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POSITIVE SOLUTIONS OF THE p-LAPLACE EMDEN-FOWLER EQUATION IN HOLLOW THIN SYMMETRIC DOMAINS

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Abstract. We study the existence of positive solutions for the p-Laplace Emden-Fowler equation. Let H and G be closed subgroups of the orthogonal group $O(N)$ such that $H \subsetneq G \subset O(N)$. We denote the orbit of G through $x \in \mathbb{R}^N$ by $G(x)$, i.e., $G(x) := \{gx\colon\ g \in G\}$. We prove that if $H(x) \subsetneq G(x)$ for all $x \in \overline{\Omega}$ and the first eigenvalue of the p-Laplacian is large enough, then no H invariant least energy solution is G invariant. Here an H invariant least energy solution means a solution which achieves the minimum of the Rayleigh quotient among all H invariant functions. Therefore there exists an H invariant G non-invariant positive solution.

Keywords: Emden-Fowler equation; group invariant solution; least energy solution; positive solution; variational method

MSC 2010: 35J20, 35J25

1. Introduction

In this paper, we study the existence of positive solutions with partial symmetry for the p-Laplace Emden-Fowler equation

$$-\Delta_p u = u^{q-1}, \qquad u > 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{1.1}$$

Here $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian and Ω is a bounded domain in $\mathbb{R}^N$ with $N \geqslant 2$. Denote the critical exponent by $p^* := Np/(N-p)$ if $p < N$ and $p^* := \infty$ if $N \leqslant p$. We assume that $2 \leqslant p < q < p^*$. We define the *Rayleigh quotient* $R(u)$

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and the *Nehari manifold* $\mathcal{N}$ by

$$R(u) := \Big(\int_\Omega |\nabla u|^p \,\mathrm{d}x\Big)\Big(\int_\Omega |u|^q \,\mathrm{d}x\Big)^{-p/q},$$

$$\mathcal{N} := \Big\{u \in W_0^{1,p}(\Omega) \setminus \{0\}\colon \int_\Omega (|\nabla u|^p - |u|^q)\,\mathrm{d}x = 0\Big\},$$

where $W_0^{1,p}(\Omega)$ denotes the Sobolev space. Let G be a closed subgroup of the orthogonal group $O(N)$. We call Ω a *G invariant domain* if $g(\Omega) = \Omega$ for any $g \in G$. We call $u(x)$ a *G invariant solution* if $u(gx) = u(x)$ for any $g \in G$ and $x \in \Omega$. Then (1.1) has a G invariant positive solution. However, we are looking for an H invariant G non-invariant solution under a certain assumption on H and G, where H and G are closed subgroups of $O(N)$ such that $H \subsetneq G \subset O(N)$. When Ω is a G invariant domain, we denote the set of G invariant functions in $W_0^{1,p}(\Omega)$ by $W_0^{1,p}(\Omega, G)$. Define $\mathcal{N}(G) := \mathcal{N} \cap W_0^{1,p}(\Omega, G)$ and put

$$R_G := \inf\{R(u)\colon\ u \in W_0^{1,p}(\Omega, G) \setminus \{0\}\} = \inf\{R(u)\colon\ u \in \mathcal{N}(G)\}. \tag{1.2}$$

We call R_G a *G invariant least energy* and u a *G invariant least energy solution* if $u \in \mathcal{N}(G)$ and $R(u) = R_G$. Such a minimizer exists and becomes a G invariant positive solution of (1.1). For $x \in \mathbb{R}^N$, we define the orbit $G(x)$ through x by

$$G(x) := \{gx\colon\ g \in G\}. \tag{1.3}$$

Let $\lambda_p(\Omega)$ denote the first eigenvalue of the p-Laplace eigenvalue problem

$$-\Delta_p u = \lambda |u|^{p-2} u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{1.4}$$

It is well known that the first eigenvalue is simple and the corresponding eigenfunction is positive (see [7]). We state the main result of this paper.

Theorem 1.1. *Assume that $2 \leqslant p < q < p^*$. Let G and H be closed subgroups of $O(N)$ and let U be a G invariant bounded domain in $\mathbb{R}^N$ such that $H \subsetneq G$ and $H(x) \subsetneq G(x)$ for all $x \in \overline{U}$. Then there exists a constant $C > 0$ depending only on G, H, U, p and q such that if Ω is a G invariant subdomain of U and if $\lambda_p(\Omega) > C$, then $R_H < R_G$. Therefore no H invariant least energy solution is G invariant.*

The existence of multiple positive solutions of (1.1) on the sphere has been obtained by Kristály [6] also, in which the nonlinear term is asymptotically critical. We observe the Faber-Krahn inequality (see [1]), $\lambda_p(\Omega) \geqslant C_{N,p}|\Omega|^{-p/N}$, where $C_{N,p} > 0$ is a constant independent of Ω and $|\Omega|$ denotes the volume of Ω. Then we obtain the next corollary.

Corollary 1.2. *Under the assumption of Theorem* 1.1, *there exists a constant* $\delta > 0$ *depending only on* G, H, U, p *and* q *such that if* Ω *is a* G *invariant subdomain of* U *and if* $|\Omega| < \delta$, *then* $R_H < R_G$.

We give a simple example of H, G and Ω. A subgroup H of $O(N)$ is said to be transitive on the sphere S^{N-1} if $H(x) = S^{N-1}$ for $x \in S^{N-1}$. All transitive Lie groups were classified by Montgomery and Samelson [8] and Borel [2].

Example 1.3. Let $G := O(N)$ and let H be any non-transitive closed subgroup of $O(N)$. Let Ω be an annulus $1 < |x| < 1 + \varepsilon$ with $\varepsilon > 0$. If $\varepsilon > 0$ is small enough, then no H invariant least energy solution is radially symmetric.

2. Least energy solutions

Let $L^r(\Omega, G)$ denote the set of G invariant functions in $L^r(\Omega)$. Define the $L^2(\Omega)$ inner product and the $H_0^1(\Omega)$ inner product by

$$(u,v)_{L^2} := \int_\Omega uv \,\mathrm{d}x, \quad (u,v)_{H_0^1} := \int_\Omega \nabla u \nabla v \,\mathrm{d}x.$$

We define the orthogonal complements of $L^2(\Omega, G)$ and $H_0^1(\Omega, G)$ by

$$\begin{aligned} L^2(\Omega,G)^\perp &:= \{u \in L^2(\Omega) \colon (u,v)_{L^2} = 0 \text{ for all } v \in L^2(\Omega,G)\},\\ H_0^1(\Omega,G)^\perp &:= \{u \in H_0^1(\Omega) \colon (u,v)_{H_0^1} = 0 \text{ for all } v \in H_0^1(\Omega,G)\}. \end{aligned}$$

Lemma 2.1 ([3], Lemma 3.2)**.** *We have the following assertions.*

(i) $H_0^1(\Omega,G)^\perp \subset L^2(\Omega,G)^\perp$.

(ii) *Let* $1 \leqslant r, s \leqslant \infty$ *with* $1/r + 1/s = 1$. *If* $u \in L^r(\Omega) \cap L^2(\Omega,G)^\perp$ *and* $v \in L^s(\Omega, G)$, *then* $\int_\Omega uv\,\mathrm{d}x = 0$.

Since $p \geqslant 2$, the Rayleigh quotient R is twice differentiable in the sense of the Fréchet derivative. Then $R''(u)vw$ is a bilinear form of v and w. We need the formula of the special case $R''(u)w^2$ only.

Lemma 2.2. *Let u be a positive solution of* (1.1)*. For $w \in W_0^{1,p}(\Omega)$, we have*

$$
\begin{aligned}
R''(u)w^2 &= p(p-2)\Big(\int |\nabla u|^p \,\mathrm{d}x\Big)^{-p/q} \int |\nabla u|^{p-4}(\nabla u \cdot \nabla w)^2 \,\mathrm{d}x \\
&\quad + p\Big(\int |\nabla u|^p \,\mathrm{d}x\Big)^{-p/q} \int |\nabla u|^{p-2}|\nabla w|^2 \,\mathrm{d}x \\
&\quad + p(q-p)\Big(\int |\nabla u|^p \,\mathrm{d}x\Big)^{-(p+q)/q} \Big(\int u^{q-1} w \,\mathrm{d}x\Big)^2 \\
&\quad - p(q-1)\Big(\int |\nabla u|^p \,\mathrm{d}x\Big)^{-p/q} \int u^{q-2} w^2 \,\mathrm{d}x.
\end{aligned}
\tag{2.1}
$$

Here all integrals are taken over Ω.

Proof. Multiplying (1.1) by u or w and integrating it over Ω, we have

$$\int |\nabla u|^p \,\mathrm{d}x = \int u^q \,\mathrm{d}x, \quad \int |\nabla u|^{p-2} \nabla u \nabla w \,\mathrm{d}x = \int u^{q-1} w \,\mathrm{d}x.$$

Using the above identities and differentiating $R(u+tw)$ twice at $t=0$, we obtain (2.1). □

The next proposition plays the most important role in the paper.

Proposition 2.3. *Let u be a G invariant least energy solution of* (1.1) *and let Ω_1 be a G invariant bounded open set such that $\Omega \subset \Omega_1$. Let φ be a function in $H_0^1(\Omega_1, G)^{\perp} \cap W^{1,\infty}(\Omega_1)$ which satisfies*

$$\int_\Omega |\nabla u|^{p-2} u^2 |\nabla \varphi|^2 \,\mathrm{d}x < \frac{q-p}{2(2q-p-1)} \int_\Omega |\nabla u|^p \varphi^2 \,\mathrm{d}x. \tag{2.2}$$

Then $R((1+\varepsilon\varphi)u) < R(u)$ for $\varepsilon > 0$ small enough.

Proof. Set $v := (1+\varepsilon\varphi)u$ and define $w := \varphi u$. Then $v = u + \varepsilon w$. Since $u \in C^1(\overline{\Omega}) \cap H_0^1(\Omega)$, w and v belong to $H_0^1(\Omega)$. Since u is a solution of (1.1), $R'(u)$ vanishes. The Taylor theorem ensures that

$$R(v) = R(u) + (\varepsilon^2/2) R''(u) w^2 + o(\varepsilon^2),$$

as $\varepsilon \to 0$. Here $o(\varepsilon^2)/\varepsilon^2 \to 0$ as $\varepsilon \to 0$. To prove $R(v) < R(u)$ for $\varepsilon > 0$ small enough, we have only to show that $R''(u)w^2 < 0$. We substitute $w = \varphi u$ in (2.1) and compute all terms on the right hand side. We extend u by setting $u(x) = 0$ outside Ω. By Lemma 2.1, we see that

$$u^q \in L^2(\Omega_1, G), \quad \varphi \in H_0^1(\Omega_1, G)^{\perp} \subset L^2(\Omega_1, G)^{\perp}.$$

Consequently,

$$\int_\Omega u^{q-1} w \, \mathrm{d}x = \int_{\Omega_1} u^q \varphi \, \mathrm{d}x = 0.$$

It is easy to see that

$$\nabla u \cdot \nabla w = |\nabla u|^2 \varphi + u \nabla u \cdot \nabla \varphi,$$
$$|\nabla w|^2 = |\nabla u|^2 \varphi^2 + 2u\varphi \nabla u \cdot \nabla \varphi + u^2 |\nabla \varphi|^2.$$

Substituting the above identities in (2.1) and putting

$$A := \left(\int |\nabla u|^p \, \mathrm{d}x \right)^{-p/q},$$

we have

$$\begin{aligned} R''(u)w^2 = {} & p(p-1)A \int |\nabla u|^p \varphi^2 \, \mathrm{d}x \\ & + 2p(p-1)A \int |\nabla u|^{p-2} u\varphi \nabla u \cdot \nabla \varphi \, \mathrm{d}x \\ & + p(p-2)A \int |\nabla u|^{p-4} u^2 (\nabla u \cdot \nabla \varphi)^2 \, \mathrm{d}x \\ & + pA \int |\nabla u|^{p-2} u^2 |\nabla \varphi|^2 \, \mathrm{d}x - p(q-1)A \int u^q \varphi^2 \, \mathrm{d}x. \end{aligned} \tag{2.3}$$

Now, multiplying (1.1) by $u\varphi^2$ and integrating over Ω, we see that

$$\int u^q \varphi^2 \, \mathrm{d}x = \int (|\nabla u|^p \varphi^2 + 2|\nabla u|^{p-2} u\varphi \nabla u \cdot \nabla \varphi) \, \mathrm{d}x.$$

Substituting the above identity in (2.3), we obtain

$$\begin{aligned} R''(u)w^2 = {} & -p(q-p)A \int |\nabla u|^p \varphi^2 \, \mathrm{d}x \\ & - 2p(q-p)A \int |\nabla u|^{p-2} u\varphi \nabla u \cdot \nabla \varphi \, \mathrm{d}x \\ & + p(p-2)A \int |\nabla u|^{p-4} u^2 (\nabla u \cdot \nabla \varphi)^2 \, \mathrm{d}x \\ & + pA \int |\nabla u|^{p-2} u^2 |\nabla \varphi|^2 \, \mathrm{d}x. \end{aligned} \tag{2.4}$$

We use the Schwarz inequality

$$|u\varphi \nabla u \cdot \nabla \varphi| \leqslant \frac{1}{4} |\nabla u|^2 \varphi^2 + u^2 |\nabla \varphi|^2$$

in the second integral on the right hand side of (2.4) and employ $|\nabla u\cdot\nabla\varphi| \leqslant |\nabla u||\nabla\varphi|$ in the third integral. Then we obtain

$$\begin{aligned} R''(u)w^2 \leqslant & -\frac{1}{2}p(q-p)A\int |\nabla u|^p\varphi^2\,\mathrm{d}x \\ & + p(2q-p-1)A\int |\nabla u|^{p-2}u^2|\nabla\varphi|^2\,\mathrm{d}x. \end{aligned}$$

The right hand side is negative because of (2.2). The proof is complete. □

To prove the main theorems, we need the Haar measure. Since G is a compact Lie group, it has a unique Haar measure $\mathrm{d}g$. It is a positive Lebesgue measure which satisfies

$$\int_G f(hg)\,\mathrm{d}g = \int_G f(gh)\,\mathrm{d}g = \int_G f(g^{-1})\,\mathrm{d}g = \int_G f(g)\,\mathrm{d}g,$$
$$\int_G f(g)\,\mathrm{d}g > 0 \quad \text{if } f \geqslant 0,\ f \not\equiv 0, \qquad \int_G 1\,\mathrm{d}g = 1,$$

for any $h \in G$ and any real valued integrable function f on G (see [9] for more details).

Let $M(N)$ be a linear space consisting of all $N\times N$ real matrices, which is equipped with the norm

$$\|g\| := \max_{|x|\leqslant 1}|gx| \quad \text{for } g\in M(N).$$

For $g_0 \in G$ and $r > 0$ we define a ball $B(g_0, r; G)$ in G by

$$B(g_0, r; G) := \{g \in G\colon\ \|g-g_0\| < r\}.$$

Then the volume of $B(g_0, r; G)$ is defined by

$$|B(g_0, r; G)| := \int_{B(g_0,r;G)} 1\,\mathrm{d}g.$$

Using the invariance of the Haar measure, we have the next lemma.

Lemma 2.4 ([4], Lemma 5.6)**.** *Let G be a closed subgroup of $O(N)$. Then the volume $|B(g_0, r; G)|$ does not depend on $g_0 \in G$ but does on r only.*

3. Proof of the main results

In this section, we prove the main theorem. Let H and G be as in Theorem 1.1. Since G and H are compact groups, we can define

$$Q(x, g) := \min_{h\in H}|gx - hx|, \quad P(x) := \max_{g\in G} Q(x, g).$$

Lemma 3.1. *We have*

$$|P(x) - P(y)| \leqslant 2|x - y| \quad \text{for } x, y \in \mathbb{R}^N.$$

P r o o f. By the same computation as in our paper [3], Lemma 2.1 or [4], Lemma 5.5, we obtain the lemma. □

Recall the assumption of Theorem 1.1 that $H(x) \subsetneq G(x)$ for all $x \in \overline{U}$. This implies that $P(x) > 0$ for $x \in \overline{U}$. Since $P(x)$ is continuous by Lemma 3.1, the minimum of $P(x)$ on $\overline{U}$ is positive. We define

$$\delta := \tfrac{1}{4} \min_{\overline{U}} P(x) > 0. \tag{3.1}$$

Then for any $x \in \overline{U}$ there exists a $g \in G$ such that

$$|gx - hx| \geqslant 4\delta > 0 \quad \text{for any } h \in H. \tag{3.2}$$

To prove Theorem 1.1, we shall construct a function φ which satisfies (2.2) and belongs to $H_0^1(\Omega_1, H)$. Let $\delta > 0$ be defined by (3.1). Choose $\Phi \in C^1(\mathbb{R})$ which satisfies $0 \leqslant \Phi(r) \leqslant 1$ in $\mathbb{R}$, $\Phi(r) = 1$ for $r \leqslant \delta$, $\Phi(r) = 0$ for $r \geqslant 2\delta$ and $-2/\delta \leqslant \Phi'(r) \leqslant 0$ in $(\delta, 2\delta)$. Put $r = |x|$. Then $\Phi(|x|)$ is a radial function whose support is in $|x| \leqslant 2\delta$.

Definition 3.2. We denote the Haar measures on H and G by $\mathrm{d}h$ and $\mathrm{d}g$, respectively. Let $x_0 \in \Omega$ be determined later on. We define

$$\begin{aligned} \varphi(x) &:= \int_G \Phi(|x - gx_0|)\,\mathrm{d}g - \int_H \Phi(|x - hx_0|)\,\mathrm{d}h, \\ \operatorname{dist}(x, \Omega) &:= \inf\{|x - y| \colon\ y \in \Omega\}, \\ \Omega_1 &:= \{x \in \mathbb{R}^N \colon\ \operatorname{dist}(x, \Omega) < 2\delta\}. \end{aligned}$$

Lemma 3.3 ([4], [5]). *Function φ belongs to $H_0^1(\Omega_1, G)^\perp \cap H_0^1(\Omega_1, H)$.*

Since U is bounded, we define $M := \sup_{x \in U} |x|$ and $\mu := \delta/M$. Then μ depends only on G, H and U. We denote the volume of $B(g_0, \mu; G)$ by c_0, i.e.,

$$c_0 := |B(g_0, \mu; G)| = \int_{B(g_0, \mu; G)} 1\,\mathrm{d}g. \tag{3.3}$$

By Lemma 2.4, c_0 depends not on g_0 but on μ, hence it depends only on G, H and U. Let $B(x, r)$ denote the ball in $\mathbb{R}^N$ which is centered at x with radius $r > 0$.

Lemma 3.4 ([4], [5]). *For any $x_0 \in \Omega$, there exists a $g_0 \in G$ such that*

$$\varphi(x) \geqslant c_0 > 0 \quad \text{for } x \in B(g_0 x_0, \delta/2). \tag{3.4}$$

In particular, $\varphi \not\equiv 0$ in Ω.

Let δ be defined by (3.1). We choose a finite covering $B(y_i, \delta/4)$ with $y_1, \ldots, y_k \in \overline{U}$ such that

$$\overline{U} \subset \bigcup_{i=1}^{k} B(y_i, \delta/4) \quad \text{with some } k \in \mathbb{N}. \tag{3.5}$$

Hereafter we fix k and $y_1, \ldots, y_k$ which satisfy the above inclusion.

Lemma 3.5. *Let Ω be a G invariant subdomain of U and let u be a G invariant least energy solution. Extend u by setting $u(x) = 0$ outside Ω. Then there exists an $x_0 \in \Omega$ such that*

$$\int_\Omega |\nabla u|^p \, \mathrm{d}x \leqslant k \int_{B(x_0, \delta/2)} |\nabla u|^p \, \mathrm{d}x.$$

P r o o f. Choose $i \in \{1, 2, \ldots, k\}$ such that

$$\int_{B(y_i, \delta/4)} |\nabla u|^p \, \mathrm{d}x = \max_j \int_{B(y_j, \delta/4)} |\nabla u|^p \, \mathrm{d}x.$$

Then we have

$$\int_\Omega |\nabla u|^p \, \mathrm{d}x \leqslant k \int_{B(y_i, \delta/4)} |\nabla u|^p \, \mathrm{d}x.$$

Observe that $\Omega \cap B(y_i, \delta/4) \neq \emptyset$. Otherwise the right hand side vanishes. We choose an $x_0 \in \Omega \cap B(y_i, \delta/4)$. Then we have

$$\int_{B(y_i, \delta/4)} |\nabla u|^p \, \mathrm{d}x \leqslant \int_{B(x_0, \delta/2)} |\nabla u|^p \, \mathrm{d}x.$$

Combining the two above inequalities, we obtain the conclusion. $\square$

Lemma 3.6. *Let λ_p be the first eigenvalue of* (1.4). *Then*

$$\int_\Omega |\nabla v|^{p-2} v^2 \,\mathrm{d}x \leqslant \lambda_p^{-2/p} \|\nabla v\|_p^p \quad \textit{for any } v \in W_0^{1,p}(\Omega).$$

Proof. From the variational characterization of the first eigenvalue, it follows that for $v \in W_0^{1,p}(\Omega)$,

$$\lambda_p \int_\Omega |v|^p \,\mathrm{d}x \leqslant \int_\Omega |\nabla v|^p \,\mathrm{d}x,$$

or equivalently

$$\|v\|_p \leqslant \lambda_p^{-1/p} \|\nabla v\|_p.$$

Using this inequality with the Hölder inequality, we get

$$\int_\Omega |\nabla v|^{p-2} v^2 \,\mathrm{d}x \leqslant \|\nabla v\|_p^{p-2} \|v\|_p^2 \leqslant \lambda_p^{-2/p} \|\nabla v\|_p^p.$$

□

Define δ, c_0 and k by (3.1), (3.3) and (3.5), respectively, and then determine x_0 by Lemma 3.5. Thus $\varphi(x)$ is well defined by Definition 3.2. To prove Theorem 1.1, we define

$$C := [32\delta^{-2} k c_0^{-2} (2q - p - 1)/(q - p)]^{p/2},$$

which depends only on G, H, U, p and q. We conclude this paper by proving Theorem 1.1.

Proof of Theorem 1.1. Let C be as above. Suppose that $\lambda_p(\Omega) > C$. We shall show that φ satisfies (2.2). Since $|\Phi'(r)| \leqslant 2/\delta$ by the definition of Φ, we have $|\nabla\varphi| \leqslant 4/\delta$. This inequality and Lemmas 3.6 and 3.5 show that

$$\begin{aligned}\int_\Omega |\nabla u|^{p-2} u^2 |\nabla\varphi|^2 \,\mathrm{d}x &\leqslant 16\delta^{-2} \lambda_p^{-2/p} \|\nabla u\|_p^p \\ &\leqslant 16\delta^{-2} \lambda_p^{-2/p} k \int_{B(x_0,\delta/2)} |\nabla u|^p \,\mathrm{d}x.\end{aligned}$$

By Lemma 3.4, we choose $g_0 \in G$ satisfying (3.4). Since u is G invariant, the last integral is estimated as

$$\int_{B(x_0,\delta/2)} |\nabla u|^p \,\mathrm{d}x = \int_{B(g_0 x_0,\delta/2)} |\nabla u|^p \,\mathrm{d}x \leqslant c_0^{-2} \int_{B(g_0 x_0,\delta/2)} |\nabla u|^p \varphi^2 \,\mathrm{d}x.$$

Combining the two above inequalities, we have

$$\int_\Omega |\nabla u|^{p-2} u^2 |\nabla\varphi|^2 \,\mathrm{d}x \leqslant 16\delta^{-2} \lambda_p^{-2/p} k c_0^{-2} \int_{B(g_0 x_0,\delta/2)} |\nabla u|^p \varphi^2 \,\mathrm{d}x.$$

Since $\lambda_p(\Omega) > C$, we obtain (2.2). Since $\varphi \in H_0^1(\Omega_1, H)$ by Lemma 3.3, $v := (1+\varepsilon\varphi)u$ belongs to $H_0^1(\Omega, H)$. By Proposition 2.3, we conclude that $R_H \leqslant R(v) < R(u) = R_G$. The proof is complete. □

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