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ON THE DIRICHLET AND NEUMANN PROBLEMS IN MULTI-DIMENSIONAL CONE

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Abstract. We consider an elliptic pseudodifferential equation in a multi-dimensional cone, and using the wave factorization concept for an elliptic symbol we describe a general solution of such equation in Sobolev-Slobodetskii spaces. This general solution depends on some arbitrary functions, their quantity being determined by an index of the wave factorization. For identifying these arbitrary functions one needs some additional conditions, for example, boundary conditions. Simple boundary value problems, related to Dirichlet and Neumann boundary conditions, are considered. A certain integral representation for this case is given.

Keywords: wave factorization; pseudodifferential equation; boundary value problem; integral equation

MSC 2010: 35J40, 42A38

1. Introduction

The key point in the theory of pseudodifferential equations and boundary value problems is studying invertibility of the model operator in a canonical domain [1]. Such model operator can be treated as a convolution operator in a well known sense. The author has used the idea to consider a cone as a canonical domain serving the theory of pseudodifferential equations on manifolds with non-smooth boundary [3]. Existence of such special wave factorization for symbols of elliptic pseudodifferential equations has permitted to obtain full solvability picture for model pseudodifferential equations in the two-dimensional case [4]. The author hopes that the consideration in more detail will allow to transfer the main results to spaces of arbitrary dimension. A pseudodifferential operator A denoted by $A(\xi)$, $\xi \in \mathbb{R}^m$, is defined by the formula

$$(Au)(x) = \int_{\mathbb{R}^m} A(\xi)\tilde{u}(\xi)\mathrm{e}^{\mathrm{i}x\xi}\,\mathrm{d}\xi, \tag{1.1}$$

where $\tilde{u}$ denotes the Fourier transform.

This is a model operator. Generally speaking one considers pseudodifferential operators depending on the space variable x. Following the tradition we call the variable x the space variable, and ξ the co-variable or impulse variable. The operator denoted by $A(x,\xi)$ is defined as in (1.1) with help of the formula

$$u(x) \longmapsto \int_{\mathbb{R}^m} A(x,\xi)\tilde{u}(\xi)\mathrm{e}^{\mathrm{i}x\xi}\,\mathrm{d}\xi$$

by "freezing" the space variable x.

Here we will consider the class of symbols independent of the space variable x and satisfying the following condition: $\exists\, c_1, c_2$,

$$c_1 \leqslant |A(\xi)(1+|\xi|)^{-\alpha}| \leqslant c_2, \quad \forall\, \xi \in \mathbb{R}^m. \tag{1.2}$$

The number $\alpha \in \mathbb{R}$ we call the order of the pseudodifferential operator A.

We will denote by P_α the symbol class satisfying the condition (1.2).

Let us define the Sobolev-Slobodetskii functional space $H^s(\mathbb{R}^m)$ as the Hilbert space of distributions [1] with the norm

$$\|u\|_s^2 = \int_{\mathbb{R}^m} |\tilde{u}(\xi)|^2(1+|\xi|)^{2s}\,\mathrm{d}\xi.$$

It is well-known that an operator from P_α is a linear bounded operator acting from $H^s(\mathbb{R}^m)$ into $H^{s-\alpha}(\mathbb{R}^m)$, see [1]. Everywhere below we use $\widetilde{H}^s(M)$ to denote the Fourier image of the space $H^s(M)$.

Let us go to studying solvability of pseudodifferential equations

$$(Au_+)(x) = f(x), \quad x \in C_+^a, \tag{1.3}$$

in the space $H^s(C_+^a)$, where C_+^a is the m-dimensional cone $C_+^a = \{x \in \mathbb{R}^m \colon x = (x_1,\ldots,x_{m-1},x_m),\ x_m > a|x'|,\ a > 0\}$, $x' = (x_1,\ldots,x_{m-1})$. (We write u_+ to show the solution is defined in C_+^a only.)

By definition, the space $H^s(C_+^a)$ consists of distributions from $H^s(\mathbb{R}^m)$, whose support belongs to C_+^a. The norm in the space $H^s(C_+^a)$ is induced by the norm from $H^s(\mathbb{R}^m)$. The right-hand side f is chosen from the space $H_0^{s-\alpha}(C_+^a)$; by definition the space $H_0^s(C_+^a)$ is the space of distributions on C_+^a admitting a continuation to $H^s(\mathbb{R}^m)$. The norm in the space $H_0^s(C_+^a)$ is defined by

$$\|f\|_s^+ = \inf \|lf\|_s,$$

where the *infimum* is taken over all continuations lf on the whole $\mathbb{R}^m$.

The symbol $\overset{*}{C}{}^a_+$ denotes the conjugate cone for C^a_+:

$$\overset{*}{C}{}^a_+ = \{x \in \mathbb{R}^m\colon\ x = (x', x_m),\ ax_m > |x'|\},$$

$C^a_- \equiv -C^a_+$; $T(C^a_+)$ denotes the radial tube domain over the cone C^a_+, i.e., the domain in the complex space $\mathbb{C}^m$ of type $\mathbb{R}^m + \mathrm{i}C^a_+$.

Further, let us define a special multi-dimensional singular integral by the formula

$$(G_m u)(x) = \lim_{\tau\to 0+} \int_{\mathbb{R}^m} \frac{u(y', y_m)\,\mathrm{d}y'\,\mathrm{d}y_m}{(|x'-y'|^2 - a^2(x_m - y_m + \mathrm{i}\tau)^2)^{m/2}}.$$

To describe the solvability picture for the equation (1.3) we will introduce the following

Definition. By wave factorization for $A(\xi)$, satisfying the condition (1.2), we mean its representation in the form

$$A(\xi) = A_{\neq}(\xi)A_{=}(\xi)$$

where the factors $A_{\neq}(\xi)$, $A_{=}(\xi)$ satisfy the following conditions:

1) $A_{\neq}(\xi), A_{=}(\xi)$ are defined for all admissible values $\xi \in \mathbb{R}^m$, without, may be, the points $\{\xi \in \mathbb{R}^m\colon\ |\xi'|^2 = a^2\xi_m^2\}$;

2) $A_{\neq}(\xi), A_{=}(\xi)$ admit analytical continuations into the radial tube domains $T(\overset{*}{C}{}^a_+), T(\overset{*}{C}{}^a_-)$, respectively, with estimates

$$|A_{\neq}^{\pm 1}(\xi + \mathrm{i}\tau)| \leqslant c_1(1 + |\xi| + |\tau|)^{\pm æ},$$
$$|A_{=}^{\pm 1}(\xi - \mathrm{i}\tau)| \leqslant c_2(1 + |\xi| + |\tau|)^{\pm(\alpha - æ)}, \quad \forall\, \tau \in \overset{*}{C}{}^a_+.$$

The number $æ \in \mathbb{R}$ is called the index of wave factorization.

Everywhere below we will suppose that the wave factorization mentioned exists.

2. Solving procedure

Now we will consider the equation (1.3) for the case $æ - s = n + \delta$, $n \in \mathbb{N}$, $|\delta| < 1/2$, only. A general solution can be constructed as follows. After wave factorization for the symbol with preliminary Fourier transform we write

$$A_{\neq}(\xi)\tilde{u}_+(\xi) + A_{=}^{-1}(\xi)\tilde{u}_-(\xi) = A_{=}^{-1}(\xi)\widetilde{lf}(\xi),$$

where $u_-(x) = lf(x) - u_+(x)$, lf is an arbitrary continuation of f to the whole $\mathbb{R}^m$.

One can see that $A_{=}^{-1}(\xi)\widetilde{lf}(\xi)$ belongs to the space $\widetilde{H}^{s-æ}(\mathbb{R}^m)$, and if we choose the polynomial $Q(\xi)$ satisfying the condition

$$|Q(\xi)| \sim (1+|\xi|)^n,$$

then $Q^{-1}(\xi)A_{=}^{-1}(\xi)\widetilde{lf}(\xi)$ will belong to the space $\widetilde{H}^{-\delta}(\mathbb{R}^m)$.

Further, according to the theory of the multi-dimensional Riemann problem [2], we can decompose the last function into two summands (jump problem):

$$Q^{-1}A_{=}^{-1}\widetilde{lf} = f_+ + f_-,$$

where $f_+ \in \widetilde{H}(C_+^a)$, $f_- \in \widetilde{H}(\mathbb{R}^m \setminus C_+^a)$.

So, we have

$$Q^{-1}A_{\neq}\tilde{u}_+ + Q^{-1}A_{=}^{-1}\tilde{u}_- = f_+ + f_-,$$

or

$$Q^{-1}A_{\neq}\tilde{u}_+ - f_+ = f_- - Q^{-1}A_{=}^{-1}\tilde{u}_-.$$

In other words,

$$A_{\neq}\tilde{u}_+ - Qf_+ = Qf_- - A_{=}^{-1}\tilde{u}_-.$$

The left-hand side of the equality belongs to the space $\widetilde{H}^{-n-\delta}(C_+^a)$, and the right-hand side belongs to $\widetilde{H}^{-n-\delta}(\mathbb{R}^m \setminus C_+^a)$, hence

$$F^{-1}(A_{\neq}\tilde{u}_+ - Qf_+) = F^{-1}(Qf_- - A_{=}^{-1}\tilde{u}_-), \tag{2.1}$$

where the left-hand side belongs to $H^{-n-\delta}(C_+^a)$, and the right-hand side belongs to $H^{-n-\delta}(\mathbb{R}^m \setminus C_+^a)$, therefore we conclude immediately that (2.1) is a distribution supported on ∂C_+^a.

The main tool now is to define the form of the distribution.

Let us denote by T_a the transformation $\mathbb{R}^m \longrightarrow \mathbb{R}^m$ of the type

$$\begin{cases} t_1 = x_1, \\ \quad\vdots \\ t_{m-1} = x_{m-1}, \\ t_m = x_m - a|x'| \end{cases}$$

(obviously, it one-to-one transforms ∂C_+^a into the hyperplane $x_m = 0$).

Then the function

$$T_aF^{-1}(A_{\neq}\tilde{u}_+ - Qf_+)$$

will be supported on the hyperplane $t_m = 0$ and belong to $H^{-n-\delta}(\mathbb{R}^m)$. Such distribution is the linear span of the Dirac mass-function and its derivatives [2] and looks as

$$\sum_{k=0}^{n-1} c_k(t')\delta^{(k)}(t_m).$$

It is left to find out, what is the Fourier image of the operator T_a. Explicit calculations give simple answer:

$$FT_a u = V_a \tilde{u},$$

where V_a is a special operator (roughly speaking it is a pseudodifferential operator denoted by $\mathrm{e}^{-\mathrm{i}a|\xi'|\xi_m}$), and further one can construct the general solution for our pseudodifferential equation(1.3).

Lemma 2.1. $V_a = FT_aF^{-1}$.

P r o o f. It follows from the relations

$$\begin{aligned}(FT_a u)(\xi) &= \int_{\mathbb{R}^m} \mathrm{e}^{-\mathrm{i}x\cdot\xi} u(x_1, \dots, x_{m-1}, x_m - a|x'|)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^m} \mathrm{e}^{-\mathrm{i}y'\xi'} \mathrm{e}^{-\mathrm{i}(y_m + a|y'|)\xi_m} u(y_1, \dots, y_{m-1}, y_m)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^{m-1}} \mathrm{e}^{-\mathrm{i}a|y'|\xi_m} \mathrm{e}^{-\mathrm{i}y'\xi'}\, \hat{u}(y_1, \dots, y_{m-1}, \xi_m)\,\mathrm{d}y',\end{aligned}$$

where $\hat{u}$ denotes the Fourier transform with respect to the last variable, and the Jacobian of T_a is equal to 1 everywhere except the origin and bounded.

According to the properties of the Fourier transform the product of two functions becomes their convolution. Roughly speaking the operator V_a is a convolution for $m-1$ variables, and a multiplier for the last variable. This proves the theorem. □

Lemma 2.2. $T_a^{-1} = T_{-a}$, $V_a^{-1} = V_{-a}$.

P r o o f. It follows immediately from the definition of T_a and the previous lemma. □

Notice that distributions supported on conical surface and their Fourier transforms were considered in [2], but the author did not find the multi-dimensional analogue of the theorem on a distribution supported at a single point in all issues of his book.

3. Construction

The following result is valid.

Theorem 3.1. *The Fourier image of the general solution of the equation* (1.3) *is given by the formula*

$$\tilde{u}(\xi) = A_{\neq}^{-1}(\xi)Q(\xi)G_m Q^{-1}(\xi)A_{=}^{-1}(\xi)\widetilde{lf}(\xi) + A_{\neq}^{-1}(\xi)V_{-a}F\bigg(\sum_{k=1}^{n} c_k(x')\delta^{(k-1)}(x_m)\bigg),$$

where $c_k(x') \in H^{s_k}(\mathbb{R}^{m-1})$ *are arbitrary functions,* $s_k = s - æ + k - 1/2$, $k = 1, 2, \ldots, n$, lf *is an arbitrary continuation of* f *on* $H^{s-\alpha}(\mathbb{R}^m)$.

P r o o f. Let us go back to equality (2.1). If we now apply the operator T_a to both the left and right hand sides then these will be distributions supported on the hyper-plane $x_m = 0$. The form of such a distribution is well-known, see [1], and the operator V_a does not change the order of the H^s-space. Thus,

$$T_a F^{-1}(A_{\neq}\tilde{u}_+ - Qf_+) = \sum_{k=1}^{n} c_k(x')\delta^{(k-1)}(x_m),$$

and after Fourier transform

$$FT_a F^{-1}(A_{\neq}\tilde{u}_+ - Qf_+) = F\bigg(\sum_{k=1}^{n} c_k(x')\delta^{(k-1)}(x_m)\bigg).$$

Further, taking into account Lemmas 2.1 and 2.2 we complete the proof. □

Starting from this representation one can suggest different statements of boundary value problems for the equation (1.3).

4. Boundary conditions

Let us consider a very simple case, when $f \equiv 0$, $a = 1$, $n = 1$. Then the formula above takes the form

$$\tilde{u}(\xi) = A_{\neq}^{-1}(\xi)V_{-1}\tilde{c}_0(\xi').$$

We consider separately the following construction. According to the Fourier transform our solution is

$$u(x) = F^{-1}\{A_{\neq}^{-1}(\xi)V_{-1}\tilde{c}_0(\xi')\}. \tag{4.1}$$

Let us suppose we put the Dirichlet boundary condition on ∂C_+^1, i.e.,

$$(Pu)(y) = g(y),$$

where g is a given function on ∂C_+^1, P is the restriction operator on the boundary, so we know the solution on the boundary ∂C_+^1.

Thus,

$$Tu(x) = TF^{-1}\{A_{\neq}^{-1}(\xi)V_{-1}\tilde{c}_0(\xi')\} = V_1\{A_{\neq}^{-1}(\xi)V_{-1}\tilde{c}_0(\xi')\},$$

and we know that $(P'Tu)(x') \equiv v(x')$, where P' is the restriction operator on the hyperplane $x_m = 0$.

The relation between the operators P' and F is well-known [1]:

$$(FP'u)(\xi') = \int_{-\infty}^{\infty} \tilde{u}(\xi', \xi_m)\,\mathrm{d}\xi_m.$$

Returning to the formula (4.1) we obtain

$$\tilde{v}(\xi') = \int_{-\infty}^{\infty} \{V_1\{A_{\neq}^{-1}(\xi)V_{-1}\tilde{c}_0(\xi')\}\}(\xi', \xi_m)\,\mathrm{d}\xi_m, \tag{4.2}$$

where $\tilde{v}(\xi')$ is a given function. Hence, the equation (4.2) is an integral equation for determining $c_0(x')$.

Let us note that for the two-dimensional case the author earlier obtained certain integral equations for determining the unknown functions, and study these equations by Mellin transform reducing them to a system of linear difference equations [5].

The Neumann boundary condition leads to an analogous integral equation.

Studying these integral equations will be the topic of forthcoming papers of the author.

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