathbb{R}}))$ -measurable for each n .

Let

$$D^n := \left\{ (x, y, z) \in X \times Y \times Y : z \in N_{\frac{1}{n}}(y) \right\}.$$

The map g^n is $(\mathcal{A}^u \otimes \mathcal{B}(Y), \mathcal{B}(\overline{\mathbb{R}}))$ -measurable if for $a \in \mathbb{R}$,

$$(1) \quad \{(x, y) \in X \times Y : g^n(x, y) < a\} \in \mathcal{A}^u \otimes \mathcal{B}(Y).$$

Given $a \in \mathbb{R}$ we have

$$(2) \quad \{(x, y) \in X \times Y : g^n(x, y) < a\} = \text{Proj}_{X \times Y}(E^n),$$

where

$$E^n := \{(x, y, z) \in D^n : f(x, z) < a\}$$

and $\text{Proj}_{X \times Y}(E^n)$ represents the projection of E^n onto $X \times Y$. Thus, to establish (1) it suffices to show that $\text{Proj}_{X \times Y}(E^n)$ belongs to $\mathcal{A}^u \otimes \mathcal{B}(Y)$.

Because Y is a Souslin space, Y is a Lindelöf space, and since Y is in addition metrizable, Y is separable. Because Y is separable, there is a countable, dense subset Q of Y . Let $\{y^1, y^2, \dots\}$ be an enumeration of this set. For $\alpha > 0$ and $y \in Y$, define

$$A^{(\alpha, y)} := \{(x, z) \in X \times N_\alpha(y) : f(x, z) < a\}.$$

Let $\text{Proj}_X(A^{(\alpha, y)})$ be the projection of $A^{(\alpha, y)}$ onto X . Let \mathbb{Q} denote the set of rational numbers in $(0, \frac{1}{n})$. Define

$$S^n := \{(\alpha, \beta) \in \mathbb{Q} \times \mathbb{Q} : \alpha + \beta \leq \frac{1}{n}\}.$$

We have

$$(3) \quad \text{Proj}_{X \times Y}(E^n) = \bigcup_{(m, (\alpha, \beta)) \in \mathbb{N} \times S^n} \left[\text{Proj}_X(A^{(\alpha, y^m)}) \times N_\beta(y^m) \right].$$

To see this, observe that given $(x, y) \in \text{Proj}_{X \times Y}(E^n)$, there exists z such that $(x, y, z) \in D^n$ (i.e., $(x, y, z) \in X \times Y \times Y$ and $z \in N_{\frac{1}{n}}(y)$) and $f(x, z) < a$. Let d be a compatible metric on Y , and fix

$$\epsilon \in \left(0, \frac{1}{3} \left(\frac{1}{n} - d(y, z)\right)\right).$$

For $y' \in N_\epsilon(y)$ we have

$$d(y', z) \leq d(y', y) + d(y, z) < \epsilon + d(y, z) < \frac{1}{3} \left(\frac{1}{n} - d(y, z) \right) + d(y, z),$$

so there is a rational number

$$\beta \in \left(\frac{1}{3} \left(\frac{1}{n} - d(y, z) \right), \frac{1}{2} \left(\frac{1}{n} - d(y, z) \right) \right)$$

such that $d(y', z) < \beta + d(y, z)$ for all $y' \in N_\epsilon(y)$, and hence there is a rational number

$$\alpha \in \left(\beta + d(y, z), \frac{1}{2} \left(\frac{1}{n} - d(y, z) \right) + d(y, z) \right)$$

such that $d(y', z) < \alpha$ for all $y' \in N_\epsilon(y)$. Consequently, since by denseness of Q in Y one may choose m such that $y^m \in N_\epsilon(y)$, we have $z \in N_\alpha(y^m)$. It follows that $(x, z) \in X \times N_\alpha(y^m)$ and $f(x, z) < a$ (so that $x \in \text{Proj}_X(A^{(\alpha, y^m)})$) and, since

$$d(y, y^m) < \epsilon < \frac{1}{3} \left(\frac{1}{n} - d(y, z) \right) < \beta,$$

we have $y \in N_\beta(y^m)$. We conclude that $(x, y) \in \text{Proj}_X(A^{(\alpha, y^m)}) \times N_\beta(y^m)$ with $(\alpha, \beta) \in \mathbb{Q} \times \mathbb{Q}$ and

$$\alpha + \beta \leq \frac{1}{2} \left(\frac{1}{n} - d(y, z) \right) + d(y, z) + \frac{1}{2} \left(\frac{1}{n} - d(y, z) \right) \leq \frac{1}{n}.$$

Conversely, if $(x, y) \in \text{Proj}_X(A^{(\alpha, y^m)}) \times N_\beta(y^m)$ for some $(m, (\alpha, \beta)) \in \mathbb{N} \times S^n$, then there exists z such that $(x, z) \in X \times N_\alpha(y^m)$ and $f(x, z) < a$. In addition,

$$d(y, z) \leq d(y, y^m) + d(y^m, z) < \beta + \alpha \leq \frac{1}{n}.$$

Consequently, $(x, y, z) \in X \times Y \times Y$ and $z \in N_{\frac{1}{n}}(y)$ (so that $(x, y, z) \in D^n$) and $f(x, z) < a$, which implies that $(x, y) \in \text{Proj}_{X \times Y}(E^n)$.

Because f is $(\mathcal{A}^u \otimes \mathcal{B}(Y), \mathcal{B}(\overline{\mathbb{R}}))$ -measurable, we have $A^{(\alpha, y)} \in \mathcal{A}^u \otimes \mathcal{B}(Y)$ for every $\alpha > 0$ and $y \in Y$. Therefore, because Y is a Souslin space, the measurable projection theorem (e.g., Sainte-Beuve [6, Theorem 4]) gives $\text{Proj}_X(A^{(\alpha, y)}) \in \mathcal{A}^u$ for $\alpha > 0$ and $y \in Y$.¹ In light of (3), therefore, we conclude that $\text{Proj}_{X \times Y}(E^n) \in \mathcal{A}^u \otimes \mathcal{B}(Y)$.

Because $\text{Proj}_{X \times Y}(E^n) \in \mathcal{A}^u \otimes \mathcal{B}(Y)$, g^n is $(\mathcal{A}^u \otimes \mathcal{B}(Y), \mathcal{B}(\overline{\mathbb{R}}))$ -measurable (recall (2) and (1)). It only remains to observe that

$$\underline{f}(x, y) = \sup_{n \in \mathbb{N}} \inf_{z \in N_{\frac{1}{n}}(y)} f(x, z) = \sup_{n \in \mathbb{N}} g^n(x, y),$$

so \underline{f} is $(\mathcal{A}^u \otimes \mathcal{B}(Y), \mathcal{B}(\overline{\mathbb{R}}))$ -measurable. □

In the remainder of the paper we present an application of the above result. Let (X, \mathcal{A}, μ) be a finite measure space with $\mathcal{A} = \mathcal{A}^u$. Let Y be a metrizable Lusin space (i.e., a metrizable topological space which is homeomorphic to a Borel subset

¹For the case when Y is Polish, the measurable projection theorem can also be found in Cohn [5, Proposition 8.4.4].

of a compact metrizable space). A *transition probability* with respect to (X, \mathcal{A}) and $(Y, \mathcal{B}(Y))$ is a function $\sigma : \mathcal{B}(Y) \times X \rightarrow [0, 1]$ satisfying the following:

- $\sigma(\cdot|x)$ is a probability measure on $(Y, \mathcal{B}(Y))$ for every $x \in X$;
- $\sigma(B|\cdot)$ is $(\mathcal{A}, \mathcal{B}([0, 1]))$ -measurable for every $B \in \mathcal{B}(Y)$.

The set of transition probabilities with respect to (X, \mathcal{A}) and $(Y, \mathcal{B}(Y))$ is denoted by \mathcal{S} .

A *normal integrand* on $X \times Y$ is a map $f : X \times Y \rightarrow \overline{\mathbb{R}}$ satisfying the following:

- $f(x, \cdot)$ is lower semicontinuous on Y for every $x \in X$;
- f is $(\mathcal{A} \otimes \mathcal{B}(Y), \mathcal{B}(\overline{\mathbb{R}}))$ -measurable.

Let $L_1(X, \mathcal{A}, \mu)$ represent the set of $(\mathcal{A}, \mathcal{B}(\mathbb{R}))$ -measurable functions $\xi : X \rightarrow \mathbb{R}$ such that

$$\int_X |\xi(x)| \mu(dx) < \infty.$$

The set of all normal integrands f on $X \times Y$ for which there exists $\xi \in L_1(X, \mathcal{A}, \mu)$ such that $\xi(x) \leq f(x, y)$ for all $(x, y) \in X \times Y$ is denoted by \mathcal{F} .

For $f \in \mathcal{F}$, the functional $I_f : \mathcal{S} \rightarrow \overline{\mathbb{R}}$ is defined by

$$I_f(\sigma) := \int_X \int_Y f(x, y) \sigma(dy|x) \mu(dx).$$

The *narrow topology* on \mathcal{S} is the coarsest topology that makes the functionals in $\{I_f : f \in \mathcal{F}\}$ lower semicontinuous. This topology has been studied by Balder [1], [2], [3] and applied to the theory of games with incomplete information (*e.g.*, Balder [2] and Carbonell-Nicolau and McLean [4]).

Suppose that the map $f : X \times Y \rightarrow \overline{\mathbb{R}}$ is $(\mathcal{A} \otimes \mathcal{B}(Y), \mathcal{B}(\overline{\mathbb{R}}))$ -measurable. Suppose further that there exists $\xi \in L_1(X, \mathcal{A}, \mu)$ such that $\xi(x) \leq f(x, y)$ for all $(x, y) \in X \times Y$. Then \underline{f} satisfies $\varphi(x) \leq \underline{f}(x, y)$ for all $(x, y) \in X \times Y$ and for some $\varphi \in L_1(X, \mathcal{A}, \mu)$. In addition, $\underline{f}(x, \cdot)$ is lower semicontinuous on Y for every $x \in X$, and, by virtue of Theorem, \underline{f} is $(\mathcal{A} \otimes \mathcal{B}(Y), \mathcal{B}(\overline{\mathbb{R}}))$ -measurable. Consequently, $\underline{f} \in \mathcal{F}$. It follows that if \mathcal{S} is endowed with the narrow topology, for each $\epsilon > 0$ and every $\sigma \in \mathcal{S}$ there exists an open set V in \mathcal{S} containing σ such that

$$I_{\underline{f}}(\nu) \geq I_{\underline{f}}(\sigma) - \epsilon, \quad \text{for all } \nu \in V.$$

REFERENCES

- [1] Balder E.J. *A general approach to lower semicontinuity and lower closure in optimal control theory*, SIAM J. Control Optim. **22** (1984), 570–598.
- [2] Balder E.J. *Generalized equilibrium results for games with incomplete information*, Math. Oper. Res. **13** (1988), 265–276.
- [3] Balder E.J. *On us-convergence of product measures*, Math. Oper. Res. **26** (2001), 494–518.
- [4] Carbonell-Nicolau O., McLean R.P. *On the existence of Nash equilibrium in Bayesian games*, mimeograph, 2013.
- [5] Cohn D.L. *Measure Theory*, Second edition, Birkhäuser/Springer, New York, 2013.

- [6] Sainte-Beuve M.-F. *On the extension of von Neumann-Aumann's theorem*, J. Functional Analysis **17** (1974), 112–129.

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