Hiroshi Watanabe; Ken Shirakawa

Stability for approximation methods of the one-dimensional Kobayashi-Warren-Carter system

Mathematica Bohemica, Vol. 139 (2014), No. 2, 381--389

Persistent URL: `http://dml.cz/dmlcz/143863`

139 (2014) MATHEMATICA BOHEMICA No. 2, 381–389

STABILITY FOR APPROXIMATION METHODS OF THE ONE-DIMENSIONAL KOBAYASHI-WARREN-CARTER SYSTEM

Hiroshi Watanabe, Tokyo, Ken Shirakawa, Chiba

(Received September 30, 2013)

Abstract. A one-dimensional version of a gradient system, known as "Kobayashi-Warren-Carter system", is considered. In view of the difficulty of the uniqueness, we here set our goal to ensure a "stability" which comes out in the approximation approaches to the solutions. Based on this, the Main Theorem concludes that there is an admissible range of approximation differences, and in the scope of this range, any approximation method leads to a uniform type of solutions having a certain common features. Further, this is specified by using the notion of "energy-dissipative solution", proposed in a relevant previous work.

Keywords: approximation method; stability; energy-dissipative solution

MSC 2010: 35K87, 35R06, 35K67

1. Introduction

Let $0 < L$, $T < \infty$ be fixed constants, and let $\Omega := (-L, L)$ be a one-dimensional bounded domain with the boundary points $\pm L \in \mathbb{R}$. Let $Q := (0, T) \times \Omega$ be the product set of the time-interval $(0, T)$ and the spatial domain Ω.

In this paper, the one dimensional version of a gradient system, known as "Kobayashi-Warren-Carter system", is considered. This system is denoted by (S) and is derived from the following free-energy:

$$\mathscr{F}(\eta, \theta) := \frac{1}{2}\int_\Omega |\eta_x|^2 \, \mathrm{d}x + \int_\Omega \hat{g}(\eta) \, \mathrm{d}x + \int_\Omega \alpha(\eta)|D_x\theta|, \quad \forall\, [\eta, \theta] \in H^1(\Omega) \times BV(\Omega),$$

including an unknown-dependent total variation $\int_\Omega \alpha(\eta)|D_x\theta|$, where $\hat{g}$ is a given nonnegative function, and α is a given positive convex function. Accordingly, the system (S) is formally described as follows.

The research is supported by Grant-in-Aid Nos. 25800086, 23540206 and 24740099, JSPS.

(S):

$$
(1.1)\qquad \begin{cases} \eta_t - \eta_{xx} + g(\eta) + \alpha'(\eta)|D_x\theta| = 0 \quad \text{in } Q, \\ \eta_x(t, \pm L) = 0, \quad t \in (0,T), \\ \eta(0,x) = \eta_0(x), \quad x \in \Omega; \end{cases}
$$

$$
(1.2)\qquad \begin{cases} \alpha_0(\eta)\theta_t - \Big(\alpha(\eta)\dfrac{D_x\theta}{|D_x\theta|}\Big)_x = 0 \quad \text{in } Q, \\ \alpha(\eta(t,\pm L))\dfrac{D_x\theta}{|D_x\theta|}(t,\pm L) = 0, \quad t \in (0,T), \\ \alpha_0(\eta(0,x))\theta(0,x) = \alpha_0(\eta_0(x))\theta_0(x), \quad x \in \Omega, \end{cases}
$$

where g is the derivative of $\hat{g}$, α' is the differential of α, and α_0 is a given nonnegative function.

System (S) is based on a phase-field model of a planar grain boundary motion in a polycrystal, proposed by Kobayashi-Warren-Carter [5]. In the context, the physical situation is reproduced by using two order parameters $\eta = \eta(t,x)$ and $\theta = \theta(t,x)$, which indicate, respectively, the orientation order and the orientation angle of the grain. In particular, η is supposed to satisfy the range constraint $0 \leqslant \eta \leqslant 1$ on Q, and the threshold values 1 and 0 are supposed to indicate, respectively, the completely oriented phase and the disordered phase of orientation.

Recently, the existence results relative to the Kobayashi-Warren-Carter system were reported in several literatures [2], [3], [6], [8], [9], [10], and in each of these, the forerunners adopted some approximation problems, configured as gradient systems of the following types of relaxed free-energies:

$$
\begin{aligned} \mathscr{F}_\nu(\eta,\theta) := \frac{1}{2}\int_\Omega |\eta_x|^2\,\mathrm{d}x + \int_\Omega \hat{g}(\eta)\,\mathrm{d}x + \int_\Omega \alpha(\eta)\beta_\nu(\theta_x)\,\mathrm{d}x \\ + \frac{\nu}{2}\int_\Omega |\theta_x|^2\,\mathrm{d}x,\ \forall\,[\eta,\theta] \in H^1(\Omega)\times H^1(\Omega), \quad \forall\,\nu \in (0,1). \end{aligned}
$$

Here, $\nu \in (0,1)$ is an index of the parabolic regularization for (1.2), and hence this index approximates the original situation as $\nu \downarrow 0$. Further $\{\alpha_\nu\,;\ \nu \in (0,1)\}$ and $\{\beta_\nu\,;\ \nu \in (0,1)\}$ are sequences of functions to relax, respectively, the possibly degenerate situation of α_0 and the nonsmoothness of the absolute-value function $|\cdot|$.

Meanwhile, with regard to the uniqueness, the standard analytic technique is not applicable, due to the unknown dependence of the weights $\alpha_0(\eta)$ and $\alpha(\eta)$ in (1.2). This difficulty has cost us satisfactory answers for the uniqueness question, except for a quite restrictive case (cf. [2], Theorem 2.2). Hence, we must still worry about that the system (S) may include some “instability” for the approximation methods, since we cannot deny the possibility of obtaining different types of solutions relative to the choices of the approximation components $\{\alpha_\nu\}$ and $\{\beta_\nu\}$.

In view of this, we here set our goal to ensure the "stability" for the approximation methods. To this end, a class $\mathscr{A}$ of pairs $A = [\{\alpha_\nu\}, \{\beta_\nu\}]$ of the approximation components will be defined to prescribe an admissible range of the approximation differences (oscillations). Based on this, the Main Theorem of this paper will be to conclude that any approximation method associated with any $A = [\{\alpha_\nu\}, \{\beta_\nu\}] \in \mathscr{A}$ leads to a uniform type of solutions to $(\mathrm{S})_\nu$, having certain commonalities. Furthermore, the commonalities will be specified on the basis of the notion of the "energy-dissipative solution", proposed in the relevant previous work [8], Definition 3.1.

2. Main Theorem

We begin with confirming the assumptions and notation in this study.

Assumptions. Here are the assumptions in the study of the system (S).

(A1) $g \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$, $g(0) < 0$, $g(1) \geqslant 0$, and g has a nonnegative primitive $\hat{g} \in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R})$.

(A2) $\alpha_0 \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$, $\alpha_0 \geqslant 0$ on $\mathbb{R}$, and $(\alpha_0)^{-1}(0) = \{0\}$.

(A3) $\alpha \in C^1(\mathbb{R})$, $\alpha > 0$ on $\mathbb{R}$, and it is a convex function such that the derivative $\alpha' \in C(\mathbb{R})$ satisfies $\alpha'(0) = 0$. Moreover, $\delta_\alpha := \alpha(0) = \min_{\tilde{\eta} \in \mathbb{R}} \alpha(\tilde{\eta}) > 0$.

(A4) The initial data $[\eta_0, \theta_0]$ belongs to a class $D_0 \subset C(\overline{\Omega}) \times L^\infty(\Omega)$, defined as

$$D_0 := \{[w, z] \in H^1(\Omega) \times BV(\Omega);\ 0 \leqslant w \leqslant 1 \ \text{ on } \overline{\Omega}\}.$$

Specific notation. For any $w \in C(\overline{\Omega})$, we define a functional $\Phi(w; \cdot)$ on $L^2(\Omega)$ by putting

$$\Phi(w, z) := \begin{cases} \displaystyle\int_\Omega \alpha(w)|D_x z|, & \text{if } z \in BV(\Omega), \\ \infty, & \text{otherwise}, \end{cases} \qquad \text{for any } z \in L^2(\Omega). \tag{2.1}$$

Furthermore, for any open interval $I \subset (0, T)$ and any $\xi \in C(\overline{I \times \Omega})$, we define a functional $\widehat{\Phi}(\xi; \cdot)_I$ on $L^2(I; L^2(\Omega))$ by putting

$$\widehat{\Phi}(\xi; \zeta)_I := \int_I \Phi(\xi(t); \zeta(t))\, \mathrm{d}t, \quad \text{for any } \zeta \in L^2(I; L^2(\Omega)). \tag{2.2}$$

As is easily seen, the functional given in (2.1) is a proper lower semicontinuous and convex function on $L^2(\Omega)$. We denote by $\partial\Phi(w; \cdot)$ the subdifferential of the convex function $\Phi(w; \cdot)$ in the topology of $L^2(\Omega)$, for any $w \in C(\overline{\Omega})$. Also, for any open interval $I \subset (0,T)$, it follows from (A3) and [9], Lemma 3 that the functional given by (2.2) is a proper lower semicontinuous and convex function on $L^2(I;L^2)$, and

$$D(\widehat{\Phi}(\xi;\cdot)_I) = \{\tilde{\zeta} \in L^2(I;L^2(\Omega));\ |D_x\tilde{\zeta}(\cdot)|(\Omega) \in L^1(I)\}, \quad \forall\, \xi \in C(\overline{I \times \Omega}).$$

Next, we propose (AP1)–(AP3) as the conditions to prescribe an admissible range of approximations.

(AP1) $\{\alpha_\nu;\ \nu \in (0,1)\} \subset W^{1,\infty}_{\rm loc}(\mathbb{R})$ such that $\alpha_\nu > 0$ on $\mathbb{R}$, $\forall\, \nu \in (0,1)$, and $\alpha_\nu \to \alpha_0$ in $C_{\rm loc}(\mathbb{R})$ as $\nu \downarrow 0$.

(AP2) $\{\beta_\nu;\ \nu \in (0,1)\} \subset W^{1,\infty}(\mathbb{R}) \cap W^{2,\infty}_{\rm loc}(\mathbb{R})$ is a sequence of convex functions such that $\beta_\nu \geqslant 0$ on $\mathbb{R}$ and $\beta_\nu(0) = 0$, $\forall\, \nu \in (0,1)$.

(AP3) There exist bounded functions $q_k\colon (0,1) \mapsto (0,\infty)$, $r_k\colon (0,1) \mapsto [0,\infty)$, $k = 0,1$, such that $\lim\limits_{\nu\downarrow 0} q_k(\nu) = 1$, $\lim\limits_{\nu\downarrow 0} r_k(\nu) = 0$, $k = 0,1$, and $q_0(\nu)|\tau| - r_0(\nu) \leqslant \beta_\nu(\tau) \leqslant q_1(\nu)|\tau| + r_1(\nu)$, $\forall\, \tau \in \mathbb{R}$, $\forall\, \nu \in (0,1)$.

Now, we define the class $\mathscr{A}$, mentioned in Introduction, as follows:

$$\mathscr{A} := \{A = [\{\alpha_\nu\},\{\beta_\nu\}];\ \{\alpha_\nu\} \text{ and } \{\beta_\nu\} \text{ fulfill (AP1)–(AP3)}\}. \tag{2.3}$$

Subsequently, for any $A = [\{\alpha_\nu\},\{\beta_\nu\}] \in \mathscr{A}$ and any $w \in L^2(\Omega)$, we define a sequence $\{\Phi_\nu(w;\cdot);\ \nu \in (0,1)\}$ of relaxed convex functions for $\Phi(w;\cdot)$ by putting

$$\Phi_\nu(w;z) := \begin{cases} \displaystyle\int_\Omega \alpha(w)\beta_\nu(z_x)\,\mathrm{d}x + \frac{\nu}{2}\int_\Omega |z_x|^2\,\mathrm{d}x, \\ \qquad\qquad \text{if } z \in H^1(\Omega),\ \forall\, z \in L^2(\Omega),\ \forall\, \nu \in (0,1). \\ \infty, \qquad \text{otherwise,} \end{cases} \tag{2.4}$$

Moreover, for any $A = [\{\alpha_\nu\},\{\beta_\nu\}] \in \mathscr{A}$, any open interval $I \subset (0,T)$ and any $\xi \in L^2(I;L^2(\Omega))$, we define a sequence $\{\widehat{\Phi}_\nu(\xi;\cdot)_I;\ \nu \in (0,1)\}$ of relaxed convex functions for $\widehat{\Phi}(\xi;\cdot)_I$ by putting

$$\widehat{\Phi}_\nu(\xi;\zeta)_I := \int_I \Phi_\nu(\xi(t);\zeta(t))\,\mathrm{d}t, \quad \forall\, \zeta \in L^2(I;L^2(\Omega)),\ \forall\, \nu \in (0,1). \tag{2.5}$$

As is easily seen, the functionals given in (2.4) are proper lower semicontinuous and convex functions on $L^2(\Omega)$. Also, for any open interval $I \subset (0,T)$, it follows from (A3) that the functionals given in (2.5) are proper lower semicontinuous and convex functions on $L^2(I;L^2(\Omega))$, and all of their effective domains uniformly coincide with the space $L^2(I;H^1(\Omega))$.

R e m a r k 2.1. The conditions (AP2)–(AP3) cover various regularization methods of the absolute-value function $|\cdot|$, e.g., approximations by using hyperbolic graphs, Yosida-regularizations, primitives of tanh and arctan, etc., although the approximation by $|\cdot|^p$ with $p > 1$ is not in the applicable scope of these conditions.

Now, for any pair $A = [\{\alpha_\nu\}, \{\beta_\nu\}] \in \mathscr{A}$, we can consider a sequence of functional pairs $\{[\eta_\nu, \theta_\nu];\ \nu \in (0,1)\}$ consisting of solutions $[\eta_\nu, \theta_\nu]$, $\nu \in (0,1)$, to the following approximation problems, denoted by $(\mathrm{AP})_\nu$:

$(\mathrm{AP})_\nu$: for a fixed index $\nu \in (0,1)$, find a functional pair $[\eta_\nu, \theta_\nu]$ fulfilling

$$(2.6)\quad \begin{cases} \eta_\nu \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)), \\ \qquad \text{and } 0 \leqslant \eta_\nu \leqslant 1 \quad \text{on } \overline{Q}, \\ \theta_\nu \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)), \text{ and } |\theta_\nu|_{C(\overline{Q})} \leqslant |\theta_{0,\nu}|_{C(\overline{\Omega})}, \end{cases}$$

$$(\eta_\nu)_t(t) + \Lambda_N \eta_\nu(t) + g(\eta_\nu(t)) + \alpha'(\eta_\nu(t))\beta_\nu((\theta_\nu)_x(t)) = 0 \quad \text{in } L^2(\Omega),\ t \in (0,T),$$
$$\alpha_\nu(\eta_\nu(t))(\theta_\nu)_t(t) + \partial\Phi_\nu(\eta_\nu(t);\theta_\nu(t)) \ni 0 \quad \text{in } L^2(\Omega),\ t \in (0,T),$$
$$[\eta_\nu(0), \theta_\nu(0)] = [\eta_{0,\nu}, \theta_{0,\nu}] \quad \text{in } L^2(\Omega) \times L^2(\Omega),$$

where $[\eta_{0,\nu}, \theta_{0,\nu}] \in D_0 \cap H^1(\Omega)^2$ is the relaxed initial data satisfying $|\theta_{0,\nu}|_{C(\overline{\Omega})} \leqslant |\theta_0|_{L^\infty(\Omega)}$ for all $\nu \in (0,1)$, and Λ_N is an operator defined as

$$\Lambda_N\colon\ z \in \{\tilde{z} \in H^2(\Omega);\ \tilde{z}_x(\pm L) = 0\} \mapsto \Lambda_N z := -z_{xx} \in L^2(\Omega),$$

and for any $\nu \in (0,1)$ and any $w \in L^2(\Omega)$, $\partial\Phi_\nu(w,\cdot)$ denotes the subdifferential of the convex function $\Phi_\nu(w;\cdot)$ on $L^2(\Omega)$.

For any $\nu \in (0,1)$, the existence and uniqueness of the solution to $(\mathrm{AP})_\nu$ can be verified by referring to the analytic method as in [2], [3], [4]. Furthermore, as another consequence, we can derive the following energy identity:

$$(2.7)\qquad \int_s^t (|(\eta_\nu)_t(t)|^2_{L^2(\Omega)} + |\sqrt{\alpha_\nu(\eta_\nu(t))}(\theta_\nu)_t(t)|^2_{L^2(\Omega)})\,\mathrm{d}t + \mathscr{F}_\nu(\eta_\nu(t), \theta_\nu(t))$$
$$= \mathscr{F}_\nu(\eta_\nu(s), \theta_\nu(s)), \quad \forall\, s, \forall\, t \in [0,T],\ \forall\, \nu \in (0,1).$$

Now, our Main Theorem is stated as follows.

Main Theorem (Stability for approximation methods)**.** *Let us assume* (A1)–(A4). *Then, for any $A \in \mathscr{A}$, the limiting set, defined as*

$$\omega(A) := \left\{ [\tilde{\eta}, \tilde{\theta}] \in L^\infty(Q)^2 \;\middle|\; \begin{array}{l} \exists\{\nu_n\} \subset (0,1) \textit{ such that the sequence of} \\ \textit{the approximate solutions } [\eta_n, \theta_n] := [\eta_{\nu_n}\theta_{\nu_n}] \\ (\mathrm{AP})_{\nu_n}, \textit{ for } n \in \mathbb{N}, \textit{ converges to } [\tilde{\eta}, \tilde{\theta}] \\ \textit{in the weak-}{*}\textit{ topology of } L^\infty(Q)^2 \textit{ as } n \to \infty \end{array} \right\}$$

is nonempty, and the union $\bigcup_{A \in \mathscr{A}} \omega(A)$ *is a subset of the class of functional pairs* $[\eta, \theta]$, *fulfilling the following conditions:*

(S1) $\eta \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega))$, $0 \leqslant \eta \leqslant 1$ *on* $\overline{Q}$, $\theta \in L^\infty(Q)$, $|\theta(\cdot)|_{BV(\Omega)} \in L^\infty(0,T)$, $\theta_t \in L^2_{\rm loc}(\overline{Q} \setminus \eta^{-1}(0))$, $|\theta|_{L^\infty(Q)} \leqslant |\theta_0|_{L^\infty(\Omega)}$, *and* $\alpha_0(\eta)\theta \in W^{1,2}(0,T;L^2(\Omega)) \cap BV(Q)$.

(S2) η *solves* (1.1) *in the following variational sense:*

$$\int_\Omega (\eta_t(t) + g(\eta(t)))w \,\mathrm{d}x + \int_\Omega \eta_x(t) w_x \,\mathrm{d}x + \int_\Omega w\alpha'(\eta(t))|D_x\theta(t)| = 0,$$
$$\forall\, w \in H^1(\Omega),\ \text{a.e. } t \in (0,T),\ \ \text{with } \eta(0) = \eta_0 \text{ in } L^2(\Omega).$$

(S3) θ *solves* (1.2) *in the following variational sense:*

$$\int_\Omega \vartheta^*(t)(\theta(t) - z)\,\mathrm{d}x + \int_\Omega \alpha(\eta(t))|D_x\theta(t)| \leqslant \int_\Omega \alpha(\eta(t))|D_x z|,$$
$$\forall\, z \in BV(\Omega),\ \text{a.e. } t \in (0,T),\ \text{with } \alpha_0(\eta(0))\theta(0) = \alpha_0(\eta_0)\theta_0 \text{ in } L^2(\Omega),$$

where $\vartheta^* := [\alpha_0(\eta)\theta]_t - [\alpha_0(\eta)]_t\theta \in L^2(0,T;L^2(\Omega))$.

(S4) $[\eta,\theta]$ *satisfies the following energy inequality:*

$$\int_s^t |\eta_t(\tau)|^2_{L^2(\Omega)}\,\mathrm{d}\tau + \frac{1}{1+|\alpha_0|_{C[0,1]}}\int_s^t |\vartheta^*(\tau)|^2_{L^2(\Omega)}\,\mathrm{d}\tau + \mathscr{F}(\eta(t),\theta(t))$$
$$\leqslant \mathscr{F}(\eta(s),\theta(s)), \quad \text{a.e. } 0 < s \leqslant t < T.$$

R e m a r k 2.2. Note that (S1)–(S4) are based on the conditions defining the "energy-dissipative solution" proposed in [8], Definition 3.1. Hence, our Main Theorem is to conclude that if the approximation method is taken in the range of $\mathscr{A}$, then the approximation stably leads to a uniform type of solutions belonging to the category of energy-dissipative solutions.

3. Proof of Main Theorem

Let us fix any $A = [\{\alpha_\nu\},\{\beta_\nu\}] \in \mathscr{A}$ to consider a sequence $\{[\eta_\nu,\theta_\nu];\ \nu \in (0,1)\}$ of solutions $[\eta_\nu,\theta_\nu]$ to $(\mathrm{AP})_\nu$ for $\nu \in (0,1)$. Then, on account of (2.6) and (2.7), it will be possible to show the existence of an approximation limit $[\eta,\theta] \in \omega(A)$. Also, an analytic method similar to that in [10], Section 5 will be applicable for the verification of (S1)–(S4), if we can prove the following lemma concerned with the Mosco-convergences of convex functions.

Lemma 3.1 (cf. Mosco [7]). *Under (A3), the following two items hold:*

(I) *If $w_0 \in C(\overline{\Omega})$, $\{w_\nu;\ \nu \in (0,1)\} \subset C(\overline{\Omega})$, and $w_\nu \to w_0$ in $C(\overline{\Omega})$ as $\nu \downarrow 0$, then the sequence of convex functions $\{\Phi_\nu(w_\nu;\cdot);\ \nu \in (0,1)\}$ converges to the convex function $\Phi(w_0;\cdot)$ on $L^2(\Omega)$, in the sense of Mosco, as $\nu \downarrow 0$.*

(II) *For any open interval $I \subset (0,T)$, if $\xi_0 \in C(\overline{I\times\Omega})$, $\{\xi_\nu;\ \nu \in (0,1)\} \subset C(\overline{I\times\Omega})$, and $\xi_\nu \to \xi_0$ in $C(\overline{I\times\Omega})$ as $\nu \downarrow 0$, then the sequence of convex functions $\{\widehat{\Phi}_\nu(\xi_\nu;\cdot)_I;\ \nu \in (0,1)\}$ converges to the convex function $\widehat{\Phi}(\xi_0;\cdot)_I$ on $L^2(I;L^2(\Omega))$, in the sense of Mosco, as $\nu \downarrow 0$.*

Now, all we have to do is reduced to giving the proof of Lemma 3.1.

P r o o f of Lemma 3.1. We prove only the item (II), because the other item (I) can be obtained similarly and more simply. Then, according to the general theory of Mosco [7], we need to verify the following two items.

(M1) (Lower bound.) $\liminf\limits_{\nu\downarrow 0} \widehat{\Phi}_\nu(\xi_\nu;\check{\zeta}_\nu)_I \geqslant \widehat{\Phi}(\xi_0;\check{\zeta}_0)_I$, if $\check{\zeta}_0 \in L^2(I;L^2(\Omega))$, $\{\check{\zeta}_\nu;\ \nu \in (0,1)\} \subset L^2(I;L^2(\Omega))$ and $\check{\zeta}_\nu \to \check{\zeta}_0$ weakly in $L^2(I;L^2(\Omega))$ as $\nu \downarrow 0$.

(M2) (Optimality.) $\forall\,\hat{\zeta}_0 \in D(\widehat{\Phi}(\xi_0;\cdot)_I)$, $\exists\{\hat{\zeta}_\nu;\ \nu \in (0,1)\} \subset L^2(I;H^1(\Omega))$ such that $\hat{\zeta}_\nu \to \hat{\zeta}_0$ in $L^2(I;L^2(\Omega))$ and $\widehat{\Phi}_\nu(\xi_\nu;\hat{\zeta}_\nu)_I \to \widehat{\Phi}(\xi_0;\hat{\zeta}_0)_I$ as $\nu \downarrow 0$.

For the verification of the lower bound, let us take a function $\check{\zeta}_0 \in L^2(I;L^2(\Omega))$ and a sequence $\{\check{\zeta}_\nu;\ \nu \in (0,1)\} \subset L^2(I;L^2(\Omega))$ as in (M1). Here, if we suppose the nontrivial case, i.e., if $\liminf\limits_{\nu\downarrow 0} \widehat{\Phi}_\nu(\xi_\nu,\check{\zeta}_\nu)_I < \infty$, then it follows from (A3) and (AP3) that

$$
\begin{aligned}
\liminf_{\nu\downarrow 0} \widehat{\Phi}_\nu(\xi_\nu,\check{\zeta}_\nu)_I &\geqslant \liminf_{\nu\downarrow 0}(\widehat{\Phi}_\nu(\xi_0;\check{\zeta}_\nu)_I - |\widehat{\Phi}_\nu(\xi_\nu;\check{\zeta}_\nu)_I - \widehat{\Phi}_\nu(\xi_0;\check{\zeta}_\nu)_I|) \\
&\geqslant \liminf_{\nu\downarrow 0} \int_I\int_\Omega \alpha(\xi_0(t))\beta_\nu((\check{\zeta}_\nu)_x)(t))\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \liminf_{\nu\downarrow 0}(-|\alpha(\xi_\nu)-\alpha(\xi_0)|_{C(\overline{I\times\Omega})}|\beta_\nu((\check{\zeta}_\nu)_x)|_{L^1(I;L^1(\Omega))}) \\
&\geqslant \liminf_{\nu\downarrow 0}\left(q_0(\nu)\int_I\int_\Omega \alpha(\xi_0(t))|(\check{\zeta}_\nu)_x(t)|\,\mathrm{d}x\,\mathrm{d}t\right) - 2LT|\alpha(\xi_0)|_{C(\overline{I\times\Omega})}\lim_{\nu\downarrow 0} r_0(\nu) \\
&\quad + \frac{1}{\delta_\alpha}\lim_{\nu\downarrow 0}(-|\alpha(\xi_\nu)-\alpha(\xi_0)|_{C(\overline{I\times\Omega})})\liminf_{\nu\downarrow 0}\widehat{\Phi}_\nu(\xi_\nu;\check{\zeta}_\nu)_I \geqslant \widehat{\Phi}(\xi_0;\zeta_0).
\end{aligned}
$$

Thus, the condition (M1) of the lower bound is verified.

Next, for the verification of the optimality, we fix any $\hat{\zeta}_0 \in D(\widehat{\Phi}(\xi_0;\cdot)_I)$ as in (M2), and apply [9], Lemma 7 to take $\{\hat{\psi}_i;\ i \in \mathbb{N}\} \subset C^\infty(\mathbb{R}^2)$ such that

$$
(3.1)\qquad \begin{cases} \hat{\psi}_i \to \hat{\zeta}_0 \quad \text{in } L^2(I;L^2(\Omega)), \quad \displaystyle\int_I\int_\Omega |(\hat{\psi}_i)_x(t)|\,\mathrm{d}x\,\mathrm{d}t \to \int_I\int_\Omega |D_x\hat{\zeta}_0(t)|\,\mathrm{d}t, \\ \hat{\psi}_i(t) \to \hat{\zeta}_0(t) \quad \text{in } L^2(\Omega) \text{ and strictly in } BV(\Omega), \text{ a.e. } t \in I, \text{ as } i \to \infty. \end{cases}
$$

On that basis, let us take a decreasing sequence $\{\hat{\nu}_i;\ i \in \mathbb{N}\} \subset (0,1)$ such that

$$\text{(3.2)} \qquad \hat{\nu}_{i+1} < \hat{\nu}_i \quad \text{and} \quad \frac{\nu}{2}\int_I\int_\Omega |(\hat{\psi}_i)_x(t)|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{1}{i}, \quad \forall\, i \in \mathbb{N},\ \forall\, \nu \in (0, \hat{\nu}_i),$$

and define a sequence $\{\hat{\zeta}_\nu;\ \nu \in (0,1)\} \subset L^2(I; H^1(\Omega))$ by putting

$$\text{(3.3)} \qquad \hat{\zeta}_\nu := \begin{cases} \hat{\psi}_i \text{ in } L^2(I; H^1(\Omega)), & \text{if } \ \nu \in [\hat{\nu}_{i+1}, \hat{\nu}_i) \text{ for some } i \in \mathbb{N}, \\ \hat{\psi}_1 \text{ in } L^2(I; H^1(\Omega)), & \text{if } \ \nu \in [\hat{\nu}_1, 1),\ \forall\, \nu \in (0,1). \end{cases}$$

Then, from (3.1)–(3.3), it is easily checked that

$$\text{(3.4)} \qquad \begin{cases} \hat{\zeta}_\nu \to \hat{\zeta}_0 \ \text{ in } L^2(I; L^2(\Omega)), \quad \frac{\nu}{2}\int_I\int_\Omega |(\hat{\zeta}_\nu)_x(t)|^2 \,\mathrm{d}x\,\mathrm{d}t \to 0, \\ \displaystyle\int_I\int_\Omega q_1(\nu)|(\hat{\zeta}_\nu)_x(t)| \,\mathrm{d}x\,\mathrm{d}t \to \int_I\int_\Omega |D_x\hat{\zeta}_0(t)| \,\mathrm{d}t, \end{cases} \qquad \text{as } \nu \downarrow 0,$$

$$\begin{aligned} \text{(3.5)} \qquad & \liminf_{\nu\downarrow 0} \int_U q_1(\nu)|(\hat{\zeta}_\nu)_x(t)| \,\mathrm{d}x\,\mathrm{d}t \\ & \geqslant \liminf_{\nu\downarrow 0} q_1(\nu) \int_I\int_{U\cap\{(t,x);\ x\in\Omega\}} |(\hat{\zeta}_\nu)_x(t)| \,\mathrm{d}x\,\mathrm{d}t \\ & \geqslant \int_I\int_{U\cap\{(t,x);\ x\in\Omega\}} |D_x\hat{\zeta}_0(t)| \,\mathrm{d}t, \quad \text{for any open set } U \subset I\times\Omega, \end{aligned}$$

and

$$\begin{aligned} \text{(3.6)} \qquad M_0(I) &:= 1 + \sup_{\nu\in(0,1)} \left\{ \int_I\int_\Omega \beta_\nu((\hat{\zeta}_\nu)_x(t)) \,\mathrm{d}x\,\mathrm{d}t + \int_I\int_\Omega |(\hat{\zeta}_\nu)_x(t)| \,\mathrm{d}x\,\mathrm{d}t \right\} \\ &\leqslant 1 + \sup_{\nu\in(0,1)} \left\{ \int_I\int_\Omega (q_1(\nu)+1)|(\hat{\zeta}_\nu)_x(t)| \,\mathrm{d}x\,\mathrm{d}t + 2TL\, r_1(\nu) \right\} < \infty. \end{aligned}$$

Taking into account (AP3), (3.1), (3.4)–(3.6) and [1], Proposition 1.80 (see also [8], Lemma 4.4, the condition (M2) of optimality is verified as follows:

$$\begin{aligned} &|\widehat{\Phi}_\nu(\xi_\nu; \hat{\zeta}_\nu)_I - \widehat{\Phi}(\xi_0; \hat{\zeta}_0)_I| \\ &\ \leqslant \left| \int_I\int_\Omega [\alpha(\xi_\nu(t)) - \alpha(\xi_0(t))]\beta_\nu((\hat{\zeta}_\nu)_x(t)) \,\mathrm{d}x\,\mathrm{d}t \right| \\ &\quad + \left| \int_I\int_\Omega \alpha(\xi_0(t))[\beta_\nu((\hat{\zeta}_\nu)_x(t)) - |(\hat{\zeta}_\nu)_x(t)|] \,\mathrm{d}x\,\mathrm{d}t \right| \\ &\quad + \left| \int_I\int_\Omega \alpha(\xi_0(t))[|(\hat{\zeta}_\nu)_x(t)| \,\mathrm{d}x - |D_x\hat{\zeta}_0(t)|] \,\mathrm{d}t \right| + \frac{\nu}{2}\int_I\int_\Omega |(\hat{\zeta}_\nu)_x(t)|^2 \,\mathrm{d}x\,\mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&\leqslant M_0(I)\Bigg[|\alpha(\xi_\nu)-\alpha(\xi_0))|_{C(\overline{I\times\Omega})}+|\alpha(\xi_0)|_{C(\overline{I\times\Omega})}\sum_{k=0}^{1}(|q_k(\nu)-1|+2TL\,r_k(\nu))\Bigg]\\
&\quad+\bigg|\int_I\int_\Omega \alpha(\xi_0)|(\hat{\zeta}_\nu)_x|\,\mathrm{d}x\,\mathrm{d}t-\int_I\int_\Omega \alpha(\xi_0(t))|D_x\hat{\zeta}_0(t)|\,\mathrm{d}t\bigg|\\
&\quad+\frac{\nu}{2}\int_I\int_\Omega |(\hat{\zeta}_\nu)_x(t)|^2\,\mathrm{d}x\,\mathrm{d}t\to 0\quad\text{as }\nu\downarrow 0.
\end{aligned}
$$

□

A c k n o w l e d g m e n t. The authors wolud like to express their hearty gratitude to the anonymous referee for valuable comments and suggestions.

References

[1] *L. Ambrosio, N. Fusco, D. Pallara*: Functions of Bounded Variation and Free Discontinuity Problems. Oxford Mathematical Monographs, Clarendon Press, Oxford, 2000.
[2] *A. Ito, N. Kenmochi, N. Yamazaki*: A phase-field model of grain boundary motion. Appl. Math. *53* (2008), 433–454.
[3] *A. Ito, N. Kenmochi, N. Yamazaki*: Weak solutions of grain boundary motion model with singularity. Rend. Mat. Appl., VII. Ser. *29* (2009), 51–63.
[4] *A. Ito, N. Kenmochi, N. Yamazaki*: Global solvability of a model for grain boundary motion with constraint. Discrete Contin. Dyn. Syst., Ser. S *5* (2012), 127–146.
[5] *R. Kobayashi, J. A. Warren, W. C. Carter*: A continuum model of grain boundary. Physica D *140* (2000), 141–150.
[6] *S. Moll, K. Shirakawa*: Existence of solutions to the Kobayashi-Warren-Carter system. Calc. Var. Partial Differ. Equ (2013), 1–36. DOI 10.1007/s00526-013-0689-2.
[7] *U. Mosco*: Convergence of convex sets and of solutions of variational inequalities. Adv. Math. *3* (1969), 510–585.
[8] *K. Shirakawa, H. Watanabe*: Energy-dissipative solution to a one-dimensional phase field model of grain boundary motion. Discrete Contin. Dyn. Syst., Ser. S *7* (2014), 139–159.
[9] *K. Shirakawa, H. Watanabe, N. Yamazaki*: Solvability of one-dimensional phase field systems associated with grain boundary motion. Math. Ann. *356* (2013), 301–330.
[10] *H. Watanabe, K. Shirakawa*: Qualitative properties of a one-dimensional phase-field system associated with grain boundary. GAKUTO Internat. Ser. Math. Sci. Appl. *36* (2013), 301–328.

Authors' addresses: *Hiroshi Watanabe*, Department of General Education, Salesian Polytechnic, 4-6-8, Oyamagaoka, Machida-city, Tokyo, 194–0215, Japan, e-mail: `h-watanabe@salesio-sp.ac.jp`; *Ken Shirakawa*, Department of Mathematics, Faculty of Education, Chiba University, 1-33 Yayoi-cho, Inage-ku, Chiba, 263–8522, Japan, e-mail: `sirakawa@faculty.chiba-u.jp`.