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CONTINUUM SPECTRUM FOR THE LINEARIZED EXTREMAL EIGENVALUE PROBLEM WITH BOUNDARY REACTIONS

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Abstract. We study the semilinear problem with the boundary reaction

\[-\Delta u + u = 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = \lambda f(u) \quad \text{on } \partial \Omega,\]

where \( \Omega \subset \mathbb{R}^N, \, N \geq 2, \) is a smooth bounded domain, \( f: [0, \infty) \to (0, \infty) \) is a smooth, strictly positive, convex, increasing function which is superlinear at \( \infty, \) and \( \lambda > 0 \) is a parameter. It is known that there exists an extremal parameter \( \lambda^* > 0 \) such that a classical minimal solution exists for \( \lambda < \lambda^*, \) and there is no solution for \( \lambda > \lambda^*. \) Moreover, there is a unique weak solution \( u^* \) corresponding to the parameter \( \lambda = \lambda^*. \) In this paper, we continue to study the spectral properties of \( u^* \) and show a phenomenon of continuum spectrum for the corresponding linearized eigenvalue problem.

Keywords: continuum spectrum; extremal solution; boundary reaction

MSC 2010: 35J25, 35J20

1. Introduction

In this paper, we consider the boundary value problem with the boundary reaction

\[-\Delta u + u = 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = \lambda f(u) \quad \text{on } \partial \Omega,\]

where \( \lambda > 0 \) and \( \Omega \subset \mathbb{R}^N, \, N \geq 2, \) is a smooth bounded domain. Throughout the paper, we assume

\[ f: [0, \infty) \to (0, \infty) \text{ is smooth, convex, increasing, } f(0) > 0, \]

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and superlinear at \( \infty \) in the sense that

\[
(1.3) \quad \lim_{t \to \infty} \frac{f(t)}{t} = \infty.
\]

Then the maximum principle implies that solutions are positive on \( \overline{\Omega} \).

It is known that there exists an extremal parameter \( \lambda^* \in (0, \infty) \) such that

(i) for every \( \lambda \in (0, \lambda^*) \), \( (1.1)_{\lambda} \) has a positive, classical, minimal solution \( u_{\lambda} \in C^2(\overline{\Omega}) \) which is strictly stable in the sense that

\[
(1.4) \quad \int_{\Omega} (|\nabla \varphi|^2 + \varphi^2) \, dx > \int_{\partial \Omega} f'(u_{\lambda})\varphi^2 \, ds
\]

for every \( \varphi \in C^1(\overline{\Omega}) \), \( \varphi \neq 0 \),

(ii) for \( \lambda = \lambda^* \), the pointwise limit

\[
(1.5) \quad u^*(x) = \lim_{\lambda \uparrow \lambda^*} u_{\lambda}(x), \quad x \in \overline{\Omega},
\]

becomes a weak solution of \( (1.1)_{\lambda^*} \),

(iii) for \( \lambda > \lambda^* \), there exists no solution of \( (1.1)_{\lambda} \), not even in the weak sense.

Here, we call a function \( u = (u_1, u_2) \in L^1(\Omega) \times L^1(\partial \Omega) \) a weak solution to \( (1.1)_{\lambda} \) if \( f(u_2) \in L^1(\partial \Omega) \) and

\[
(1.6) \quad \int_{\Omega} (-\Delta \zeta + \zeta) u_1 \, dx = \int_{\partial \Omega} \left( \lambda f(u_2)\zeta - \frac{\partial \zeta}{\partial \nu} u_2 \right) \, ds
\]

holds for any \( \zeta \in C^2(\overline{\Omega}) \). The statement (ii) says that, under the assumption (1.3), \( u^* = (u^*|_\Omega, u^*|_{\partial \Omega}) \) is a weak solution in the above sense. We proved in [10] Theorem 11 that \( u^* \in W^{1,\gamma}(\Omega) \) for all \( \gamma \in [1, N/(N-2)) \) when \( N \geq 3 \) (for any \( \gamma \in [1, \infty) \) when \( N = 2 \), so \( u^*|_{\partial \Omega} \in W^{1-1/\gamma,\gamma}(\partial \Omega) \subset L^{(N-1)\gamma/(N-\gamma)}(\partial \Omega) \) is the usual trace of the \( W^{1,\gamma} \) function \( u^* \) on \( \partial \Omega \). For the facts (ii), (iii), we refer the reader to [10]. In the following, we call \( u^* \) the extremal solution of \( (1.1) \). In [10], the author obtained several properties such as regularity and uniqueness of the extremal solution \( u^* \). This paper is a sequel to [10]. For related elliptic problems with boundary reaction terms, see, e.g., [4], [6], [9]. For a well-studied problem

\[
-\Delta u = \lambda f(u) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega
\]

where \( f \) satisfies (1.2), (1.3), see [1], [2], [3], [5], [7], [8], and the references therein.

For \( \lambda \in (0, \lambda^*) \), we denote by \( \mu_1(\lambda f'(u_{\lambda})) \) the first eigenvalue of the eigenvalue problem

\[
-\Delta \varphi + \varphi = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial \varphi}{\partial \nu} = \lambda f'(u_{\lambda})\varphi + \mu \varphi \quad \text{on} \quad \partial \Omega.
\]

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By the variational characterization, we have
\[
\mu_1(\lambda f'(u_\lambda)) = \inf_{\varphi \in C^1(\Omega), \varphi \neq 0} \frac{\int_\Omega (|\nabla \varphi|^2 + \varphi^2) \, dx - \int_{\partial \Omega} \lambda f'(u_\lambda) \varphi^2 \, ds_x}{\int_{\partial \Omega} \varphi^2 \, ds_x}.
\]
Note that \(\mu_1(\lambda f'(u_\lambda)) > 0\) since the minimal solution \(u_\lambda\) is strictly stable, and decreases as \(\lambda \uparrow \lambda^*\). Denote
\[
(1.7) \quad \mu^*_1 = \lim_{\lambda \uparrow \lambda^*} \mu_1(\lambda f'(u_\lambda)).
\]
If \(u^*\) is classical, it must hold that \(\mu^*_1 = 0\) by considering (iii) above. However, if \(u^* = (u^*|_{\Omega}, u^*|_{\partial \Omega}) \notin L^\infty(\Omega) \times L^\infty(\partial \Omega)\), it could happen that \(\mu^*_1\) is positive. In [10], we proved that even when \(\mu^*_1 > 0\), there exists a nonnegative weak solution of
\[
(1.8) \quad -\Delta \varphi + \varphi = 0 \quad \text{in } \Omega, \quad \frac{\partial \varphi}{\partial \nu} = \lambda^* f'(u^*) \varphi + \mu \varphi \quad \text{on } \partial \Omega
\]
for \(\mu = 0\). This is a phenomenon of the existence of \((L^1)\)-zero eigenvalue for the eigenvalue problem (1.8). The main purpose of this paper is to prove the following result, which is a generalization of the result by Cabré and Martel [3] to our setting, and may be seen as a phenomenon of the existence of \((L^1)\)-continuum spectrum for the eigenvalue problem (1.8).

**Theorem 1.1.** Let \(\mu^*_1\) be defined by (1.7). Then for any \(\mu \in [0, \mu^*_1]\) there exists a weak solution \(\varphi\) to (1.8), \(\varphi \in W^{1,q}(\Omega)\) \((1 \leq q < N/(N-1))\), \(\varphi \geq 0\), in the sense that \(f'(u^*)\varphi|_{\partial \Omega} \in L^1(\partial \Omega)\) and
\[
\int_\Omega (-\Delta \zeta + \zeta) \varphi \, dx = \int_{\partial \Omega} \left\{ (\lambda^* f'(u^*) \varphi + \mu \varphi) \zeta - \frac{\partial \zeta}{\partial \nu} \varphi \right\} \, ds_x
\]
for all \(\zeta \in C^2(\Omega)\). Here \(\varphi|_{\partial \Omega}\) is the usual trace of \(\varphi \in W^{1,q}(\Omega)\).

**2. Proof of Theorem 1.1**

In this section we prove Theorem 1.1. We need the uniqueness theorem from [10], which is an analogue of the result by Y. Martel [8].
Theorem 2.1 ([10], Theorem 14). Assume \((1.1)_\lambda\) has a weak supersolution \(w = (w_1, w_2) \in L^1(\Omega) \times L^1(\partial \Omega)\), in the sense that \(f(w_2) \in L^1(\partial \Omega)\) and

\[
\int_{\Omega} (-\Delta \zeta + \zeta) w_1 \, dx \geq \int_{\partial \Omega} \left\{ \lambda^* f(w_2) \zeta - \frac{\partial \zeta}{\partial \nu} w_2 \right\} \, ds_x
\]

for any \(\zeta \in C^2(\overline{\Omega}), \zeta \geq 0\) on \(\overline{\Omega}\). Then \((w_1, w_2) = (u^*_|\Omega, u^*_|\partial \Omega)\), where \(u^*\) is defined by \((1.5)\).

The following is Lemma 17 in [10].

Lemma 2.2. Let \(\{u_n\} \subset C^2(\overline{\Omega})\) be a sequence of functions such that

\[-\Delta u_n + u_n = 0 \quad \text{in} \ \Omega, \quad \frac{\partial u_n}{\partial \nu} \geq 0 \quad \text{on} \ \partial \Omega.\]

Assume \(\|u_n\|_{L^1(\partial \Omega)} \leq C\) for some \(C > 0\) independent of \(n\). Then there exists a subsequence (denoted again by \(u_n\)) and \(u \in W^{1,q}(\Omega)\) such that

\[u_n \rightharpoonup u \quad \text{weakly in} \ W^{1,q}(\Omega), \quad 1 < q < \frac{N}{N-1}.\]

Moreover, for any \(1 \leq p < (N-1)/(N-2)\) there exists a constant \(C_p > 0\) depending only on \(p\) such that

\[\|u_n\|_{L^p(\partial \Omega)} \leq C_p \|u_n\|_{L^1(\partial \Omega)} \quad \text{for any} \ n \in \mathbb{N}.\]

Now, we prove Theorem 1.1.

Proof. We follow the argument by X. Cabré and Y. Martel [3].

Step 1. For \(n \in \mathbb{N}\), define a sequence of functions \(f_n\) as

\[f_n(s) = \begin{cases} f(s) & \text{if} \ s \leq n, \\ f(n) + f'(n)(s-n) & \text{if} \ s > n, \end{cases}\]

and consider the approximated problem

\[(2.1) \quad -\Delta u + u = 0 \quad \text{in} \ \Omega, \quad \frac{\partial u}{\partial \nu} = \lambda f_n(u) \quad \text{on} \ \partial \Omega.\]

Denote \(\lambda^*_n = \sup \{\lambda > 0 : (2.1)_\lambda \text{ admits a minimal solution } \in C^2(\overline{\Omega})\}\), and let \(u_{n,\lambda} \in C^2(\overline{\Omega})\) be the classical minimal solution to \((2.1)_\lambda\) for \(\lambda < \lambda^*_n\). Since \(f_n \leq f_{n+1} \leq f\), we have \(u_{n,\lambda} \leq u_{n+1,\lambda} \leq u_\lambda\) and \(\lambda^* \leq \lambda^*_n \leq \lambda^*_n\) for any \(n \in \mathbb{N}\). Define

\[(2.2) \quad \mu_1(\lambda f_n'(u_{n,\lambda})) = \inf_{\varphi \in C^1(\overline{\Omega}), \varphi \neq 0} \frac{\int_{\Omega} (|\nabla \varphi|^2 + \varphi^2) \, dx - \int_{\partial \Omega} \lambda f_n'(u_{n,\lambda}) \varphi^2 \, ds_x}{\int_{\partial \Omega} \varphi^2 \, ds_x}.
\]
Note that \( \mu_1(\lambda f'_n(u_{n,\lambda})) \) is continuous with respect to \( \lambda \) by (2.2). Take \( 0 \leq \mu \leq \mu_1^* \) where \( \mu_1^* \) is defined by (1.7). Since \( u_{n,\lambda_n} \) is classical (which is because \( f_n \) is asymptotically linear) and there is no classical solution of (2.1) for \( \lambda > \lambda^*_n \), the linearized problem around \((\lambda^*_n, u_{n,\lambda_n})\) must have zero eigenvalue. Thus

\[
\mu_1(\lambda^*_n f'_n(u_{n,\lambda}^*)) = 0 \leq \mu \leq \mu_1^* \leq \mu_1(\lambda f'_n(u_{n,\lambda}^*));
\]

here we have used the fact that \( f'_n \leq f' \) and \( u_{n,\lambda} \leq u_{\lambda} \), which implies \( \mu_1(\lambda f'_n(u_{n,\lambda})) \leq \mu_1(\lambda f'_n(u_{n,\lambda}^*)) \). By the Intermediate Value Theorem, there exists \( \lambda_n \in [\lambda^*, \lambda^*_n] \) such that

\[
\mu_1(\lambda_n f'_n(u_{n,\lambda_n})) = \mu,
\]

which in turn implies there exists \( \varphi_n > 0 \) with \( \int_{\partial \Omega} \varphi_n \, ds_x = 1 \) such that

(2.3) \[-\Delta \varphi_n + \varphi_n = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial \varphi_n}{\partial \nu} = \lambda_n f'_n(u_{n,\lambda_n})\varphi_n + \mu \varphi_n \quad \text{on} \quad \partial \Omega.\]

Recall also that \( u_{n,\lambda_n} \) satisfies

(2.4) \[-\Delta u_{n,\lambda_n} + u_{n,\lambda_n} = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial u_{n,\lambda_n}}{\partial \nu} = \lambda_n f_n(u_{n,\lambda_n}) \quad \text{on} \quad \partial \Omega.
\]

We claim there exists \( n_0 \in \mathbb{N} \) such that

(2.5) \[\|u_{n,\lambda_n}\|_{L^1(\partial \Omega)} \leq C \quad \text{for any} \quad n \geq n_0.\]

Indeed, let \( \psi_1 \) be the first eigenfunction of the Steklov type eigenvalue problem

(2.6) \[-\Delta \psi_1 + \psi_1 = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial \psi_1}{\partial \nu} = \kappa_1 \psi_1 \quad \text{on} \quad \partial \Omega
\]

with the first eigenvalue \( \kappa_1 \), which is normalized as \( \int_{\partial \Omega} \psi_1 \, ds_x = 1 \). Multiplying (2.4) by \( \psi_1 \) and using Jensen’s inequality for \( f_n \), we obtain

\[
\kappa_1 \int_{\partial \Omega} \psi_1 u_{n,\lambda_n} \, ds_x = \lambda_n \int_{\partial \Omega} f_n(u_{n,\lambda_n}) \psi_1 \, ds_x \\
\geq \lambda_n f_n \left( \int_{\partial \Omega} \psi_1 u_{n,\lambda_n} \, ds_x \right) \\
\geq \lambda^* f_n \left( \int_{\partial \Omega} \psi_1 u_{n,\lambda_n} \, ds_x \right).
\]

Put \( a_n = \int_{\partial \Omega} \psi_1 u_{n,\lambda_n} \, ds_x \). Then we have

(2.7) \[a_n \geq \frac{\lambda^*}{\kappa_1} f_n(a_n).\]
Assume by contradiction that \( f_n(a_n) = f'(n)(a_n - n) + f(n) \) for some \( n \in \mathbb{N} \) sufficiently large. Then, since \( a_n > n \) and \( f(n) > (\kappa_1/\lambda^*)n \), \( f'(n) > (\kappa_1/\lambda^*) \) for \( n \) sufficiently large by (1.2) and (1.3), we have, by (2.7),

\[
\begin{align*}
\lambda^* f_n(a_n) &= \frac{\lambda^*}{\kappa_1} f'(n)(a_n - n) + f(n) \\
&> a_n - n + n = a_n,
\end{align*}
\]

which is a contradiction. Thus we conclude there exists \( n_0 \in \mathbb{N} \) such that \( f_n(a_n) = f(a_n) \) for any \( n \geq n_0 \). Again, this and (2.7) imply \( a_n \geq (\lambda^*/\kappa_1)f(a_n) \) for any \( n \geq n_0 \). Now, by the assumption on \( f_n \), we have \( C > 0 \) such that \( f(s) \geq (2\kappa_1/\lambda^*)s - C \) holds for any \( s > 0 \). From this and the former estimate, we have \( a_n \leq (\lambda^*/\kappa_1)C \) for \( n \geq n_0 \). This implies the claim (2.5).

Step 2. By (2.5), we have \( \| u_{n,\lambda_n} \|_{L^q(\partial \Omega)} \leq C \) for some \( C \) independent of \( n \). Also recall that \( \| \varphi_n \|_{L^1(\partial \Omega)} = 1 \) for a solution \( \varphi_n \) of (2.3). Thus we can apply Lemma 2.2 and the trace Sobolev embedding to obtain \( w, \varphi \in L^1(\Omega), \varphi \geq 0 \) a.e. satisfying

\[
\begin{align*}
u_n, \lambda_n &\rightarrow w, \quad \varphi_n \rightharpoonup \varphi \quad \text{weakly in } W^{1,q}(\Omega), \\
u_n, \lambda_n &\rightarrow w, \quad \varphi_n \rightarrow \varphi \quad \text{strongly in } L^p(\partial \Omega) \text{ and a.e. on } \partial \Omega
\end{align*}
\]

for any \( 1 < q < N/(N-1) \) and \( 1 \leq p < (N-1)/(N-2) \). Since \( \int_{\partial \Omega} \varphi \, ds_x = 1 \), we see \( \varphi \neq 0 \) on \( \partial \Omega \).

In the following, we prove that \( \lambda_n \downarrow \lambda^* \) as \( n \rightarrow \infty \) and \( w = u^* \). We will show that \( w \in W^{1,q}(\Omega) \) is a weak supersolution in the sense of Theorem 2.1. Then the conclusion is obtained by Theorem 2.1. To prove that \( w \) is a weak supersolution, put \( \bar{\lambda} = \inf_{n \in \mathbb{N}} \lambda_n \). Since \( \lambda_n \geq \lambda^* \), we have \( \bar{\lambda} \geq \lambda^* \). We observe that

\[
\begin{align*}
\int_{\Omega} (-\Delta \zeta + \zeta) u_{n,\lambda_n} \, dx &= \lambda_n \int_{\partial \Omega} f_n(u_{n,\lambda_n}) \zeta \, ds_x - \int_{\partial \Omega} \frac{\partial \zeta}{\partial \nu} u_{n,\lambda_n} \, ds_x \\
&\geq \bar{\lambda} \int_{\partial \Omega} f_n(u_{n,\lambda_n}) \zeta \, ds_x - \int_{\partial \Omega} \frac{\partial \zeta}{\partial \nu} u_{n,\lambda_n} \, ds_x
\end{align*}
\]

holds for all \( \zeta \in C^2(\overline{\Omega}), \zeta \geq 0 \). Using the fact that \( u_{n,\lambda_n} \rightharpoonup w \) in \( L^1(\Omega) \) or \( L^1(\partial \Omega) \), respectively, and Fatou’s lemma, we have

\[
\begin{align*}
\int_{\Omega} (-\Delta \zeta + \zeta) w \, dx &\geq \bar{\lambda} \int_{\partial \Omega} f(w) \zeta \, ds_x - \int_{\partial \Omega} \frac{\partial \zeta}{\partial \nu} w \, ds_x \\
&\geq \lambda^* \int_{\partial \Omega} f(w) \zeta \, ds_x - \int_{\partial \Omega} \frac{\partial \zeta}{\partial \nu} w \, ds_x, \quad \forall \zeta \in C^2(\overline{\Omega}), \zeta \geq 0.
\end{align*}
\]

This implies also \( f(w) \in L^1(\partial \Omega) \) if we take \( \zeta \equiv 1 \). Thus, we conclude that \( w \) is a weak supersolution to (1.1)_{\lambda^*}.
Step 3. Let $\varphi_n, \varphi$ be as in Step 2. We claim that

$$\lambda_n f_n'(u_{n,\lambda_n}) \varphi_n \to \lambda^* f'(u^*) \varphi$$ strongly in $L^1(\partial \Omega)$

as $n \to \infty$. For the proof, we invoke Vitali’s Convergence Theorem. First, by (2.8), we see

$$\lambda_n f_n'(u_{n,\lambda_n}(x)) \varphi_n(x) \to \lambda^* f'(u^*(x)) \varphi(x) \quad \text{a.e. } x \in \partial \Omega$$

for a subsequence. Next, we prove the uniformly absolute continuity property of the sequence $\{\lambda_n f_n'(u_{n,\lambda_n}) \varphi_n\}_{n \in \mathbb{N}}$. For that purpose, let $A \subset \partial \Omega$ be measurable and $\varepsilon > 0$ be given arbitrary. Since $f_n$ is convex, we have

$$f_n\left(\frac{X_A(x)}{\varepsilon}\right) \geq f_n(u_{n,\lambda_n}(x)) + f'_n(u_{n,\lambda_n}(x))\left(\frac{X_A(x)}{\varepsilon} - u_{n,\lambda_n}(x)\right)$$

a.e. $x \in \partial \Omega$; here $\chi_A$ is the characteristic function of $A$. By (2.3) and (2.4), we have

$$\lambda_n \int_{\partial \Omega} f_n(u_{n,\lambda_n}) \varphi_n \, ds_x = \lambda_n \int_{\partial \Omega} f'_n(u_{n,\lambda_n}) u_{n,\lambda_n} \varphi_n \, ds_x + \mu \int_{\partial \Omega} u_{n,\lambda_n} \varphi_n \, ds_x$$

$$\geq \lambda_n \int_{\partial \Omega} f'_n(u_{n,\lambda_n}) u_{n,\lambda_n} \varphi_n \, ds_x.$$

Also an easy consideration shows that

$$\left\{ f_n\left(\frac{X_A(x)}{\varepsilon}\right) - f(0) \right\} \varphi_n(x) \leq f\left(\frac{1}{\varepsilon}\right) \varphi_n(x) \chi_A(x) \quad \text{a.e. on } \partial \Omega.$$

Thus by (2.10), (2.11) and (2.12), we have

$$\int_{\partial \Omega} f'_n(u_{n,\lambda_n}) \frac{X_A}{\varepsilon} \varphi_n \, ds_x$$

$$\leq \int_{\partial \Omega} f_n\left(\frac{X_A}{\varepsilon}\right) \varphi_n \, ds_x + \int_{\partial \Omega} f'_n(u_{n,\lambda_n}) u_{n,\lambda_n} \varphi_n \, ds_x - \int_{\partial \Omega} f_n(u_{n,\lambda_n}) \varphi_n \, ds_x$$

$$\leq \int_{\partial \Omega} f_n\left(\frac{X_A}{\varepsilon}\right) \varphi_n \, ds_x = \int_{\partial \Omega} \left\{ f_n\left(\frac{X_A}{\varepsilon}\right) - f(0) \right\} \varphi_n \, ds_x + \int_{\partial \Omega} f(0) \varphi_n \, ds_x$$

$$\leq \int_{\partial \Omega} f\left(\frac{1}{\varepsilon}\right) \varphi_n \chi_A \, ds_x + f(0) \leq f\left(\frac{1}{\varepsilon}\right) |A|^{1/p'} \| \varphi_n \|_{L^p(\partial \Omega)} + f(0)$$

$$\leq Cf\left(\frac{1}{\varepsilon}\right) |A|^{1/p'} + f(0)$$

for any $1 \leq p < (N - 1)/(N - 2)$, where $|A|$ denotes the $(N - 1)$ dimensional Hausdorff measure of $A \subset \partial \Omega$ and $p' = p/(p - 1)$. In (2.13) we have used $\| \varphi_n \|_{L^p(\partial \Omega)} \leq C$ for some $C > 0$ independent of $n$ by (2.8). Define

$$\delta(\varepsilon) = \left(\frac{f(0)}{f(1/\varepsilon)C}\right)^{p'}.$$
Then for any \( \varepsilon > 0 \) we obtain \( \int_A f'_n(u_{n,\lambda_n}) \varphi_n \, ds_x \leq 2f(0)\varepsilon \) if \( A \subset \partial \Omega \) satisfies that \( |A| < \delta(\varepsilon) \) by (2.13). This implies the uniform absolute continuity of the sequence \( \{\lambda_n f'_n(u_{n,\lambda_n}) \varphi_n\}_{n \in \mathbb{N}} \). Also for any \( \varepsilon > 0 \), if we take \( E \subset \partial \Omega \) such that \( |\partial \Omega \setminus E| < \delta(\varepsilon) \) where \( \delta(\varepsilon) \) is as above, we obtain that \( \int_{\partial \Omega \setminus E} \lambda_n f'_n(u_{n,\lambda_n}) \varphi_n \, ds_x \leq C \varepsilon \). This implies the uniform integrability of \( \{\lambda_n f'_n(u_{n,\lambda_n}) \varphi_n\}_{n \in \mathbb{N}} \). Therefore, Vitali’s Convergence Theorem ensures the claim (2.9).

By (2.9), we pass to the limit \( n \to \infty \) in the weak formulation of (2.3):

\[
\int_\Omega (-\Delta \zeta + \zeta) \varphi_n \, dx = \int_{\partial \Omega} (\lambda_n f'_n(u_{n,\lambda_n}) + \mu) \varphi_n \zeta - \frac{\partial \zeta}{\partial \nu} \varphi_n \, ds_x, \quad \forall \zeta \in C^2(\overline{\Omega}),
\]

and conclude that \( \varphi \) is a weak solution of

\[
-\Delta \varphi + \varphi = 0 \quad \text{in } \Omega, \quad \frac{\partial \varphi}{\partial \nu} = \lambda^* f'(u^*) \varphi + \mu \varphi \quad \text{on } \partial \Omega.
\]

Recall \( \varphi \in W^{1,q}(\Omega) \) for any \( 1 \leq q < N/(N-1) \). The proof of Theorem 1.1 is completed. \( \square \)

References


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