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A NOTE ON THE CAHN-HILLIARD EQUATION IN $H^1(\mathbb{R}^N)$ INVOLVING CRITICAL EXPONENT

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Abstract. We consider the Cahn-Hilliard equation in $H^1(\mathbb{R}^N)$ with two types of critically growing nonlinearities: nonlinearities satisfying a certain limit condition as $|u| \to \infty$ and logistic type nonlinearities. In both situations we prove the $H^2(\mathbb{R}^N)$-bound on the solutions and show that the individual solutions are suitably attracted by the set of equilibria. This complements the results in the literature; see J. W. Cholewa, A. Rodriguez-Bernal (2012).

Keywords: initial value problem for higher order parabolic equations; asymptotic behavior of solutions; critical exponent

MSC 2010: 35K30, 35B40, 35B33

1. Introduction

We consider the Cauchy problem for the Cahn-Hilliard equation

$$u_t + \Delta^2 u + \Delta f(x,u) = 0, \quad t > 0,\ x \in \mathbb{R}^N, \tag{1.1}$$

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}^N, \tag{1.2}$$

with initial data in $H^1(\mathbb{R}^N)$. We single out $H^1(\mathbb{R}^N)$ as a phase space for (1.1) since, under some mild assumptions on f, involving even weakly integrable potentials, the Lyapunov type functional

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - \int_{\mathbb{R}^N} F(x,u), \quad \text{where } F(x,u) = \int_0^u f(x,s)\,\mathrm{d}s \tag{1.3}$$

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is well defined in $H^1(\mathbb{R}^N)$. Our concern is the situation when the nonlinear term satisfies a certain critical growth condition.

In bounded domains, see e.g. [32], [31] and references therein, the past decades have witnessed an extensive study of the Cahn-Hilliard model, which was originally derived in [9] as a phenomenological equation describing phase transition problems in binary metallic alloys. Several variations of the model were also considered, like multi-component alloys models [20], [12], [26], models with viscosity or inertial terms [19], [30], [28], [11], [21], Cahn-Hilliard-Oono model [29] and Cahn-Hilliard-Cook equations [4], [5].

In contrast with the case of bounded domains, merely a few references seem to deal with the Cahn-Hilliard problem in unbounded domains or in $\mathbb{R}^N$; see [8], [7], [25], [6], [27], [18], [17], [15], [34] in the chronological order.

In [8] the authors were concerned with the existence of an $L^\infty(\mathbb{R}^N)$ bound for the solutions in the case when a nonlinear term grew linear at infinity.

In [7], and then in [25], stability of a particular steady state solution, the so called kink solution, was considered.

In [6] the viscous model in a channel like unbounded domain in dimensions $N = 2, 3$ was studied with the aid of weighted spaces and the existence of an attractor was proved.

In [27], and after that in [18], some temporal decay estimates of the solutions have been reported.

In [17] for the viscous model in the subcritical case the concept of the so called H-solutions was developed.

In [15] the critical exponent

$$\varrho_c := 1 + \frac{4}{N-2}, \quad N \geqslant 3, \tag{1.4}$$

naturally appeared when proving local well posedness of (1.1)–(1.2) in $H^1(\mathbb{R}^N)$ whereas the analysis of the dissipative mechanism was then carried out in the subcritical case $\varrho < \varrho_c$.

Recently, in [34], the 3-dimensional model has been studied in the so called locally uniform spaces and the existential result has been obtained under suitably chosen assumptions on f.

In this note we set forth the analysis of (1.1) in the 'energy' space $H^1(\mathbb{R}^N)$. Our goal is to extend from the subcritical to the critical case some of the results obtained in [15] for the unperturbed Cahn-Hilliard equation (1.1). This includes the global well posedness in $H^1(\mathbb{R}^N)$ and a suitable notion of dissipativeness.

Following [15] we will assume that $f\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ in (1.1) is of the form

$$f(x,u) = g(x) + m(x)u + f_0(x,u), \quad x \in \mathbb{R}^N, \ u \in \mathbb{R}, \tag{1.5}$$

with

$$g \in L^2(\mathbb{R}^N), \tag{1.6}$$

$$f_0(x,0) = 0, \quad x \in \mathbb{R}^N, \tag{1.7}$$

$$f_0\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \quad \text{locally Lipschitz in } u \in \mathbb{R} \text{ uniformly for } x \in \mathbb{R}^N \tag{1.8}$$

and

$$m \in L^r_U(\mathbb{R}^N) \quad \text{for some } r > \frac{N}{2},\ r \geqslant 2, \tag{1.9}$$

where the above space $L^r_U(\mathbb{R}^N)$ is defined, for $1 \leqslant r \leqslant \infty$, as

$$L^r_U(\mathbb{R}^N) \overset{\text{def}}{=} \{\varphi \in L^r_{\text{loc}}(\mathbb{R}^N)\colon \|\varphi\|_{L^r_U(\mathbb{R}^N)} = \sup_{y\in\mathbb{R}^N} \|\varphi\|_{L^r(B(y,1))} < \infty\}$$

(see e.g. [24], [3]). Also the growth of f will be restricted by the condition

$$\begin{aligned} &\exists c > 0\, \exists \varrho,\ 1 < \varrho \leqslant \varrho_c\ \forall u_1, u_2 \in \mathbb{R}\ \forall x \in \mathbb{R}^N \\ &\qquad |f_0(x,u_1) - f_0(x,u_2)| \leqslant c|u_1 - u_2|(1 + |u_1|^{\varrho-1} + |u_2|^{\varrho-1}), \end{aligned} \tag{1.10}$$

where ϱ_c is as in (1.4).

Taking into account the form of f in (1.5) one can also require $(\partial f_0/\partial u)(\cdot,0) = 0$ (see [15]) however here we omit this because it is not necessary for further consideration.

We remark that with the above assumptions one can view (1.1)–(1.2) as the abstract parabolic problem

$$\begin{cases} \dot{u} + P_0^2 u = P_0(f(\cdot,u)) =: \mathcal{F}(u), & t > 0, \\ u(0) = u_0 \in H^1(\mathbb{R}^N), \end{cases} \tag{1.11}$$

where

$$P_0 = -\Delta\colon \operatorname{dom}(P_0) \subset L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N), \quad \operatorname{dom}(P_0) = H^2(\mathbb{R}^N), \tag{1.12}$$

and that the following local well posedness result then holds (see [15], Corollary 2.3).

Proposition 1.1. *Assume* (1.4)–(1.10). *Then, for any* $u_0 \in H^1(\mathbb{R}^N)$, *there is a unique solution* $u \in C([0,\tau_0), H^1(\mathbb{R}^N))$ *of* (1.1)–(1.2) *defined on a maximal interval of existence* $[0,\tau_{u_0})$ *and satisfying*

$$u(t) = \mathrm{e}^{-\Delta^2 t}u_0 + \int_0^t \mathrm{e}^{-\Delta^2(t-s)}(-\Delta)(f(\cdot,u(s)))\,\mathrm{d}s,\ t \in [0,\tau_0),$$

or, equivalently,

$$u(t) = \mathrm{e}^{-\Delta^2 t}u_0 + \int_0^t (-\Delta)\mathrm{e}^{-\Delta^2(t-s)}(f(\cdot,u(s)))\,\mathrm{d}s,\ t \in [0,\tau_0).$$

Furthermore,

$$\begin{gathered} u \in C((0,\tau_0), H^2(\mathbb{R}^N)) \cap C^1((0,\tau_0), H^s(\mathbb{R}^N)), \quad s<2, \\ \Delta u(t) + f(\cdot,u(t)) \in H^2(\mathbb{R}^N), \quad t \in (0,\tau_{u_0}), \end{gathered} \tag{1.13}$$

and for $t \in (0,\tau_{u_0})$ *we have*

$$u_t + (-\Delta)(-\Delta u - f(x,u)) = 0 \quad \textit{in} \ \ L^2(\mathbb{R}^N).$$

As for reaction-diffusion equations and problems with higher order diffusion (see [14], [2]) in the studies of global solutions of (1.1)–(1.2) the cooperation between diffusion and reaction terms in the equation becomes crucial. This cooperation is exhibited in the structure condition

$$vf(x,v) \leqslant C(x)v^2 + D(x)|v|, \quad x \in \mathbb{R}^N,\ v \in \mathbb{R}, \tag{1.14}$$

where

$$\begin{gathered} C \in L^\sigma(\mathbb{R}^N), \quad \sigma > \max\Big\{\frac{N}{2},1\Big\}, \\ 0 \leqslant D \in L^s(\mathbb{R}^N), \quad \max\Big\{1,\frac{2N}{N+2}\Big\} \leqslant s \leqslant 2, \end{gathered} \tag{1.15}$$

and C is such that

$$\exists \omega_0 > 0\ \forall \varphi \in H^1(\mathbb{R}^N)\ \int_{\mathbb{R}^N} (|\nabla\varphi|^2 - C(x)\varphi^2) \geqslant \omega_0\|\varphi\|^2_{L^2(\mathbb{R}^N)}; \tag{1.16}$$

that is, the solutions of the linear problem

$$\begin{cases} u_t = \Delta u + C(x)u, & t>0,\ x \in \mathbb{R}^N, \\ u(0) = u_0 \in L^2(\mathbb{R}^N) \end{cases}$$

are uniformly exponentially decaying as $t \to \infty$ (see [15], Appendix A).

Note that assuming (1.4)–(1.10), (1.14)–(1.16) and using the energy (1.3) we obtain for the solutions of (1.1)–(1.2) the a priori bound of the form

$$\|u\|^2_{H^1(\mathbb{R}^N)} \leqslant c_1 E(u) + c_2 \leqslant c_1 E(u_0) + c_2 \tag{1.17}$$

where c_1, c_2 are some positive constants (see [15], Lemma 3.5). If, in addition, the growth of f is restricted to a subcritical case, that is for

$$\varrho < \varrho_c, \tag{1.18}$$

then the following result holds (see [15]).

Theorem 1.2. *For the Cahn-Hilliard problem* (1.1)–(1.2)

(i) *the local solution through $u_0 \in H^1(\mathbb{R}^N)$ as in Proposition* 1.1 *exists globally in time and the associated semigroup $\{S(t)\colon\ t \geqslant 0\}$ in $H^1(\mathbb{R}^N)$,*

$$S(t)u_0 = u(t; u_0), \quad t \geqslant 0,\ u_0 \in H^1(\mathbb{R}^N),$$

has bounded positive orbits of bounded sets,

(ii) *the positive orbit $\gamma^+(B)$ of any set B bounded in $H^1(\mathbb{R}^N)$ is immediately bounded in $H^2(\mathbb{R}^N)$; that is,*

$$S(t)\gamma^+(B) \text{ is bounded in } H^2(\mathbb{R}^N)$$

for any $t > 0$,

(iii) *the set $\mathcal{E}$ of equilibria of $\{S(t)\colon\ t \geqslant 0\}$,*

$$\mathcal{E} = \{\varphi \in H^2(\mathbb{R}^N)\colon\ -\Delta\varphi = f(\cdot, \varphi) \text{ in } \mathbb{R}^N\},$$

is nonvoid and for each $u_0 \in H^1(\mathbb{R}^N)$ and any sequence $t_n \to \infty$ there is a subsequence $\{t_{n_k}\}$ and an equilibrium $\psi \in \mathcal{E}$ such that

$$u(t_{n_k}) \overset{k\to\infty}{\longrightarrow} \psi \quad \text{in } H^s_{\mathrm{loc}}(\mathbb{R}^N) \text{ and in } H^s_{\varphi}(\mathbb{R}^N) \text{ for any } s < 2,$$

where the weight is given by $\varphi(x) = (1 + |x|^2)^{-\nu}$ with $\nu > N/2$.

Furthermore, if in (1.14) *we have, in addition to* (1.15),

$$D \in L^\sigma(\mathbb{R}^N) \cap L^s(\mathbb{R}^N), \quad \sigma > N/2, \quad \max\left\{1, \frac{2N}{N+2}\right\} \leqslant s \leqslant 2,$$

then

(iv) *there are two ordered extremal equilibria* φ_m, φ_M *in* $H^1(\mathbb{R}^N)$, *minimal and maximal, respectively, such that any equilibrium* ψ *in* $H^1(\mathbb{R}^N)$ *satisfies*

$$\varphi_m(x) \leqslant \psi(x) \leqslant \varphi_M(x), \quad x \in \mathbb{R}^N,$$

(v) *the order interval* $[\varphi_m, \varphi_M]_{H^1(\mathbb{R}^N)}$ *attracts 'pointwise asymptotic dynamics' of* (1.1) *in the sense that for each* $u_0 \in H^1(\mathbb{R}^N)$ *and any sequence* $t_n \to \infty$, *there is a subsequence* $\{t_{n_k}\}$ *such that*

$$\varphi_m(x) \leqslant \lim_{k\to\infty} u(t_{n_k}, x; u_0) \leqslant \varphi_M(x)$$

for a.e. $x \in \mathbb{R}^N$.

Note that Theorem 1.2 concerns some weak form of dissipativity and that a strong dissipativity, the one which relies on the asymptotic compactness in $H^1(\mathbb{R}^N)$, cannot be obtained in general (see [15]). In particular, a global attractor in the sense of [22] will not generally exist in $H^1(\mathbb{R}^N)$ (see [15], Proposition 4.6).

In what follows our concern will be to ensure validity of Theorem 1.2 in the case when the critical exponent $\varrho = \varrho_c$ appears in the nonlinear term in the equation. We will thus assume (1.4)–(1.10), (1.14)–(1.16) but not (1.18).

In Section 2 we will consider critical nonlinearities $f(x,u)$ satisfying a certain limit condition as $|u| \to \infty$.

In Section 3 we will discuss critically growing logistic type nonlinearities of the form $f(x,u) = g(x) + m(x)u - u|u|^{\varrho_c - 1}$.

In the closing Section 4 we will include some final comments concerning the critical exponent.

2. Critical nonlinearities satisfying a limit condition as $|u| \to \infty$

In this section we will consider nonlinearities satisfying uniformly for $x \in \mathbb{R}^N$ the *limit condition*

$$\lim_{|u|\to\infty} \frac{|(\partial f_0/\partial u)(x,u)|}{|u|^{\varrho_c - 1}} = 0. \tag{2.1}$$

Theorem 2.1. *Theorem* 1.2 *applies for critical nonlinearities satisfying the limit condition as in* (2.1).

In the proof of Theorem 2.1 we will use the following three lemmas. To avoid too many technicalities, instead of (1.9) we will use some stronger local integrability property of m requiring that

$$m \in L^r_U(\mathbb{R}^N), \quad r \geqslant \frac{N\varrho_c}{\varrho_c + 1} = \frac{N}{2} + 1. \tag{2.2}$$

Then we will come back to the original assumption (1.9) in Remark 2.5, which closes this section.

Lemma 2.2. *If* (1.8), (2.1) *hold and* $f_0(\cdot, 0) = 0$ *then, given* $\eta > 0$, *there exist a positive constant* C_η *and maps* f_{01}^η, f_{02}^η *such that*

$$f_{01}^\eta(\cdot, 0) = f_{02}^\eta(\cdot, 0) = 0,$$

$$f_0(x, v) = f_{01}^\eta(x, v) + f_{02}^\eta(x, v), \quad x \in \mathbb{R}^N, \ v \in \mathbb{R}, \tag{2.3}$$

$$f_{01}^\eta(x, \cdot) \text{ is a Lipschitz map uniformly for } x \in \mathbb{R}^N, \tag{2.4}$$

and

$$|f_{02}^\eta(x, v_1) - f_{02}^\eta(x, v_2)| \leqslant \eta |v_1 - v_2| (|v_1|^{\varrho_c - 1} + |v_2|^{\varrho_c - 1}), \quad v_1, v_2 \in \mathbb{R}. \tag{2.5}$$

P r o o f. Fix $\eta > 0$, choose $s_\eta > 1$ such that

$$\Big| \frac{\partial f_0}{\partial s}(x, s) \Big| \leqslant \eta |s|^{\varrho_c - 1} \quad \text{for } |s| > s_\eta \tag{2.6}$$

and define

$$f_{01}^\eta(x, v) = \begin{cases} f_0(x, v), & x \in \mathbb{R}^N, \ |v| \leqslant s_\eta, \\ f_0(x, s_\eta), & x \in \mathbb{R}^N, \ v > s_\eta, \\ f_0(x, -s_\eta), & x \in \mathbb{R}^N, \ v < -s_\eta, \end{cases}$$
$$f_{02}^\eta(x, v) = f_0(x, v) - f_{01}^\eta(x, v), \quad x \in \mathbb{R}^N, \ v \in \mathbb{R}.$$

Since $f_0(x, v)$ is locally Lipschitz in $v \in \mathbb{R}$ uniformly for $x \in \mathbb{R}^N$, choosing $L_\eta > 0$ as a Lipschitz constant for f_0 restricted to $\mathbb{R}^N \times [-s_\eta, s_\eta]$ and using the definition of f_{01}^η, we have that

$$|f_{01}^\eta(x, v_1) - f_{01}^\eta(x, v_2)| \leqslant L_\eta |v_1 - v_2|, \quad x \in \mathbb{R}^N, \ v_1, v_2 \in \mathbb{R}.$$

Considering then the difference $f_{02}^\eta(x, v_1) - f_{02}^\eta(x, v_2)$, using the definition of f_{02}^η, the mean value theorem and (2.6) we get (2.5). □

Lemma 2.3. *Besides* (1.4)–(1.10) *assume the limit condition as in* (2.1) *and* (2.2). *Then for any* $\eta > 0$ *there exists* $C_\eta > 0$ *such that* $\mathcal{F}$ *in* (1.11) *satisfies*

$$\begin{aligned}\|\mathcal{F}(v) - \mathcal{F}(w)\|_{H^{-2}(\mathbb{R}^N)} \\ \leqslant c\|v-w\|_{H^{1+1/\varrho_c}(\mathbb{R}^N)}(C_\eta + \eta\|v\|^{\varrho-1}_{H^{1+1/\varrho_c}(\mathbb{R}^N)} + \eta\|w\|^{\varrho-1}_{H^{1+1/\varrho_c}(\mathbb{R}^N)})\end{aligned}$$

for any $v, w \in H^{1+1/\varrho_c}(\mathbb{R}^N)$.

Proof. Due to (1.5), (2.3), given $\eta > 0$ we have that

$$\mathcal{F}(u) = \mathcal{F}_1^\eta(u) + \mathcal{F}_2^\eta(u) + \mathcal{F}_3(u),$$

where

$$\mathcal{F}_1^\eta(u) = P_0(f_{01}^\eta(\cdot,u) + g), \quad \mathcal{F}_2^\eta(u) = P_0(f_{02}^\eta(\cdot,u)) \quad \text{and} \quad \mathcal{F}_3(u) = P_0(m(\cdot)u).$$

Using (2.5) and repeating the proof of [15], Lemma B.2, we obtain

$$\begin{aligned}\|\mathcal{F}_2^\eta(v_1) - \mathcal{F}_2^\eta(v_2)\|_{H^{-2}(\mathbb{R}^N)} \\ \leqslant c\eta\|v_1 - v_2\|_{H^{1+1/\varrho_c}(\mathbb{R}^N)}(\|v_1\|^{\varrho_c-1}_{H^{1+1/\varrho_c}(\mathbb{R}^N)} + \|v_2\|^{\varrho_c-1}_{H^{1+1/\varrho_c}(\mathbb{R}^N)}).\end{aligned}$$

On the other hand, from (2.4) we get

$$\begin{aligned}\|\mathcal{F}_1^\eta(v_1) - \mathcal{F}_1^\eta(v_2)\|_{H^{-2}(\mathbb{R}^N)} &= \|(P_0 + \mathrm{Id})^{-1}P_0(f_{01}^\eta(\cdot,v_1) - f_{01}^\eta(\cdot,v_2))\|_{L^2(\mathbb{R}^N)} \\ &\leqslant c\|f_{01}^\eta(\cdot,v_1) - f_{01}^\eta(\cdot,v_2)\|_{L^2(\mathbb{R}^N)} \\ &\leqslant cL_\eta\|v_1 - v_2\|_{L^2(\mathbb{R}^N)} \leqslant \tilde{c}L_\eta\|v_1 - v_2\|_{H^{1+1/\varrho_c}(\mathbb{R}^N)} \\ &\leqslant \|v_1 - v_2\|_{H^{1+1/\varrho_c}(\mathbb{R}^N)}\big(\tilde{c}L_\eta + c\eta\|v_1\|^{\varrho_c-1}_{H^{1+1/\varrho_c}(\mathbb{R}^N)} + c\eta\|v_2\|^{\varrho_c-1}_{H^{1+1/\varrho_c}(\mathbb{R}^N)}\big).\end{aligned}$$

To deal with $\mathcal{F}_3$ let us denote by Q_i the open cube in $\mathbb{R}^N$ centered at $i \in \mathbb{Z}^N$ and having unitary edges parallel to the axes. Then $\mathbb{R}^N = \bigcup\limits_{i\in\mathbb{Z}^N} \overline{Q_i}$, $Q_i \cap Q_j = \emptyset$ for $i \neq j$ and we have

$$\begin{aligned}\int_{\mathbb{R}^N} |m(x)|^2|v_1 - v_2|^2 &= \sum_{i\in\mathbb{Z}^N}\int_{Q_i}|m(x)|^2|v_1 - v_2|^2 \\ &\leqslant \sum_{i\in\mathbb{Z}^N}\|m\|^2_{L^r(Q_i)}\|v_1 - v_2\|^2_{L^{2r/(r-2)}(Q_i)} \leqslant c\|m\|^2_{L^r_U(\mathbb{R}^N)}\sum_{i\in\mathbb{Z}^N}\|v_1 - v_2\|^2_{H^{1+1/\varrho_c}(Q_i)},\end{aligned}$$

where, following (2.2), Hölder's inequality and the embedding $H^{1/2+1/(2\varrho_c)}(Q_i) \hookrightarrow L^{2r/(r-2)}(Q_i)$ were used. Due to [2], Lemma 2.4, $\sum\limits_{i\in\mathbb{Z}^N}\|v_1 - v_2\|^2_{H^{1+1/\varrho_c}(Q_i)}$ can be bounded by a multiple of $\|v_1 - v_2\|^2_{H^{1+1/\varrho_c}(\mathbb{R}^N)}$ and hence we get

$$\begin{aligned}\|\mathcal{F}_3(v_1) - \mathcal{F}_3(v_2)\|_{H^{-2}(\mathbb{R}^N)} &= \|(P_0 + \mathrm{Id})^{-1}P_0(mv_1 - mv_2)\|_{L^2(\mathbb{R}^N)}\\ &\leqslant c\|m(v_1 - v_2)\|_{L^2(\mathbb{R}^N)} \leqslant \hat{c}\|v_1 - v_2\|_{H^{1+1/\varrho_c}(\mathbb{R}^N)}\\ &\leqslant \|v_1 - v_2\|_{H^{1+1/\varrho_c}(\mathbb{R}^N)}(\hat{c} + \tilde{c}L_\eta + c\eta\|v_1\|^{\varrho_c-1}_{H^{1+1/\varrho_c}(\mathbb{R}^N)} + c\eta\|v_2\|^{\varrho_c-1}_{H^{1+1/\varrho_c}(\mathbb{R}^N)}).\end{aligned}$$

The result now follows easily. □

Lemma 2.4. *If the assumptions of Lemma 2.3 hold then Proposition 1.1 applies and the solution therein satisfies the blow-up $H^1(\mathbb{R}^N)$-alternative; namely,*

$$\text{either} \quad \tau_{u_0} = \infty \quad \text{or} \quad \limsup_{t\to\tau_{u_0}^-} \|u(t;u_0)\|_{H^1(\mathbb{R}^N)} = \infty.$$

Furthermore, given $R > 0$, a ball $B_{H^1(\mathbb{R}^N)}(v_0, R)$ in $H^1(\mathbb{R}^N)$ of radius $R > 0$ around $v_0 \in H^1(\mathbb{R}^N)$ and any $0 < \theta < \varrho_c\varepsilon = 1/4$, $u_0 \in B_{H^1(\mathbb{R}^N)}(v_0, R)$ we have that

$$t^\theta\|u(t;u_0)\|_{H^{1+4\theta}(\mathbb{R}^N)} \to 0 \quad \text{as } t \to 0^+.$$

Also,

$$t^\theta\|u(t;u_{01}) - u(t;u_{02})\|_{H^{1+4\theta}(\mathbb{R}^N)} \leqslant C'\|u_{01} - u_{02}\|_{H^1(\mathbb{R}^N)}$$

whenever $t \in [0, \tau_0]$, $0 \leqslant \theta < \varrho_c\varepsilon = 1/4$ and $u_{01}, u_{02} \in B_{H^1(\mathbb{R}^N)}(v_0, R)$.

Proof. By Lemma 2.3 there are $c > 0$, $\varepsilon = 1/(4\varrho_c)$ and, given $\eta > 0$, there exists $C_\eta > 0$ such that

$$\begin{aligned}&\|\mathcal{F}(v) - \mathcal{F}(w)\|_{H^{-3+4\varrho_c\varepsilon}(\mathbb{R}^N)}\\ &\qquad\leqslant c\|v - w\|_{H^{1+4\varepsilon}(\mathbb{R}^N)}(C_\eta + \eta\|v\|^{\varrho-1}_{H^{1+4\varepsilon}(\mathbb{R}^N)} + \eta\|w\|^{\varrho-1}_{H^{1+4\varepsilon}(\mathbb{R}^N)})\end{aligned}$$

for any $v, w \in H^{1+4\varepsilon}(\mathbb{R}^N)$. Recall also that the scale of fractional power spaces, $\{E^\alpha, \alpha \in \mathbb{R}\}$, associated with the linear main part operator P_0^2 in (1.11)–(1.12), is given by

$$E^\alpha = H^{4\alpha}(\mathbb{R}^N) \quad \text{for } -1 \leqslant \alpha \leqslant 1.$$

Following [10], Definition 2.1, the right hand side $\mathcal{F}$ in (1.11) can be thus viewed as an almost critical ε-regular map relative to spaces $X^1 = E^{1/4}$ and $X = E^{-3/4}$. Consequently, application of [10], Theorem 2.1, gives the result. □

P r o o f of Theorem 2.1. Part (i) follows from (1.17) and Lemma 2.4.

Proceeding next as in [15], Lemma 5.1, we observe via Lemma 2.4 that the positive orbits of bounded subsets of $H^1(\mathbb{R}^N)$ are immediately bounded in $H^s(\mathbb{R}^N)$ for any $s < 2$. Similarly to [15], Lemma 5.2, we then obtain the $H^2(\mathbb{R}^N)$-bound of the orbits stated in part (ii).

Once we have the $H^2(\mathbb{R}^N)$-bound of the orbits, the proof of parts (iii)–(v) in Theorem 1.2 follows the same lines as in [15]. □

R e m a r k 2.5. Condition (2.2) is not necessary and can be replaced by (1.9). This requires however a longer argument involving the decomposition of $\mathcal{F}$ into a sum of ε_i-regular maps $\mathcal{F}_i$, $i = 1, 2, 3$, with different parameters ε_i (see [1], Theorem 2.2). This is a matter we do not pursue here.

3. Critically growing logistic type nonlinearities

In this section we consider *critically growing logistic type nonlinearities*

$$f(x,u) = g(x) + m(x)u - u|u|^{\varrho_c - 1}, \quad x \in \mathbb{R}^N,\ u \in \mathbb{R}. \tag{3.1}$$

We thus focus now on the problem (1.1)–(1.2) with f as in (3.1).

Theorem 3.1. *Theorem* 1.2 *applies for* (1.1)–(1.2) *with critically growing logistic type nonlinearities.*

The following observation concerning the abstract counterpart of (1.1)–(1.2) will be useful.

Lemma 3.2. *If* (3.1) *holds with* ϱ_c, g, m *as in Section* 1, *that is*

$$\varrho_c = \frac{N+2}{N-2}, \ g \in L^2(\mathbb{R}^N) \quad \text{and} \quad m \in L^r_U(\mathbb{R}^N),\ r > \frac{N}{2},\ r \geqslant 2, \tag{3.2}$$

then, given $\varepsilon < 1/4$ *close enough to* $1/4$, *the nonlinear term* $\mathcal{F}$ *in* (1.11) *can be viewed as a Lipschitz map from* $H^{1+4\varepsilon}(\mathbb{R}^N)$ *into* $H^{-2}(\mathbb{R}^N)$, *Proposition* 1.1 *applies and the solution therein satisfies the blow up* $H^{1+4\varepsilon}(\mathbb{R}^N)$*-alternative; namely,*

$$\text{either} \quad \tau_{u_0} = \infty \quad \text{or} \quad \limsup_{t \to \tau_{u_0}^-} \|u(t;u_0)\|_{H^{1+4\varepsilon}(\mathbb{R}^N)} = \infty. \tag{3.3}$$

P r o o f. We remark that the solution in Proposition 1.1 is constructed as in [15], Appendix B, and the Lipschitz property in the statement follows from [15], Remark B.4. Since via (1.13) the solution enters $H^2(\mathbb{R}^N) \hookrightarrow H^{1+4\varepsilon}(\mathbb{R}^N)$ the blow up $H^{1+4\varepsilon}(\mathbb{R}^N)$-alternative thus holds as in [23], Theorem 3.3.4. □

Below we will derive a uniform in time $H^2(\mathbb{R}^N)$-estimate away from $t = 0$. Note that once this estimate is known then parts (i)–(ii) in Theorem 1.2 hold true due to (3.3) and (1.17). The proof of parts (iii)–(v) follows then the same lines as in [15].

Lemma 3.3. *If f is as in* (3.1)–(3.2) *then the solutions of* (1.1)–(1.2) *are a priori bounded in $H^2(\mathbb{R}^N)$ uniformly in time away from $t = 0$ and for initial conditions in bounded subsets of $H^1(\mathbb{R}^N)$; namely, given $\|u_0\|_{H^1(\mathbb{R}^N)} \leqslant R$ and $\tau > 0$ there exists a positive constant $M(R, \tau)$ such that*

$$\|u(t)\|_{H^2(\mathbb{R}^N)} \leqslant M(R, \tau), \quad t \geqslant \tau. \tag{3.4}$$

P r o o f. From (1.1) and (3.1) we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|\nabla(\Delta u + f(\cdot, u))\|^2_{L^2(\mathbb{R}^N)} &= 2\langle \nabla(\Delta u + f(\cdot, u)), \nabla(\Delta u_t + f'_u(\cdot, u)u_t)\rangle_{L^2(\mathbb{R}^N)} \\ &= 2\langle -\Delta(\Delta u + f(\cdot, u)), \Delta u_t + f'_u(\cdot, u)u_t\rangle_{L^2(\mathbb{R}^N)} \\ &= 2\langle u_t, \Delta u_t + f'_u(\cdot, u)u_t\rangle_{L^2(\mathbb{R}^N)} \\ &\leqslant -2\|\nabla u_t\|^2_{L^2(\mathbb{R}^N)} + 2\int_{\mathbb{R}^N} m(x)u_t^2. \end{aligned} \tag{3.5}$$

Applying [15], Lemma A.5, we have

$$\int_{\mathbb{R}^N} m(x)u_t^2 \leqslant \frac{1}{2}\|\nabla u_t\|^2_{L^2(\mathbb{R}^N)} + c_{1/2}\|u_t\|^2_{L^2(\mathbb{R}^N)}$$

and, using (1.1),

$$\begin{aligned} c_{1/2}\|u_t\|^2_{L^2(\mathbb{R}^N)} &= c_{1/2}\langle u_t, -\Delta(\Delta u + f(\cdot, u))\rangle_{L^2(\mathbb{R}^N)} \\ &= c_{1/2}\langle \nabla u_t, \nabla(\Delta u + f(\cdot, u))\rangle_{L^2(\mathbb{R}^N)} \\ &\leqslant \frac{1}{2}\|\nabla u_t\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{2}c^2_{1/2}\|\nabla(\Delta u + f(\cdot, u))\|^2_{L^2(\mathbb{R}^N)}. \end{aligned} \tag{3.6}$$

From (3.5)–(3.6) we then get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla(\Delta u + f(\cdot, u))\|^2_{L^2(\mathbb{R}^N)} \leqslant c^2_{1/2}\|\nabla(\Delta u + f(\cdot, u))\|^2_{L^2(\mathbb{R}^N)}, \tag{3.7}$$

where, due to [15] (3.9) and (3.11),

$$\int_s^t \|\nabla(\Delta u + f(\cdot, u))\|^2_{L^2(\mathbb{R}^N)} = E(u(s)) - E(u(t)) \leqslant E(u_0) + \frac{c_2}{c_1} \tag{3.8}$$

with constants c_1, c_2 as in (1.17). Since (3.7) yields that for $s < t+\tau$

$$
\begin{aligned}
&\|\nabla(\Delta u + f(\cdot,u))(t+\tau)\|^2_{L^2(\mathbb{R}^N)} \\
&\qquad \leqslant \|\nabla(\Delta u + f(\cdot,u))(s)\|^2_{L^2(\mathbb{R}^N)} + c^2_{1/2}\int_s^{t+\tau} \|\nabla(\Delta u + f(\cdot,u))\|^2_{L^2(\mathbb{R}^N)},
\end{aligned}
$$

integrating with respect to $s \in (t, t+\tau)$ and using (3.8) we obtain

$$
\text{(3.9)} \quad \|\nabla(\Delta u + f(\cdot,u))(t+\tau)\|^2_{L^2(\mathbb{R}^N)} \leqslant \frac{1+c^2_{1/2}\tau}{\tau}\Big(E(u_0) + \frac{c_2}{c_1}\Big), \quad t \geqslant 0,\ \tau > 0.
$$

On the other hand, using injectivity of P_0 in (1.12) (see [15], Lemma 3.1) and rewriting (1.1) as

$$
P_0^{-1}u_t = \Delta u + f(\cdot,u) = \Delta u + g(x) + m(x)u - u|u|^{\varrho_c - 1}
$$

we obtain with the aid of Young's inequality that

$$
\begin{aligned}
\text{(3.10)} \quad \|\Delta u\|^2_{L^2(\mathbb{R}^N)} &= \langle \Delta u, \Delta u\rangle_{L^2(\mathbb{R}^N)} \\
&= \langle \Delta u, P_0^{-1}u_t - g - mu + u|u|^{\varrho_c-1}\rangle_{L^2(\mathbb{R}^N)} \\
&= \langle \Delta u, \Delta u + f(\cdot,u)\rangle_{L^2(\mathbb{R}^N)} - \langle \Delta u, g\rangle_{L^2(\mathbb{R}^N)} \\
&\quad - \langle \Delta u, mu\rangle_{L^2(\mathbb{R}^N)} + \langle \Delta u,\ u|u|^{\varrho_c-1}\rangle_{L^2(\mathbb{R}^N)} \\
&\leqslant \frac{1}{4}\|\nabla u\|^2_{L^2(\mathbb{R}^N)} + \|\nabla(\Delta u + f(\cdot,u))\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{4}\|\Delta u\|^2_{L^2(\mathbb{R}^N)} \\
&\quad + \|g\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{4}\|\Delta u\|^2_{L^2(\mathbb{R}^N)} + \|mu\|^2_{L^2(\mathbb{R}^N)}.
\end{aligned}
$$

Concerning the last term in (3.10) we have from [15], Lemma A.1, that

$$
\|mu\|_{L^2(\mathbb{R}^N)} \leqslant c\|u\|_{H^{2\alpha}(\mathbb{R}^N)} \quad \text{for some } \alpha \in (0,1)
$$

and hence, by interpolation (see [33], §2.4.2 (11), §1.9.3 (3)) and Young's inequality,

$$
\text{(3.11)} \quad \|mu\|^2_{L^2(\mathbb{R}^N)} \leqslant c\|u\|^{2\alpha}_{H^2(\mathbb{R}^N)}\|u\|^{2(1-\alpha)}_{L^2(\mathbb{R}^N)} \leqslant \frac{1}{4}\|\Delta u\|^2_{L^2(\mathbb{R}^N)} + c_{1/4,\alpha}\|u\|^2_{L^2(\mathbb{R}^N)}.
$$

Combining (1.17), (3.10) and (3.11) we get

$$
\begin{aligned}
\frac{1}{4}\|\Delta u\|^2_{L^2(\mathbb{R}^N)} &\leqslant \Big(\frac{1}{4} + c_{1/4,\alpha}\Big)(c_1E(u_0) + c_2) \\
&\qquad + \|\nabla(\Delta u + f(x,u))\|^2_{L^2(\mathbb{R}^N)} + \|g\|^2_{L^2(\mathbb{R}^N)}.
\end{aligned}
$$

Application of (3.9) now leads to the estimate

$$\frac{1}{4}\|\Delta u(t+\tau)\|^2_{L^2(\mathbb{R}^N)} \leqslant \Big(\frac{1}{4}+c_{1/4,\alpha}\Big)(c_1E(u_0)+c_2) + \frac{1+c_{1/2}^2\tau}{\tau}\Big(E(u_0)+\frac{c_2}{c_1}\Big)+\|g\|^2_{L^2(\mathbb{R}^N)},$$

which together with (1.17) gives the result. □

Let us now discuss briefly that the solution in Proposition 1.1 will often possesses some more regularity properties than mentioned therein. For example, if for $\mathcal{F}$ in (1.11) we have that

(3.12) $\quad \mathcal{F}$ is Lipschitz continuous on bounded sets from $H^2(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$

then, following [23], [13], the solution u of (1.1)–(1.2) through $u_0 \in H^2(\mathbb{R}^N)$ will exist in the class $C([0,\tau_{u_0}),H^2(\mathbb{R}^N))$ and, in addition,

$$u \in C((0,\tau_{u_0}),H^4(\mathbb{R}^N)) \cap C^1((0,\tau_{u_0}),H^s(\mathbb{R}^N)), \quad s<4. \tag{3.13}$$

Since the solution through $u_0 \in H^1(\mathbb{R}^N)$ in Proposition 1.1 enters $H^2(\mathbb{R}^N)$, assuming (3.12) we obtain that (3.13) holds for such solution as well.

Concerning f as in (3.1) we finally remark that (3.12) will hold if e.g. $N=3$, $g \in H^2(\mathbb{R}^3)$ and $m \in BUC(\mathbb{R}^3)$ has bounded partial derivatives of order less or equal two. In this sample case the solutions through $u_0 \in H^1(\mathbb{R}^N)$ will then possess the regularity properties stated in (3.13).

4. Closing remarks

In the case of the critical exponent $\varrho = \varrho_c$ it remains generally unknown whether the blow up $H^1(\mathbb{R}^N)$-alternative holds; that is whether an $H^1(\mathbb{R}^N)$-estimate of the solutions implies their global existence. This was shown to hold true for nonlinearities satisfying the limit condition (2.1) but not for logistic type nonlinearities as in (3.1).

Due to the properties of the energy functional (1.3) observe that for arbitrarily fast growing logistic type nonlinearities $f(x,u)=g(x)+m(x)u-u|u|^{\varrho-1}$ one can formally obtain a uniform in time $H^1(\mathbb{R}^N)\cap L^{\varrho+1}(\mathbb{R}^N)$-estimate of the solutions. Also, analogously to Lemma 3.3, one can then derive the $H^2(\mathbb{R}^N)$-bound. Nonetheless, the local well posedness in $H^1(\mathbb{R}^N)$ as in Proposition 1.1 applies only for ϱ not exceeding ϱ_c and faster growing nonlinearities necessitate a separate treatment (see e.g. [16] and [34]).

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