Costas Poulios
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The fixed point property in a Banach space isomorphic to $c_0$

Costas Poulios

Abstract. We consider a Banach space, which comes naturally from $c_0$ and it appears in the literature, and we prove that this space has the fixed point property for non-expansive mappings defined on weakly compact, convex sets.

Keywords: non-expansive mappings; fixed point property; Banach spaces isomorphic to $c_0$

Classification: Primary 47H10, 47H09, 46B25

1. Introduction

Let $K$ be a weakly compact, convex subset of a Banach space $X$. A mapping $T : K \to K$ is called non-expansive if $\|Tx - Ty\| \leq \|x - y\|$ for any $x, y \in K$. In the case where every non-expansive map $T : K \to K$ has a fixed point, we say that $K$ has the fixed point property. The space $X$ is said to have the fixed point property if every weakly compact, convex subset of $X$ has the fixed point property.

A lot of Banach spaces are known to enjoy the aforementioned property. The earlier results show that uniformly convex spaces have the fixed point property (see [3]) and this is also true for the wider class of spaces with normal structure (see [7]). The classical Banach spaces $\ell_p$, $L_p$ with $1 < p < \infty$ are uniformly convex and hence they have the fixed point property. On the contrary, the space $L_1$ fails this property (see [1]).

The proofs of many positive results depend on the notion of minimal invariant sets. Suppose that $K$ is a weakly compact, convex set, $T : K \to K$ is a non-expansive mapping and $C$ is a nonempty, weakly compact, convex subset of $K$ such that $T(C) \subseteq C$. The set $C$ is called minimal for $T$ if there is no strictly smaller weakly compact, convex subset of $C$ which is invariant under $T$. A straightforward application of Zorn’s lemma implies that $K$ always contains minimal invariant subsets. So, a standard approach in proving fixed point theorems is to first assume that $K$ itself is minimal for $T$ and then use the geometrical properties of the space to show that $K$ must be a singleton. Therefore, $T$ has a fixed point.

Although a non-expansive map $T : K \to K$ does not have to have fixed points, it is well-known that $T$ always has an approximate fixed point sequence. This means that there is a sequence $(x_n)$ in $K$ such that $\lim_{n \to \infty} \|x_n - Tx_n\| = 0$. For such sequences, the following result holds (see [6]).
Theorem 1.1. Let $K$ be a weakly compact, convex set in a Banach space, let $T : K \to K$ be a non-expansive map such that $K$ is $T$-minimal, and let $(x_n)$ be any approximate fixed point sequence. Then, for all $x \in K$, 
\[ \lim_{n \to \infty} \|x - x_n\| = \text{diam}(K). \]

Although from the beginning of the theory it became clear that the classical spaces $\ell_p, L_p$, $1 < p < \infty$ have the fixed point property, the case of $c_0$ remained unsolved for some period of time. The geometrical properties of this space are not very nice, in the sense that $c_0$ does not possess normal structure. However, it was finally proved that the geometry of $c_0$ is still good enough and it does not allow the existence of minimal sets with positive diameter, that is, $c_0$ has the fixed point property. This was done by B. Maurey [8] (see also [4]) who also proved that every reflexive subspace of $L_1$ has the fixed point property.

Theorem 1.2. The space $c_0$ has the fixed point property.

The proof of Theorem 1.2 is based on the fact that the set of approximate fixed point sequences is convex in a natural sense. More precisely, we have the following ([8], [4]).

Theorem 1.3. Let $K$ be a weakly compact, convex subset of a Banach space which is minimal for a non-expansive map $T : K \to K$. Let $(x_n)$ and $(y_n)$ be approximate fixed point sequences for $T$ such that $\lim_{n \to \infty} \|x_n - y_n\|$ exists. Then there is an approximate fixed point sequence $(z_n)$ in $K$ such that 
\[ \lim_{n \to \infty} \|x_n - z_n\| = \lim_{n \to \infty} \|y_n - z_n\| = \frac{1}{2} \lim_{n \to \infty} \|x_n - y_n\|. \]

In the present paper, we define a Banach space $X$ isomorphic to $c_0$ and we prove that this space has the fixed point property. Our interest in this space derives from several reasons. Firstly, the space $X$ comes from $c_0$ in a natural way. In fact, the Schauder basis of $X$ is equivalent to the summing basis of $c_0$. Secondly, the space $X$ is close to $c_0$ in the sense that the Banach-Mazur distance between the two spaces is equal to 2. It is worth mentioning that from the proof of Theorem 1.2 we can conclude that whenever $Y$ is a Banach space isomorphic to $c_0$ and the Banach-Mazur distance between $Y$ and $c_0$ is strictly less than 2, then $Y$ has the fixed point property. In our case, the Banach-Mazur distance is equal to 2, that is the space $X$ lies on the boundary of what is already known. This fact should also be compared with the following question in metric fixed point theory: Find a nontrivial class of Banach spaces invariant under isomorphism such that each member of the class has the fixed point property (a trivial example is the class of spaces isomorphic to $\ell_1$). We shall see that even for spaces close to $c_0$, such as the space $X$, the situation is quite complicated and this points out the difficulty of the aforementioned question. Finally, the space $X$ has been used in several places in the study of the geometry of Banach spaces (for instance see [5], [2]). More precisely, the well-known Hagler Tree space ($HT$) [5] contains
The fixed point property in a Banach space isomorphic to $c_0$ has a plethora of subspaces isometric to $X$. Nevertheless, we do not know if $HT$ has the fixed point property.

2. Definition and basic properties

We consider the vector space $c_{00}$ of all real-valued finitely supported sequences. We let $(e_n)_{n \in \mathbb{N}}$ stand for the usual unit vector basis of $c_{00}$, that is $e_n(i) = 1$ if $i = n$ and $e_n(i) = 0$ if $i \neq n$. If $S \subset \mathbb{N}$ is any interval of integers and $x = (x_i) \in c_{00}$ then we set $S^*(x) = \sum_{i \in S} x_i$. We now define the norm of $x$ as follows

$$\|x\| = \sup |S^*(x)|$$

where the supremum is taken over all finite intervals $S \subset \mathbb{N}$. The space $X$ is the completion of the normed space we have just defined.

It is easily verified that the sequence $(e_n)$ is a normalized monotone Schauder basis for the space $X$. In the following, $(e_n^*)_{n \in \mathbb{N}}$ denotes the sequence of the biorthogonal functionals and $(P_n)_{n \in \mathbb{N}}$ denotes the sequence of the natural projections associated to the basis $(e_n)$. That is, for any $x = \sum_{i=1}^{\infty} x_i e_i \in X$ we have $e_n^*(x) = x_n$ and $P_n(x) = \sum_{i=1}^{n} x_i e_i$. Furthermore, if $S \subset \mathbb{N}$ is any interval of integers (not necessarily finite), we define the functional $S^*: X \to \mathbb{R}$ by $S^*(x) = S^*(\sum_{i=1}^{\infty} x_i e_i) = \sum_{i \in S} x_i$. It is easy to see that $S^*$ is a bounded linear functional with $\|S^*\| = 1$. In the special case where $S = \mathbb{N}$, the corresponding functional is denoted by $B^*$ (instead of the confusing $N^*$). Therefore, $B^*(x) = \sum_{i=1}^{\infty} x_i$ for any $x = \sum_{i=1}^{\infty} x_i e_i \in X$.

The following proposition provides some useful properties of the space $X$ and demonstrates the relation between $X$ and $c_0$. We remind that for any pair $E, F$ of isomorphic normed spaces, the Banach-Mazur distance between $E$ and $F$ is defined as follows

$$d(E, F) = \inf \{\|T\| \cdot \|T^{-1}\| \mid T : E \to F \text{ is an isomorphism from } E \text{ onto } F\}.$$ 

**Proposition 2.1.** The following holds.

1. The space $X$ is isomorphic to $c_0$ and in particular the basis of $X$ is equivalent to the summing basis of $c_0$.

2. The subspace of $X^*$ generated by the sequence of the biorthogonal functionals has codimension 1. More precisely, $X^* = \overline{\text{span}}\{e_n^*\}_{n \in \mathbb{N}} \oplus \langle B^* \rangle$.

3. The Banach-Mazur distance $d(X, c_0)$ between $X$ and $c_0$ is equal to 2.

**Proof:** We define the linear operator

$$\Phi : X \to c_0$$

$$x = (x_i) \mapsto \left( \sum_{i=1}^{\infty} x_i, \sum_{i=2}^{\infty} x_i, \ldots \right).$$

It is easily verified that $\Phi$ is an isomorphism from $X$ onto $c_0$ with $\|\Phi\| = 1$, $\|\Phi^{-1}\| = 2$ and $\Phi$ maps the basis of $X$ to the summing basis of $c_0$. This proves...
the first assertion. The second assertion is an immediate consequence of the relation between $X$ and $c_0$ established above.

It remains to show that the Banach-Mazur distance $d = d(X, c_0)$ is equal to 2. Firstly, we observe that the isomorphism $\Phi$ defined above implies that $d \leq 2$. In order to prove the reverse inequality we fix a real number $\epsilon > 0$. Then there exists an isomorphism $T : X \to c_0$ from $X$ onto $c_0$ such that $\|x\| \leq \|Tx\|_{c_0} \leq (d + \epsilon)\|x\|$ for any $x \in X$. We now consider the normalized sequence $(x_n)$ in $X$ where $x_n = (x_n(i))_{i \in \mathbb{N}}$ is defined by

$$x_n(2n - 1) = -1, \quad x_n(2n) = 1, \quad x_n(i) = 0 \text{ otherwise.}$$

The description of $X^*$ given by the second assertion implies that any bounded sequence $(t_n)_{n \in \mathbb{N}}$ of elements of $X$ converges weakly to 0 if and only if $e_n^* (t_n) \to 0$ for every $m \in \mathbb{N}$ and $B^*(t_n) \to 0$. It follows that the sequence $(x_n)_{n \in \mathbb{N}}$ defined above is weakly null. Now we set $y_n = T(x_n)$ for any $n \in \mathbb{N}$ and we have

$$1 \leq \|y_n\|_{c_0} \leq d + \epsilon \quad \text{and} \quad (y_n)_{n \in \mathbb{N}} \text{ converges weakly to 0.}$$

Therefore, we find $k_1 \in \mathbb{N}$ such that the vectors $y_1$ and $y_{k_1}$ have essentially disjoint supports. More precisely, since $y_1 \in c_0$, there exists $N_1 \in \mathbb{N}$ such that $y_1(i) < \epsilon$ for any $i > N_1$. Since $y_n \to 0$ weakly, we find $k_1$ so that $|y_{k_1}(i)| < \epsilon$ for any $i \leq N_1$. It follows that $\|y_1 - y_{k_1}\|_{c_0} \leq \max\{\|y_1\|_{c_0}, \|y_{k_1}\|_{c_0}\} + \epsilon \leq d + 2\epsilon$. On the other hand, $\|x_1 - x_{k_1}\| = 2$. Therefore,

$$2 = \|x_1 - x_{k_1}\| \leq \|y_1 - y_{k_1}\|_{c_0} \leq d + 2\epsilon.$$

If $\epsilon$ tends to 0, we obtain $2 \leq d$ as we desire. \hfill $\square$

3. The fixed point property

This section is entirely devoted to the proof of the fixed point property for the space $X$. First we need to establish some notation. If $S, S' \subseteq \mathbb{N}$ are intervals we write $S < S'$ to mean that $\max S < \min S'$. Moreover, if $k \in \mathbb{N}$, we write $k < S$ (resp., $S < k$) to mean $k < \min S$ (resp., $\max S < k$). Finally, for any $x = (x_i) \in X$, $\text{supp}(x) = \{i \in \mathbb{N} \mid x_i \neq 0\}$ denotes the support of $x$.

**Theorem 3.1.** The space $X$ has the fixed point property.

**Proof:** We follow the standard approach. We assume that $K$ is a weakly compact, convex subset of $X$ which is minimal for a non-expansive map $T : K \to K$. Using the geometry of the space $X$, we have to show that $K$ is a singleton, that is $\text{diam}(K) = 0$. Let us suppose that $\text{diam}(K) > 0$ and now we have to reach a contradiction. Without loss of generality we may assume that $\text{diam}(K) = 1$.

Let $(x_n)_{n \in \mathbb{N}}$ be an approximate fixed point sequence for the map $T$ in the set $K$. By passing to a subsequence and then using some translation, we may assume that $0 \in K$ and $(x_n)$ converges weakly to 0. Theorem 1.1 implies that $\lim_n \|x_n\| = \text{diam}(K) = 1$. 

Furthermore, using a standard perturbation argument we may assume that
\((x_n)\) is a finitely supported approximate fixed point sequence. Indeed, we inductively construct a subsequence \((x_{q_n})\) of \((x_n)\) and integers \(l_0 = 0 < l_1 < l_2 < \ldots\) such that for every \(n \in \mathbb{N}\), \(\|P_{l_{n-1}}(x_{q_n})\| < 1/n\) and \(\|x_{q_n} - P_{l_n}(x_{q_n})\| < 1/n\). We start with \(x_{q_1} = x_1\) and \(l_0 = 0\). Suppose that \(q_1 < q_2 < \ldots < q_n\) and \(l_0 < l_1 < \ldots < l_{n-1}\) have been defined. Then there exists \(l_n > l_{n-1}\) such that \(\|x_{q_n} - P_{l_n}(x_{q_n})\| < 1/n\). Since \((x_n)\) is weakly null, it follows that \(P_{l_n}(x_n) \to 0\) for every \(m \in \mathbb{N}\). Therefore, there exists \(q_{n+1} > q_n\) such that \(\|P_{l_n}(x_{q_{n+1}})\| < \frac{1}{n+1}\). The construction of \((x_{q_n})\) and \((l_n)\) is complete. Consequently, by passing to the subsequence \((x_{q_n})\) and perturbing \((x_{q_n})\), if necessary, we may assume that for the original sequence \((x_n)\) we have \(\text{supp}(x_n) \subset (l_{n-1}, l_n)\) for every \(n \in \mathbb{N}\), that is, \((x_n)\) consists of finitely supported vectors.

We next consider the subsequences \((z_n) = (x_{2n-1})\) and \((y_n) = (x_{2n})\) and we also set \(l_{2n-1} = k_n\), \(l_{2n} = m_n\) for every \(n \in \mathbb{N}\) and \(m_0 = l_0\). The properties of the sequence \((x_n)\) imply that the following holds.

1. \((z_n)\) and \((y_n)\) are approximate fixed point sequences for the map \(T\) and
\[
\lim \|z_n\| = \lim \|y_n\| = 1.
\]
2. \((z_n)\) and \((y_n)\) converge weakly to 0.
3. \(\text{supp}(z_n) \subset (m_{n-1}, k_n)\) and \(\text{supp}(y_n) \subset (k_n, m_n)\) for every \(n \in \mathbb{N}\).
4. \(\lim \|z_n - y_n\| = 1\).

In order to justify the fourth conclusion, we first observe that \(\limsup \|z_n - y_n\| \leq \text{diam}(K) = 1\). On the other hand, by the definition of the norm of the space \(X\), for every \(n \in \mathbb{N}\) there exists a finite interval \(E_n \subset \mathbb{N}\) such that \(\|z_n\| = |E_n^r(z_n)|\). Clearly we may assume that \(E_n \subset (m_{n-1}, k_n)\). Then \(\|z_n - y_n\| \geq |E_n^r(z_n - y_n)| = \|z_n\|\). Since \(\lim \|z_n\| = 1\), it emerges that \(\liminf \|z_n - y_n\| \geq 1\) and finally \(\lim \|z_n - y_n\| = 1\).

We are ready now to apply Maurey’s theorem (Theorem 1.3). To this end, we fix a positive integer \(N \geq 4\), we set \(\epsilon = 2^{-N}\) and we iteratively use Theorem 1.3 as follows. Firstly, we consider the sequences \((z_n)\) and \((y_n)\). Applying Theorem 1.3 we obtain an approximate fixed point sequence \((v_n^1)_{n \in \mathbb{N}}\) in the set \(K\) such that \(\lim \|v_n^1 - y_n\| = \frac{1}{2} \lim \|z_n - y_n\| = \frac{1}{2}\) and \(\lim \|v_n^1 - z_n\| = \frac{1}{2} \lim \|z_n - y_n\| = \frac{1}{2}\). Assume now that in the \(i\)-th step of this procedure we find an approximate fixed point sequence \((v_n^i)_{n \in \mathbb{N}}\) satisfying \(\lim \|v_n^i - z_n\| = 2^{-i}\) and \(\lim \|v_n^i - y_n\| = 1 - 2^{-i}\). Then, Theorem 1.3 implies that “halfway” between \((z_n)\) and \((v_n^i)\) there exists an approximate fixed point sequence \((v_n^{i+1})_{n \in \mathbb{N}}\), that is, \(\lim \|v_n^{i+1} - v_n^i - z_n\| = \frac{1}{2} \lim \|v_n^i - z_n\| = 2^{-(i+1)}\) and \(\lim \|v_n^{i+1} - z_n\| = \frac{1}{2} \lim \|v_n^i - z_n\| = 2^{-(i+1)}\). Now, we estimate the distance between \(v_n^{i+1}\) and \(y_n\). We have
\[
\|v_n^{i+1} - y_n\| \leq \|v_n^{i+1} - v_n^i\| + \|v_n^i - y_n\| \quad \text{and}
\|v_n^{i+1} - y_n\| \geq \|z_n - y_n\| - \|v_n^{i+1} - z_n\|.
\]
Therefore, it follows that \(\lim \|v_n^{i+1} - y_n\| = 1 - 2^{-(i+1)}\). After \(N\) iterated applications of Theorem 1.3 we find a sequence \((v_n)_{n \in \mathbb{N}} = (v_n^N)_{n \in \mathbb{N}}\) in the set \(K\) satisfying
the following: \((v_n)\) is an approximate fixed point sequence for the map \(T\) (which implies that \(\lim \|v_n\| = 1\)) and further \(\lim \|v_n - z_n\| = \epsilon\) and \(\lim \|v_n - y_n\| = 1 - \epsilon\). Therefore, for all sufficiently large \(n \in \mathbb{N}\) the following holds:

1. \(\|v_n\| > 1 - \frac{\epsilon}{2}\);
2. \(\|v_n - z_n\| < 3\epsilon/2\) and \(\|v_n - y_n\| < 1 - \frac{\epsilon}{2}\);
3. \(|B^*(z_n)| < \epsilon/2\) (since \((z_n)\) is weakly null).

We also set \(S_n = (m_{n-1}, k_n)\) so that we have \(S_1 < S_2 < \ldots\) Concerning the sequence \((v_n)\) in the set \(K\) and the sequence of intervals \((S_n)\) we prove the following two claims.

**Claim 1.** For all sufficiently large \(n\), the support of \(v_n\) is essentially contained in the interval \(S_n\), in the sense that if \(S\) is any interval with \(S \cap S_n = \emptyset\) then \(|S^*(v_n)| < 3\epsilon/2\).

Indeed, we know that \(\text{supp}(z_n) \subset (m_{n-1}, k_n) = S_n\). Therefore, if \(S\) is any interval with \(S \cap S_n = \emptyset\) then \(S^*(z_n) = 0\) and hence

\[
|S^*(v_n)| = |S^*(v_n - z_n)| \leq |v_n - z_n| < \frac{3\epsilon}{2}.
\]

**Claim 2.** For all sufficiently large \(n\), there exist intervals \(L_n < R_n\) such that \(S_n = L_n \cup R_n\) and \(L_n^*(v_n) < -1 + 7\epsilon\), \(R_n^*(v_n) > 1 - 2\epsilon\).

We fix a sufficiently large positive integer \(n\). Since \(\|v_n\| > 1 - \frac{\epsilon}{2}\), it follows that there exists a finite interval \(F_n \subset \mathbb{N}\) such that \(|F_n^*(v_n)| > 1 - \frac{\epsilon}{2}\). If \(k_n < F_n\), we know by the previous claim that \(|F_n^*(v_n)| < 3\epsilon/2\), which is a contradiction. Moreover, if we assume that \(F_n \leq k_n\) then \(F_n \cap (k_n, m_n] = \emptyset\) and the choice of \((y_n)\) implies \(F_n^*(y_n) = 0\). Thus,

\[
|F_n^*(v_n)| = |F_n^*(v_n - y_n)| \leq |v_n - y_n| < 1 - \frac{\epsilon}{2},
\]

which is also a contradiction. By this discussion it is clear that \(\min F_n \leq k_n < \max F_n\). Now we set \(R_n = F_n \cap [1, k_n]\) and we estimate

\[
1 - \frac{\epsilon}{2} < |F_n^*(v_n)| \leq |R_n^*(v_n)| + |(F_n \setminus R_n)^*(v_n)| < |R_n^*(v_n)| + \frac{3\epsilon}{2},
\]

where the last inequality follows from Claim 1. Therefore, \(|R_n^*(v_n)| > 1 - 2\epsilon\). Passing to a subsequence, we may assume that either \(R_n^*(v_n) > 1 - 2\epsilon\) for all sufficiently large \(n\) or \(R_n^*(v_n) < -1 + 2\epsilon\) for all sufficiently large \(n\). We suppose that the first possibility happens, as the second one is treated similarly (by interchanging the roles of \(L_n\) and \(R_n\)). Consequently, for the interval \(R_n\) we have \(\max R_n = k_n\) and \(R_n^*(v_n) > 1 - 2\epsilon\).

On the other hand, we observe that

\[
|B^*(v_n)| \leq |B^*(v_n - z_n)| + |B^*(z_n)| \leq |v_n - z_n| + \frac{\epsilon}{2} < 2\epsilon.
\]
We note that the sequence \((v_n)\) is not necessarily weakly null. However, \(v_n\) is close to \(z_n\) and hence \(|B^*(v_n)|\) is very small. We next set \(G_n = [1, \min R_n]\) (possibly empty) and \(W_n = (k_n, +\infty)\). Then,

\[
2\epsilon > |B^*(v_n)| = |G_n^*(v_n) + R_n^*(v_n) + W_n^*(v_n)| \\
\geq R_n^*(v_n) - |G_n^*(v_n)| - |W_n^*(v_n)| \\
> 1 - 2\epsilon - |G_n^*(v_n)| - \frac{3\epsilon}{2}.
\]

Therefore \(G_n\) is non-empty and \(|G_n^*(v_n)| > 1 - \frac{11\epsilon}{2}\). However, if \(G_n^*(v_n) > 1 - \frac{11\epsilon}{2}\), then it would follow

\[
|B^*(v_n)| \geq R_n^*(v_n) + G_n^*(v_n) - |W_n^*(v_n)| \geq 2 - 9\epsilon,
\]

which is a contradiction. Hence, \(G_n^*(v_n) < -1 + \frac{11\epsilon}{2}\). Further, we observe that we cannot have \(G_n < S_n\), since in this case it would follow \(|G_n^*(v_n)| < \frac{4\epsilon}{2}\). Consequently, \(G_n > m_n - 1\) which clearly implies \(\min R_n > m_n - 1 + 1\). Finally, we set \(L_n = G_n \cap (m_n - 1, k_n]\) and we estimate

\[
-1 + \frac{11\epsilon}{2} > G_n^*(v_n) = L_n^*(v_n) + (G_n \setminus L_n)^*(v_n) \geq L_n^*(v_n) - \frac{3\epsilon}{2}.
\]

We deduce that \(L_n^*(v_n) < -1 + 7\epsilon\). Therefore, the intervals \(L_n < R_n\) satisfy the following: \(S_n = L_n \cup R_n\), \(R_n^*(v_n) > 1 - 2\epsilon\) and \(L_n^*(v_n) < -1 + 7\epsilon\). The proof of the claim is now complete.

Using the construction and the properties of the sequences \((v_n)\) and \((S_n)\), we can reach the final contradiction and finish the proof of the theorem. Our goal is to show that for all sufficiently large \(n \in \mathbb{N}\), \(||v_n - v_{n+1}|| \geq 5/4 > 1\), contradicting the assumption \(\text{diam}(K) = 1\). Indeed, we fix a sufficiently large \(n \in \mathbb{N}\) and we consider the intervals \(D = (k_n, m_n]\) and \(S = R_n \cup D \cup L_{n+1}\). Then, using Claim 1 and Claim 2 we have

\[
S^*(v_n) = R_n^*(v_n) + (D \cup L_{n+1})^*(v_n) > 1 - 2\epsilon - \frac{3\epsilon}{2} = 1 - \frac{7\epsilon}{2} \\
S^*(v_{n+1}) = (R_n \cup D)^*(v_n) + L_{n+1}^*(v_{n+1}) < \frac{3\epsilon}{2} - 1 + 7\epsilon = -1 + \frac{17\epsilon}{2}.
\]

Therefore,

\[
||v_n - v_{n+1}|| \geq |S^*(v_n - v_{n+1})| = |S^*(v_n) - S^*(v_{n+1})| \geq 2 - 12\epsilon.
\]

The choice of \(\epsilon\) implies that \(||v_n - v_{n+1}|| \geq 5/4 > 1\) for all sufficiently large \(n \in \mathbb{N}\), hence we obtain the desired contradiction.

\[\square\]

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ATHENS, 15784 ATHENS, GREECE
E-mail: k-poulios@math.uoa.gr

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