Andrey Andreev; Milena Racheva

Two-sided bounds of eigenvalues of second- and fourth-order elliptic operators

Applications of Mathematics, Vol. 59 (2014), No. 4, 371--390

Persistent URL: http://dml.cz/dmlcz/143870

59 (2014) APPLICATIONS OF MATHEMATICS No. 4, 371–390

TWO-SIDED BOUNDS OF EIGENVALUES OF SECOND- AND FOURTH-ORDER ELLIPTIC OPERATORS

Andrey Andreev, Milena Racheva, Gabrovo

(Received November 15, 2012)

Abstract. This article presents an idea in the finite element methods (FEMs) for obtaining two-sided bounds of exact eigenvalues. This approach is based on the combination of nonconforming methods giving lower bounds of the eigenvalues and a postprocessing technique using conforming finite elements. Our results hold for the second and fourth-order problems defined on two-dimensional domains.

First, we list analytic and experimental results concerning triangular and rectangular nonconforming elements which give at least asymptotically lower bounds of the exact eigenvalues. We present some new numerical experiments for the plate bending problem on a rectangular domain. The main result is that if we know an estimate from below by nonconforming FEM, then by using a postprocessing procedure we can obtain two-sided bounds of the first (essential) eigenvalue. For the other eigenvalues λ_l, $l = 2, 3, \ldots$, we prove and give conditions when this method is applicable. Finally, the numerical results presented and discussed in the paper illustrate the efficiency of our method.

Keywords: eigenvalue problem; nonconforming finite element method; conforming finite element method; postprocessing; lower bound

MSC 2010: 65N25, 65N30

1. Introduction and preliminaries

Let V and H be Hilbert spaces with functions defined on a polygonal domain $\Omega \subset \mathbb{R}^2$, where $V \subset H$ with a compact embedding.

Let also $a(\cdot,\cdot)$ be a symmetric, V-elliptic and continuous bilinear form on $V \times V$, and $b(\cdot,\cdot)$ a bilinear form on $H \times H$ which is continuous, symmetric, and positive definite.

The research has been supported by the Bulgarian National Science Fund under grant DFNI-I01/5.

We define inner products and norms on H by $b(\cdot,\cdot)$ and $\|\cdot\|_b = \sqrt{b(\cdot,\cdot)}$ and on V by $a(\cdot,\cdot)$ and $\|\cdot\|_a = \sqrt{a(\cdot,\cdot)}$, respectively.

Consider the weak form of eigenvalue problems of the self-adjoint second- or fourth-order elliptic differential operator: Find a number $\lambda \in \mathbb{R}$ and a function $u \in V$, $\|u\|_b = 1$, such that

$$a^{(i)}(u,v) = \lambda b(u,v), \quad \forall\, v \in V, \tag{1.1}$$

where $i = 1$ and $i = 2$ correspond to the second- and fourth-order problem, respectively. In (1.1) the form $b(\cdot,\cdot)$ is the usual L_2-inner product:

$$b(u,v) = (u,v) = \int_\Omega uv\,\mathrm{d}x, \quad \|u\|_b = \|u\|_{0,\Omega}.$$

In the case of the second-order operator, we consider $V = H_0^1(\Omega)$ and $H = L_2(\Omega)$, where $H_0^1(\Omega) = \{v \in H^1(\Omega)\colon\ v_{|\partial\Omega} = 0\}$, see e.g. ([10], p. 659), and

$$a^{(1)}(u,v) = \int_\Omega \nabla u \cdot \nabla v\,\mathrm{d}x. \tag{1.2}$$

It is well known that problem (1.1), (1.2) has a countable sequence of real eigenvalues $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots$, and the corresponding eigenfunctions $u_1, u_2, \ldots$ can be assumed to satisfy $\|u_i\|_b = 1$; $(u_i, u_j) = \delta_{ij}$, $i, j \geqslant 1$.

The eigenfunctions u_j belong to the Besov space $B_2^{1+r,\infty}(\Omega)$, and in particular to the Sobolev space $H^{1+r-\varepsilon}(\Omega)$ for the small parameter $\varepsilon > 0$, where $r = 1$ if Ω is convex and $r = \pi/\omega$ (with ω being the largest inner angle of Ω) otherwise [9].

As the fourth-order problem we consider the plate vibration problem, i.e. the eigenvalue problem (1.1) with $V = H_0^2(\Omega)$, $H = L_2(\Omega)$, where $H_0^2(\Omega) = \{v \in H^2(\Omega)\colon\ v_{|\partial\Omega} = \partial_\nu v_{|\partial\Omega} = 0\}$, see ([10], p. 660), and

$$a^{(2)}(u,v) = \int_\Omega (\sigma \Delta u \Delta v + (1-\sigma)(2\partial_{12}u\partial_{12}v + \partial_{11}u\partial_{11}v + \partial_{22}u\partial_{22}v))\,\mathrm{d}x, \tag{1.3}$$

where $\sigma \in [0, 0.5)$ is the Poisson ratio. Clearly, the bilinear form $a^{(2)}(u,v)$ is symmetric and according to [13], it is also continuous and $H_0^2(\Omega)$-elliptic. The problem (1.1), (1.3) has an infinite number of eigenvalues λ_j, all being strictly positive, having finite multiplicity and showing no finite accumulation point [10]. We arrange them as $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \to \infty$.

For the second and fourth order elliptic eigenvalue problems it is well known that the eigenvalues computed by using the standard conforming FEM are always above the exact ones. This fact comes from the minimum-maximum characterization of the eigenvalues (see for example [10], p. 699). Therefore, it is an important problem to find methods which give lower bounds of the eigenvalues. However, before the year 2000, only few results in this direction have been obtained and mainly for finite difference methods [15]. It seems natural to look among nonconforming methods ways for obtaining bounds from below and this will be our starting point. Indeed, if the finite element space is not contained in the Hilbert space where the continuous variational problem is formulated, it is not clear in advance whether the eigenvalues of the discretized problem approximate the eigenvalues of the continuous problem from below or from above.

Let us emphasize that it is valuable to find an interval as small as possible which the exact eigenvalue belongs to. But, obviously, it would be undesirably expensive to compute the eigenvalues twice—once using nonconforming FEM approximations and then by means of a conforming one.

Here, we propose a new procedure for determining bilateral finite element estimates for the eigenvalues. Our approach has two principal advantages:

(i) Usually, the exact eigenvalues are unknown. So, in numerical methods it is always valuable to obtain two-sided bound of any unknown quantity. Moreover, if the interval is small enough one could take the arithmetical mean or an extrapolation formula using the lower and upper bounds.

(ii) We use nonconforming FEM avoiding C^0- or C^1-continuity for the second- or fourth-order eigenvalue problem, respectively. Then we apply a simple postprocedure in order to obtain a bound from above.

The paper is organized as follows. In the next section we state the results concerning lower bounds of eigenvalues. They are divided into two subsections: a survey devoted to using nonconforming finite elements in eigenvalue problems of second- and fourth-order problems. Here we discuss some new numerical results. In Section 3 we prove the main result, namely, the proposed postprocessing giving an estimate from above of the eigenvalue. Finally, we illustrate the efficiency of the presented method by numerical results.

2. Lower bounds of eigenvalues using nonconforming FEMs

Let τ_h be a regular mesh of the domain Ω (see [13], p. 131) such that any two triangles (or rectangles) in τ_h share at most a vertex or an edge. Let h denote the mesh-size parameter, namely $h = \max\limits_{K\in\tau_h} h_K$, with h_K being the diameter of the triangle/rectangle K and $K \in \tau_h$ being affine equivalent to a reference element $\hat{K}$. We suppose that the family of triangulations τ_h satisfies the usual shape regularity condition, i.e., there exists a constant $\alpha > 0$ such that $h_K/\varrho_K \leqslant \alpha$ where ϱ_K is the diameter of the largest ball contained in K.

By V_h, $V_h \subset H$, $V_h \not\subset V$, we denote a nonconforming finite dimensional space.

The nonconforming finite element approximation of (1.1) states: Find a number $\lambda_h \in \mathbb{R}$ and a function $u_h \in V_h$, $\|u_h\|_b = 1$, such that

$$a_h^{(i)}(u_h, v_h) = \lambda_h b(u_h, v_h), \quad \forall\, v_h \in V_h, \tag{2.1}$$

where $a_h^{(i)}(\cdot,\cdot)$ is the mesh-dependent bilinear form defined by

$$a_h^{(i)}(u,v) = \sum_{K\in\tau_h} a_K^{(i)}(u,v), \quad i = 1,2,$$

and $a_K^{(i)}$ denotes the restriction of $a^{(i)}$ on $K \in \tau_h$.

2.1. Second-order eigenvalue problem. In this case the $a^{(1)}$-form in (2.1) is defined by (1.2). We will consider some nonconforming finite elements for which we will investigate whether and under which conditions they give lower bounds for computed eigenvalues.

Piecewise linear finite element of Crouzeix-Raviart (*C-R*) is a well-known triangular element, for which the degrees of freedom (interpolation conditions) are function values at the three midpoints of the edges. The nonconforming Crouzeix-Raviart element space, proposed by Crouzeix and Raviart [14], is defined by

$$\begin{aligned} V_h^{C\text{-}R} = \{v\colon\ v_{|K} \in P_1 \text{ is continuous at the midpoints of the edges of } K,\\ \forall\, K \in \tau_h \text{ and } v = 0 \text{ at the midpoints of the edges } l \in \partial\Omega\}, \end{aligned}$$

where P_k denotes the space of polynomials of degree less than or equal to k, $k \geqslant 0$ is integer and τ_h is a triangular mesh.

Armentano and Durán [8] proved that the Crouzeix-Raviart element results in a lower bound in the singular eigenfunction case. In [38] it was established that the use of Crouzeix-Raviart element gives lower bounds not only in the singular eigenfunction case, but also in the smooth eigenfunction case. Some numerical experiments have been also carried out on both an L-shaped domain and the unit square.

The authors proved estimates from below of this element (Figure 1(a)) for the Laplace operator eigenvalues of problems defined on convex [4] and nonconvex [6] domains.

Extension of Crouzeix-Raviart element (*EC-R*)—this element (Figure 1(b)), introduced in [7], [23] (see also [5], [26]) is an extension of the previous one. The degrees

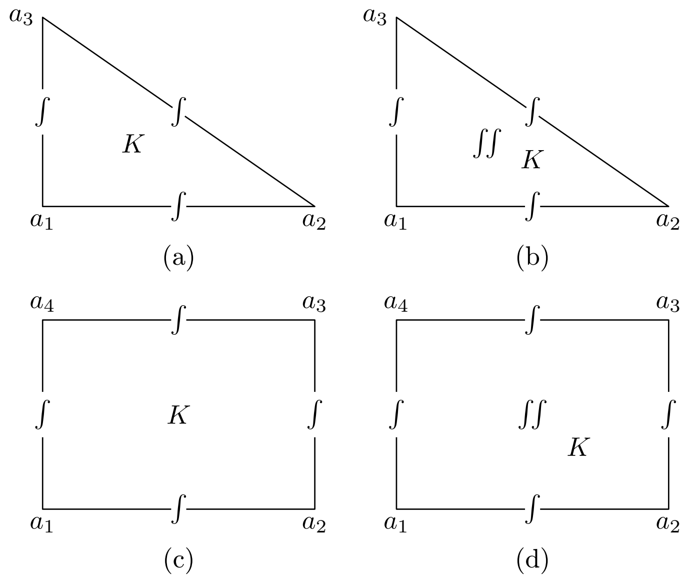

Figure 1. Nonconforming finite elements: (a) FE of Crouzeix-Raviart; (b) Extended FE of Crouzeix-Raviart; (c) Q_1^{rot}-element; (d) EQ_1^{rot}-element

of freedom of an *EC-R* element are

$$\frac{1}{|l_j|}\int_{l_j} v\,\mathrm{d}s, \quad j = 1,2,3 \text{ and } \frac{1}{|K|}\int_K v\,\mathrm{d}x,$$

for any test function v, and l_j, $j = 1,2,3$, being the edges of the triangle $K \in \tau_h$, where

$$|l_j| = \int_{l_j} \mathrm{d}s \quad \text{and} \quad |K| = \int_K \mathrm{d}x.$$

The *EC-R* finite element space is defined by (see [7], [23], [26])

$$\begin{aligned} V_h^{EC\text{-}R} = \big\{ v \in L_2(\Omega)\colon\ & v_{|K} \in \operatorname{span}\{1, x, y, x^2+y^2\},\ v \text{ is integrally continuous} \\ & \text{on the edges of } K,\ \forall\, K \in \tau_h \text{ and } \textstyle\int_l v\,\mathrm{d}s = 0 \quad \text{for any edge } l \subset \partial\Omega \big\}. \end{aligned}$$

In [5] it is proved that the eigenvalues obtained by means of this element are always less than the eigenvalues obtained using an element of Crouzeix-Raviart (the so-called effect of the enriched variational spaces). From this fact it follows that the extension of the Crouzeix-Raviart element gives lower bounds for the exact eigenvalues in case for which it is valid for the Crouzeix-Raviart element. It is to be noted here that this fact is also proved in [26].

Rotated bilinear element (Q_1^{rot}) is a rectangular element proposed by Rannacher and Turek [31] (Figure 1(c)). For any test function v, the degrees of freedom could be presented by

$$\frac{1}{|l_j|}\int_{l_j} v\,\mathrm{d}s,$$

where l_j, $j=1,2,3,4$, are the edges of any $K\in\tau_h$.

The Q_1^{rot}-element space is defined by

$$V_h^{Q_1^{\mathrm{rot}}}=\{v\in L_2(\Omega)\colon\ v_{|K}\in\operatorname{span}\{1,x,y,x^2-y^2\},\ v \text{ is integrally continuous on } \Omega\}.$$

In 2005 Liu and Yan [25] made the following numerical observation for the eigenvalue problem (1.1), (1.2) using a Q_1^{rot}-element:

▷ On the square domain, numerical eigenvalues $\lambda_{1,h}$ and $\lambda_{4,h}$ approximate the corresponding exact eigenvalues from below, while $\lambda_{2,h}$ and $\lambda_{3,h}$ approximate them from above.

▷ On the L-shaped domain, numerical eigenvalues $\lambda_{1,h}$ and $\lambda_{3,h}$ approximate the corresponding exact eigenvalues from below, while $\lambda_{2,h}$ and $\lambda_{4,h}$ approximate them from above.

Again in [25], Liu and Yan explained the phenomenon for the square domain. Until 2010, the phenomenon for the L-shaped domain [38] was not analyzed. Huang, Li, and Lin [17], [20] proposed a new expansion of approximated eigenvalues on the unit square under the uniform mesh and gave some numerical results.

Extension of rotated bilinear element (EQ_1^{rot})—this element represents an extension of the previous one. The degrees of freedom of EQ_1^{rot} are

$$\frac{1}{|l_j|}\int_{l_j} v\,\mathrm{d}s,\quad j=1,2,3,4 \text{ and } \frac{1}{|K|}\int_K v\,\mathrm{d}x,$$

where v is a test function and $K\in\tau_h$ (Figure 1(d)).

The EQ_1^{rot}-element space is defined by (see [19], [21])

$$V_h^{EQ_1^{\mathrm{rot}}}=\{v\in L_2(\Omega)\colon\ v_{|K}\in\operatorname{span}\{1,x,y,x^2,y^2\},\ v \text{ is integrally continuous on } \Omega\}.$$

The properties of an EQ_1^{rot}-element are under consideration in a number of works. In 2005, Liu and Yan [25] provided some numerical results for this element, which approximates eigenvalues of the Laplace operator from below. Lin and Lin proved in their book [19] (2006) that when Ω is a rectangular domain and τ_h is a uniform rectangular mesh, the EQ_1^{rot}-element eigenvalues give lower bounds of the exact eigenvalues for a mesh size small enough.

2.2. Fourth-order eigenvalue problem. For fourth-order elliptic problems, conforming FEMs require C^1-continuity, which usually leads to complicated element construction [12]. In order to avoid C^1-difficulty, nonconforming finite element methods are often preferred. Besides, they provide the only way how to obtain finite element approximations from below for the exact eigenvalues.

Adini element (A) is a C^0-rectangular plate element [1]. The degrees of freedom for this element are the values of the test function v and its first-order derivatives $\partial_1 v$, $\partial_2 v$ at the four vertices a_j, $j = 1, 2, 3, 4$, of any rectangle $K \in \tau_h$ (Figure 2(a)).

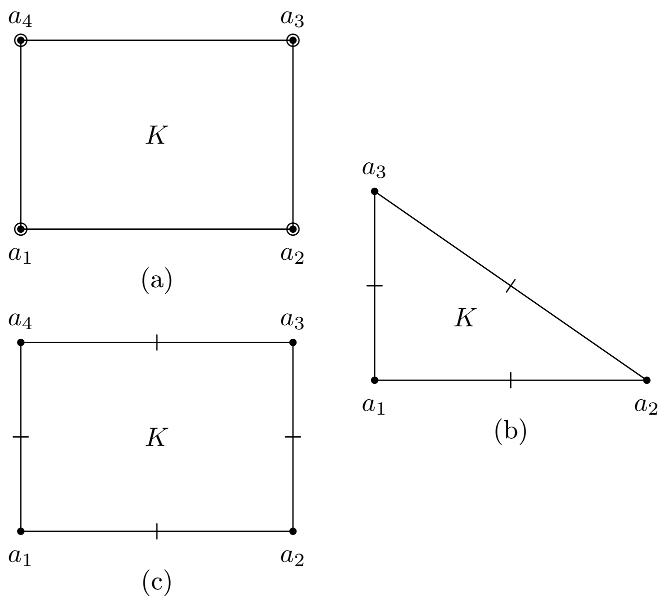

Figure 2. Nonconforming FEs for plate bending problems: (a) Adini FE; (b) Morley FE; (c) Morley rectangular FE.

The Adini finite element space is

$$V_h^A = \{v \in C^0(\Omega)\colon\ v_{|_K} \in P_3 + \operatorname{span}\{x^3y, xy^3\},\ K \in \tau_h,\ v,\ \partial_1 v \text{ and } \partial_2 v \text{ are continuous at element vertices and are equal to zero on boundary nodes}\},$$

where τ_h is a rectangular mesh.

Let us observe that $V_h^A \subset C^0$, $V_h^A \not\subset H^2(\Omega)$.

The numerical examples provided by Rannacher [30] in 1979 show that for the plate vibration problem on a rectangular domain, the Adini element approximates the exact eigenvalues from below. This fact was proved by Yang [37] in 2000 in case of uniform mesh.

However, there are some exceptions, e.g., the Adini element approximates from above under mixed boundary conditions ($u = \partial_\nu u = 0$ on one side and free boundary conditions on the other three sides) on a square domain.

For the special case of the biharmonic operator, Lin and Lin [19] proved in 2006 that the Adini element approximates exact eigenvalues from below. Numerical evidence for this case was provided by Rannacher [30].

The *Morley Element* (M) is a triangular element proposed in [27] for the plate bending problem. The nodal parameters are the function values at the vertices a_j, $j = 1, 2, 3$, of the triangle $K \in \tau_h$ and the first derivatives in normal direction at the midside nodes (Figure 2(b)):

$$v(a_j),\ \partial_\nu v\Big(\frac{a_i + a_j}{2}\Big), \quad i, j = 1, 2, 3,\ i \neq j,\ \text{for } v \in C^1(K),\ K \in \tau_h.$$

The Morley finite element space is

$$\begin{aligned} V_h^M = \{ & v \in L_2(\Omega)\colon\ v_{|K} \in P_2,\ K \in \tau_h,\ v \text{ is continuous at the vertices of } K, \\ & \partial_\nu v \text{ is continuous at the midpoints of edges of } K, \\ & v \text{ and } \partial_\nu v \text{ are equal to zero on the boundary nodes}\}. \end{aligned}$$

The values at the midside nodes could be replaced by

$$\frac{1}{|l_j|} \int_{l_j} \partial_\nu v \,\mathrm{d}s,$$

where l_j is an edge of K which is opposite to the vertex a_j, $j = 1, 2, 3$. Then $v \in V_h^M$ should be integrally continuous on the edges of the elements from τ_h.

Among the nonconforming elements for plate bending problems, the triangular Morley element is the simplest one [27], [33]. It is particularly attractive for fourth order problems because of its low number of degrees of freedom and simple structure. The Morley element and its convergence can be found in [11], [12], [18], [32]. The continuity of this element is very weak—it is not even of class C^0.

Rannacher [30] provided numerical results for plate vibration problems, which indicated that, beside the Adini element, the Morley element can be used to obtain lower bounds of eigenvalues. Recently, this theoretical result has been proved by Lin and Xie [22].

Morley Rectangular Elements (M^{rect}):

Recently, a number of rectangular analogues of Morley elements were introduced and studied (see, e.g. [34], [35], [39]).

Let τ_h consist of rectangles with edges parallel to the coordinate axes and let $K \in \tau_h$ be a rectangle with vertices a_j and edges l_j, $j = 1, 2, 3, 4$. If we choose the set of degrees of freedom to consist of the function values at the vertices a_j of $K \in \tau_h$ and the first derivatives in normal direction at the midside nodes b_j, $j = 1, 2, 3, 4$, of l_j (Figure 2(c)):

$$v(a_j),\ \partial_\nu v(b_j), \quad j = 1, 2, 3, 4,$$

for $v \in C^1(K)$, $K \in \tau_h$, then the finite element space associated with the Morley rectangle is [39]

$$\begin{aligned} V_h^{M^{\text{rect}}} = \{ & v \in L_2(\Omega)\colon\ v_{|K} \in \mathcal{P}_K,\ K \in \tau_h,\ v \text{ is continuous at the vertices of } K, \\ & \partial_\nu v \text{ is continuous at the midpoints of edges of } K, \\ & v \text{ and } \partial_\nu v \text{ are equal to zero on the boundary nodes}\}. \end{aligned}$$

It is easy to see that an interpolation-equivalent choice for degrees of freedom is (see [35])

$$v(a_j),\ \frac{1}{|l_j|}\int_{l_j} \partial_\nu v \,\mathrm{d}s, \quad j = 1, 2, 3, 4.$$

There are two variants for the polynomial space $\mathcal{P}_K$ reported in the literature:

▷ One way is to use (see, e.g. [34], [39])

$$\mathcal{P}_K^{(1)} = P_2 + \operatorname{span}\{x^3, y^3\}. \tag{2.2}$$

▷ Another variant for the polynomial space is [28]

$$\mathcal{P}_K^{(2)} = P_2 + \operatorname{span}\{x^3 - 3xy^2, y^3 - 3yx^2\}. \tag{2.3}$$

The latter choice is motivated by the fact that Δv is constant on K for any $v \in \mathcal{P}_K$, since $x^3 - 3xy^2$ and $y^3 - 3yx^2$ are the unique polynomials of degree greater than or equal to 3 which are harmonic.

There are no theoretical results showing that the Morley rectangle approximates the exact eigenvalues from below. However, here we give numerical experiments which verify convincingly this fact.

Finally, for the fourth order eigenvalue problems, both Adini and Morley elements give one and the same order of convergence (see, for example [18]). Namely, let $u \in H^4(\Omega)$ be the exact eigenfunction and let u_h be the corresponding approximate solution obtained by (2.1), (1.3), then

$$\|u - u_h\|_{2,h,\Omega} \leqslant Ch(|u|_{3,\Omega} + h|u|_{4,\Omega}),$$

where $\|\cdot\|_{2,h,\Omega}$ is the second-order mesh-dependent Sobolev norm.

Also (obviously $\|\cdot\|_{0,h,\Omega} = \|\cdot\|_{0,\Omega}$)

$$\|u - u_h\|_{0,\Omega} \leqslant Ch^2(|u|_{3,\Omega} + h|u|_{4,\Omega}).$$

In this case, the essential H^2-norm determines the eigenvalue estimates, i.e.

$$|\lambda - \lambda_h| = \mathcal{O}(h^2).$$

This order of convergence is also valid for the Morley rectangle with the polynomial space (2.2) [39]. As far as we know, there are no theoretical investigations for the Morley rectangle with the polynomial space (2.3).

For Adini element applied to the uniform mesh, there is an improved accuracy result [18]:

$$\|u - u_h\|_{2,h,\Omega} \leqslant Ch^2 |u|_{4,\Omega}. \tag{2.4}$$

As we mentioned above, here we present some new numerical results concerning estimates from below for Adini and Morley rectangles and their comparison. Moreover, the a-form (1.3) is used where different values of the Poisson ratio are taken.

Let Ω be the unit square. We solve numerically the plate bending problem (2.1), (1.3) for $\sigma = 0.1, 0.2, 0.3, 0.4$, respectively, as well as for the special case when $\sigma = 0$.

The numerical experiments are implemented by means of the Adini element and the Morley rectangular element in its two versions (2.2) and (2.3) for the finite element space, which we will denote by numbers 1 and 2, respectively.

The domain is uniformly divided into n^2 rectangles, where $n = 4, 8, 12, 16$, respectively, so that it should be mentioned here that the estimate (2.4) is valid when the Adini element is used.

In Tables 1–5 we give results for the approximations of the first three eigenvalues. The results for any eigenvalue form an increasing sequence when the mesh parameter $h = \sqrt{2}/n$ decreases. Our numerical results show that both the Morley rectangles give eigenvalues less than those obtained by the Adini element. According to (2.4) in case of a uniform mesh the Adini element approximates more accurately any eigenvalues λ_j than the Morley rectangle. The numerical test confirms that Morley rectangles give an approximation of the exact eigenvalues from below (see [22]).

n	FE	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{3,h}$
	Adini	1185.55	4944.32	4994.39
4	Morley Rect. 1	1075.86	4481.46	4481.46
	Morley Rect. 2	1003.06	4107.34	4107.34
	Adini	1254.15	5164.79	5219.99
8	Morley Rect. 1	1223.11	5017.69	5017.69
	Morley Rect. 2	1187.88	4770.16	4770.16
	Adini	1274.36	5258.87	5308.86
12	Morley Rect. 1	1261.18	5205.06	5205.06
	Morley Rect. 2	1243.09	5068.72	5068.72
	Adini	1283.20	5307.71	5342.19
16	Morley Rect. 1	1275.56	5280.65	5280.65
	Morley Rect. 2	1264.83	5197.49	5197.49

Table 1. Approximations of first three eigenvalues for plate bending problem when $\sigma = 0$

n	FE	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{3,h}$
	Adini	1176.59	4906.11	4960.54
4	Morley Rect. 1	1067.04	4441.57	4441.57
	Morley Rect. 2	965.42	3880.74	3880.74
	Adini	1250.23	5143.91	5203.53
8	Morley Rect. 1	1220.11	5002.25	5002.25
	Morley Rect. 2	1172.22	4667.86	4667.86
	Adini	1272.33	5246.43	5301.07
12	Morley Rect. 1	1259.76	5197.45	5197.45
	Morley Rect. 2	1235.16	5014.04	5014.04
	Adini	1282.02	5299.82	5337.72
16	Morley Rect. 1	1274.75	5276.20	5276.20
	Morley Rect. 2	1260.15	5164.37	5164.37

Table 2. Approximations of first three eigenvalues for plate bending problem when $\sigma = 0.1$

n	FE	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{3,h}$
	Adini	1167.39	4866.17	4926.19
4	Morley Rect. 1	1057.62	4398.95	4398.95
	Morley Rect. 2	921.45	3627.59	3627.59
	Adini	1246.19	5122.27	5186.87
8	Morley Rect. 1	1216.86	4985.49	4985.49
	Morley Rect. 2	1152.81	4544.04	4544.04
	Adini	1270.22	5233.34	5293.22
12	Morley Rect. 1	1258.22	5189.14	5189.14
	Morley Rect. 2	1225.18	4946.15	4946.15
	Adini	1280.79	5291.42	5333.23
16	Morley Rect. 1	1273.86	5271.34	5271.34
	Morley Rect. 2	1254.21	5122.82	5122.82

Table 3. Approximations of first three eigenvalues for plate bending problem when $\sigma = 0.2$

n	FE	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{3,h}$
	Adini	1157.92	4824.14	4891.26
4	Morley Rect. 1	1047.63	4353.81	4353.81
	Morley Rect. 2	869.87	3345.01	3345.01
	Adini	1242.00	5100.00	5169.97
8	Morley Rect. 1	1213.35	4967.45	4967.45
	Morley Rect. 2	1128.38	4392.29	4392.29
	Adini	1268.00	5219.44	5285.29
12	Morley Rect. 1	1256.55	5180.16	5180.16
	Morley Rect. 2	1212.33	4860.31	4860.31
	Adini	1279.50	5282.42	5328.72
16	Morley Rect. 1	1272.89	5266.07	5266.07
	Morley Rect. 2	1246.50	5069.57	5069.57

Table 4. Approximations of first three eigenvalues for plate bending problem when $\sigma = 0.3$

n	FE	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{3,h}$
	Adini	1148.14	4779.92	4855.63
4	Morley Rect. 1	1037.12	4306.33	4306.33
	Morley Rect. 2	809.01	3029.47	3029.47
	Adini	1237.63	5076.06	5152.81
8	Morley Rect. 1	1209.58	4948.14	4948.14
	Morley Rect. 2	1097.00	4203.31	4203.31
	Adini	1265.66	5204.56	5277.29
12	Morley Rect. 1	1254.75	5170.50	5170.50
	Morley Rect. 2	1195.37	4749.13	4749.13
	Adini	1278.11	5272.64	5324.17
16	Morley Rect. 1	1271.86	5260.39	5260.39
	Morley Rect. 2	1236.20	4999.41	4999.41

Table 5. Approximations of first three eigenvalues for plate bending problem when $\sigma = 0.4$

3. Main result

In this section, a method giving lower and upper bounds of exact eigenvalues is presented. Throughout the section we will drop the index $i = 1, 2$ and will use $a(\cdot,\cdot)$ and $a_h(\cdot,\cdot)$ instead of $a^{(i)}(\cdot,\cdot)$ and $a_h^{(i)}(\cdot,\cdot)$, respectively.

We look for $\lambda_h \in \mathbb{R}$ and $u_h \in V_h$ such that

$$a_h(u_h, v_h) = \lambda_h(u_h, v_h), \ \forall\, v_h \in V_h. \tag{3.1}$$

Let the eigenvalue problem (3.1) be solved by using nonconforming FEMs. Here $V_h \not\subset V$ and we adopt that mesh-dependent a-forms are determined by (1.2) or (1.3). Let λ_h approximate the corresponding exact eigenvalue λ from below, i.e. $\lambda_h < \lambda$. These cases were discussed in the previous section.

Now, we solve an additional elliptic problem with a solution $\widetilde{w}_h \in \widetilde{V}_h$ using the conforming FEM, in which the right-hand side is the approximate eigenfunction $u_h \in V_h$ obtained from (3.1):

$$a(\widetilde{w}_h, \tilde{v}_h) = (u_h, \tilde{v}_h), \quad \forall\, \tilde{v}_h \in \widetilde{V}_h. \tag{3.2}$$

Here $\widetilde{V}_h \subset V$ and consequently we replace the a-form $a_h(\cdot,\cdot)$ by $a(\cdot,\cdot)$.

If $\widetilde{w}_h \in \widetilde{V}_h$ is a solution of (3.2), we define the number

$$\tilde{\lambda}_h = \frac{1}{(u_h, \widetilde{w}_h)}. \tag{3.3}$$

Since $\widetilde{V}_h$ is the space corresponding to a conforming FEM, i.e. $\widetilde{V}_h \subset V$, we could be able to determine the solution $\widetilde{w} \in V$ of the (continuous) elliptic problem:

$$a(\widetilde{w}, v) = (u_h, v), \quad \forall\, v \in V. \tag{3.4}$$

Lemma 3.1. *Let (λ, u) be an eigenpair of problem* (1.1), (1.2) $(m=1)$ *or* (1.1), (1.3) $(m=2)$ *with* $b(u,v)=(u,v)$. *Let also* (λ_h, u_h) *be the corresponding solution of* (3.1), *the eigenfunctions being normalized:* $(u,u)=(u_h,u_h)=1$.

Then $\tilde\lambda_h$ approximates the exact eigenvalue λ. More precisely (here and further $C>0$ is independent of the mesh parameter h),

$$|\lambda-\tilde\lambda_h| \leqslant C(\|u-u_h\|^2_{0,\Omega}+\|\widetilde w-\widetilde w_h\|^2_{m,\Omega}), \quad m=1,2. \tag{3.5}$$

Proof. Consider the solution operator for the boundary value problem $\mathcal{T}$: $L_2(\Omega)\to V$, defined by $u=\mathcal{T}f$, $u\in V$, for any $f\in L_2(\Omega)$, where

$$a(u,v)=(f,v), \quad \forall\, v\in V.$$

Obviously, $a(u,v)$ and (u,v) are symmetric forms. Consequently (see, e.g. [10])

$$\begin{aligned} a(\mathcal{T}u,v)&=a(u,\mathcal{T}v), \quad \forall\, u,v\in H^m(\Omega),\ m=1,2,\\ (\mathcal{T}u,v)&=(u,\mathcal{T}v), \quad \forall\, u,v\in L_2(\Omega). \end{aligned}$$

Thus, the operator $\mathcal{T}$ is symmetric and positive. It immediately follows from the Ritz representation theorem ($a(\cdot,\cdot)$ is an inner product on V) that $\mathcal{T}$ is bounded.

By analogy with (3.3), from (3.4) we define

$$\tilde\lambda=\frac{1}{(\widetilde w,u_h)}=\frac{1}{(\mathcal{T}u_h,u_h)}.$$

From the equality

$$a(\mathcal{T}u,v)=(u,v), \quad \forall\, v\in V,$$

it follows that $a(\mathcal{T}u,u)=(u,u)=1$ where u is an exact eigenfunction. On the other hand, this function is a solution of $a(u,\mathcal{T}u)=\lambda(u,\mathcal{T}u)$. Consequently,

$$\lambda=\frac{1}{(\mathcal{T}u,u)}$$

in view of the fact that the operator $\mathcal{T}$ is symmetric.

Therefore,

$$\begin{aligned} \frac{1}{\lambda}-\frac{1}{\tilde\lambda}&=(\mathcal{T}u,u)-(\mathcal{T}u_h,u_h)\\ &=(\mathcal{T}u,u)-(\mathcal{T}u_h,u_h)+(\mathcal{T}(u-u_h),u-u_h)-(\mathcal{T}(u-u_h),u-u_h)\\ &=2(\mathcal{T}u,u-u_h)-(\mathcal{T}(u-u_h),u-u_h). \end{aligned} \tag{3.6}$$

We estimate the last two terms:

$$2(\mathcal{T}u, u-u_h) = \frac{2}{\lambda}(1-(u,u_h)) \\ = \frac{1}{\lambda}((u,u)-2(u,u_h)+(u_h,u_h)) \\ = \frac{1}{\lambda}(u-u_h, u-u_h) = \frac{1}{\lambda}\|u-u_h\|^2_{0,\Omega}. \tag{3.7}$$

Since the operator $\mathcal{T}$ is bounded, we have

$$|(\mathcal{T}(u-u_h), u-u_h)| \leqslant \|\mathcal{T}\| \|u-u_h\|^2_{0,\Omega}. \tag{3.8}$$

Finally, from (3.6), (3.7) and (3.8) we obtain

$$|\lambda - \tilde{\lambda}| \leqslant C\|u-u_h\|^2_{0,\Omega}. \tag{3.9}$$

By the same argument, it follows that

$$\frac{1}{\tilde{\lambda}} - \frac{1}{\tilde{\lambda}_h} = (\widetilde{w}, u_h) - (\widetilde{w}_h, u_h) = a(\widetilde{w}, \widetilde{w}) - a(\widetilde{w}_h, \widetilde{w}_h).$$

Since $\widetilde{w}_h \in \widetilde{V}_h$, $\widetilde{w} \in V$ are solutions of the elliptic problems (3.2) and (3.4), respectively, we have $a(\widetilde{w}-\widetilde{w}_h, \widetilde{w}_h) = 0$, because $\widetilde{V}_h \subset V$.

Consequently,

$$\frac{1}{\tilde{\lambda}} - \frac{1}{\tilde{\lambda}_h} = a(\widetilde{w}, \widetilde{w}) - a(\widetilde{w}_h, \widetilde{w}_h) - 2a(\widetilde{w}-\widetilde{w}_h, \widetilde{w}_h) \\ = a(\widetilde{w}, \widetilde{w}) - 2a(\widetilde{w}, \widetilde{w}_h) + a(\widetilde{w}_h, \widetilde{w}_h) \\ = a(\widetilde{w}-\widetilde{w}_h, \widetilde{w}-\widetilde{w}_h).$$

The continuity of the a-form on $H^m(\Omega)$ leads to the inequality

$$|\tilde{\lambda} - \tilde{\lambda}_h| \leqslant C\|\widetilde{w}-\widetilde{w}_h\|^2_{m,\Omega}, \quad m = 1, 2.$$

This estimate and (3.9) prove the lemma. □

Remark 3.1. It is important to emphasize that the estimate (3.5) could easily result in superconvergence order. Indeed, the optimal order of convergence of the eigenvalues is $\mathcal{O}(\|u-u_h\|^2_{m,\Omega})$, where u is the corresponding exact eigenfunction. Obviously $\|u-u_h\|^2_{0,\Omega}$ has higher order of accuracy and, moreover, it is easy to obtain the same high order of convergence for $\|\widetilde{w}-\widetilde{w}_h\|^2_{m,\Omega}$ (see [2], [3], [29], [36]).

Now, we will present the final step of our idea described as nonconforming-conforming approach to eigenvalues approximation.

Theorem 3.1. *Let the conditions of Lemma* 3.1 *be fulfilled. If* $\lambda_{1,h}$ *obtained by the nonconforming FEM approximates the first (essential) eigenvalue* λ_1 *from below, then* $\tilde\lambda_{1,h}$ *determined by* (3.3) *gives an upper bound of* λ_1*, i.e.*

$$\lambda_{1,h} \leqslant \lambda_1 \leqslant \tilde\lambda_{1,h}. \tag{3.10}$$

Proof. First, we present results which concern any eigenpair (λ, u). So we drop the indices $1, 2, \ldots$ Let us introduce the function $\tilde u_h = \tilde\lambda_h \widetilde w_h$, where $\widetilde w_h$ is the solution of (3.2) and $\tilde\lambda_h$ is determined by (3.3). Then

$$(u_h, \tilde u_h) = \tilde\lambda_h (u_h, \widetilde w_h) = 1. \tag{3.11}$$

Next, we have

$$a(\tilde u_h, \tilde u_h) = \tilde\lambda_h^2 a(\widetilde w_h, \widetilde w_h) = \tilde\lambda_h^2 (u_h, \widetilde w_h) = \tilde\lambda_h. \tag{3.12}$$

Let us set

$$0 \leqslant \varepsilon(h) = \|u_h - \tilde u_h\|_{0,\Omega}^2 = (u_h, u_h) - 2(u_h, \tilde u_h) + (\tilde u_h, \tilde u_h).$$

Therefore, having in mind that $(u_h, u_h) = 1$, from (3.11) we get $(\tilde u_h, \tilde u_h) = 1 + \varepsilon(h)$. By virtue of (3.12), it follows that

$$\lambda_1 = \min_{\substack{v \in V \\ v \neq 0}} \frac{a(v,v)}{(v,v)} \leqslant \frac{a(\tilde u_{1,h}, \tilde u_{1,h})}{(\tilde u_{1,h}, \tilde u_{1,h})} = \frac{a(\tilde u_{1,h}, \tilde u_{1,h})}{1 + \varepsilon(h)} \leqslant a(\tilde u_{1,h}, \tilde u_{1,h}) = \tilde\lambda_{1,h},$$

which proves the theorem. □

Theorem 3.2. *Consider the second order eigenvalue problem* (1.1), (1.2). *Let* $(\lambda_{l,h}, u_{l,h})$, $l = 2, 3, \ldots$, *be the solution of* (3.1) *obtained using some of the following nonconforming finite elements: C-R, EC-R,* $Q_1^{\rm rot}$ *and* $EQ_1^{\rm rot}$*. Let for any integer* $l \geqslant 2$ *the eigenfunctions be normalized* $(u_l, u_l) = (u_{l,h}, u_{l,h}) = 1$ *and let the partitions be quasiuniform. Then, if we solve the problem* (3.2) *using linear/bilinear conforming finite elements,*

$$\lambda_{l,h} \leqslant \lambda_l \leqslant \tilde\lambda_{l,h}, \quad l = 2, 3, \ldots \tag{3.13}$$

Proof. Let R denote the Rayleigh quotient, let $u \in V$ be an eigenfunction corresponding to λ and $\|w\|_a = \sqrt{a^{(1)}(w,w)}$. Then for any exact eigenvalue λ and any function $w \in V$ we shall use the following equality (see [10], p. 701):

$$R(w) - \lambda = \frac{\|w-u\|_a^2}{\|w\|_{0,\Omega}^2} - \lambda \frac{\|w-u\|_{0,\Omega}^2}{\|w\|_{0,\Omega}^2}. \tag{3.14}$$

For $l = 2, 3, \ldots, N_h$, $N_h = \dim V_h$, from (3.12) and $(\tilde{u}_{l,h}, \tilde{u}_{l,h}) = 1+\varepsilon(h)$ we obtain

$$\tilde{\lambda}_{l,h} - \lambda_l = a^{(1)}(\tilde{u}_{l,h}, \tilde{u}_{l,h}) - \lambda_l \geqslant \frac{a^{(1)}(\tilde{u}_{l,h}, \tilde{u}_{l,h})}{(\tilde{u}_{l,h}, \tilde{u}_{l,h})} - \lambda_l.$$

Now, in (3.14) we take $\lambda = \lambda_l$, $w = \tilde{u}_{l,h}$, and $u = u_l$. Then we get

$$\tilde{\lambda}_{l,h} - \lambda_l \geqslant R(\tilde{u}_{l,h}) - \lambda_l = \frac{\|\tilde{u}_{l,h} - u_l\|_a^2}{\|\tilde{u}_{l,h}\|_{0,\Omega}^2} - \lambda_l \frac{\|\tilde{u}_{l,h} - u_l\|_{0,\Omega}^2}{\|\tilde{u}_{l,h}\|_{0,\Omega}^2}. \tag{3.15}$$

We have to estimate the first term on the right-hand side of (3.15) from below and the second from above.

Let us introduce the elliptic projection operator $\widetilde{\mathcal{R}}_h \colon V \to \widetilde{V}_h$ defined by

$$a^{(1)}(u - \widetilde{\mathcal{R}}_h u, \tilde{v}_h) = 0, \quad \forall\, \tilde{v}_h \in \widetilde{V}_h.$$

Obviously, by Friedrichs' inequality,

$$\|\tilde{u}_{l,h} - u_l\|_{0,\Omega} \leqslant C\|\tilde{u}_{l,h} - \widetilde{\mathcal{R}}_h u_l\|_a + \|\widetilde{\mathcal{R}}_h u_l - u_l\|_{0,\Omega}. \tag{3.16}$$

We consider both terms on the right-hand side of (3.16).

For the first we obtain

$$\begin{aligned} \|\tilde{u}_{l,h} - \widetilde{\mathcal{R}}_h u_l\|_a^2 &= a^{(1)}(\tilde{u}_{l,h} - \widetilde{\mathcal{R}}_h u_l, \tilde{u}_{l,h} - \widetilde{\mathcal{R}}_h u_l) \\ &= (\tilde{\lambda}_{l,h} u_{l,h} - \lambda_l u_l, \tilde{u}_{l,h} - \widetilde{\mathcal{R}}_h u_l) \\ &\leqslant C\|\tilde{\lambda}_{l,h} u_{l,h} - \lambda_l u_l\|_{0,\Omega} \|\tilde{u}_{l,h} - \widetilde{\mathcal{R}}_h u_l\|_a \\ &\leqslant (\tilde{\lambda}_{l,h}\|u_{l,h} - u_l\|_{0,\Omega} + |\tilde{\lambda}_{l,h} - \lambda_l| \|u_l\|_{0,\Omega}) \|\tilde{u}_{l,h} - \widetilde{\mathcal{R}}_h u_l\|_a. \end{aligned} \tag{3.17}$$

Taking into account that (3.1) is solved by nonconforming finite elements specified by the conditions of the theorem, from (3.17) we conclude that [8], [16]

$$\|\tilde{u}_{l,h} - \widetilde{\mathcal{R}}_h u_l\|_a \leqslant C h^{2r} \|u_l\|_{1+r}, \tag{3.18}$$

where $r = \pi/\omega$ if Ω is a non-convex polygonal domain and ω is its maximal interior angle ($\frac{1}{2} < r < 1$) and $r = 1$ when Ω is a convex domain. It is worth noting that the last inequality gives an estimate of a superconvergent type.

For the second term on the right-hand side of (3.16) we take into account that $\widetilde{V}_h$ is constructed using linear/bilinear conforming finite elements, consequently for the elliptic projection operator $\widetilde{\mathcal{R}}_h$ the following estimate is valid [16]:

$$\|\widetilde{\mathcal{R}}_h u_l - u_l\|_{0,\Omega} \leqslant C h^{2r} \|u_l\|_{1+r,\Omega}.$$

The last inequality, (3.18) and (3.16) give the following estimate concerning the second term on the right-hand side of (3.15):

$$\|\tilde{u}_{l,h} - u_l\|_{0,\Omega} \leqslant Ch^{2r}\|u_l\|_{1+r,\Omega}.$$

On the other hand, with regard to the first term in the right-hand side of (3.15), we get

$$\begin{aligned}
\|\tilde{u}_{l,h} - u_l\|_a^2 &= a^{(1)}(\tilde{u}_{l,h} - u_l, \tilde{u}_{l,h} - u_l) = a^{(1)}((\tilde{u}_{l,h} - \widetilde{\mathcal{R}}_h u_l) \\
&\quad + (\widetilde{\mathcal{R}}_h u_l - u_l), (\tilde{u}_{l,h} - \widetilde{\mathcal{R}}_h u_l) + (\widetilde{\mathcal{R}}_h u_l - u_l)) \\
&= a^{(1)}(\tilde{u}_{l,h} - \widetilde{\mathcal{R}}_h u_l, \tilde{u}_{l,h} - \widetilde{\mathcal{R}}_h u_l) + a^{(1)}(\widetilde{\mathcal{R}}_h u_l - u_l, \widetilde{\mathcal{R}}_h u_l - u_l) \\
&\quad + 2a^{(1)}(\tilde{u}_{l,h} - \widetilde{\mathcal{R}}_h u_l, \widetilde{\mathcal{R}}_h u_l - u_l) \\
&= \|\tilde{u}_{l,h} - \widetilde{\mathcal{R}}_h u_l\|_a^2 + \|\widetilde{\mathcal{R}}_h u_l - u_l\|_a^2 \geqslant \|\widetilde{\mathcal{R}}_h u_l - u_l\|_a^2.
\end{aligned}$$

Finally, we use the estimate

$$\|\widetilde{\mathcal{R}}_h u_l - u_l\|_a \geqslant Ch,$$

which is valid for linear/bilinear conforming finite elements. This estimate is proved by Q. Lin, H. Xie, and J. Xu [24] for quasiuniform partitions of Ω.

So, we proved that

$$\|\tilde{u}_{l,h} - u_l\|_{0,\Omega} \leqslant Ch^{2r}, \quad r \in (\tfrac{1}{2}, 1],$$

and

$$\|\tilde{u}_{l,h} - u_l\|_a \geqslant Ch,$$

consequently, the first term in (3.15) is dominant and then $\tilde{\lambda}_{l,h} \geqslant \lambda_l$. □

Now we present a postprocessing algorithm which will give two-sided bounds of the exact eigenvalues.

Algorithm.

(i) Solve the eigenvalue problem (3.1) by nonconforming FEM for $(\lambda_h, u_h) \in \mathbb{R} \times V_h$ and $(u_h, u_h) = 1$ with λ_h giving lower bound of the exact eigenvalue;

(ii) Solve the elliptic problem (3.2) and find the solution $\widetilde{w}_h$;

(iii) Determine the approximate eigenvalue $\tilde{\lambda}_h$ by means of (3.3).

As a result of the algorithm given above, it will follow that

$$\lambda_l \in (\lambda_{l,h}, \tilde{\lambda}_{l,h}), \quad l = 1, 2, \ldots$$

R e m a r k 3.2. The proposed algorithm gives a simple method for obtaining two-sided bounds of the exact eigenvalues (3.10) and (3.13). The main idea is to implement a postprocessing procedure based on solving an elliptic problem by a conforming FEM instead of employing a conforming eigenproblem solver.

Finally, let us illustrate the efficiency of the presented approach for obtaining two-sided bounds for the first eigenvalue. Consider the fourth-order problem with $\sigma = 0$ and $\sigma = 0.2$. Initially, we use the Adini finite element giving a lower bound of the eigenvalue (see Section 2). Then the conforming method will be applied with Bogner-Fox-Schmidt element.

The unit square is uniformly divided into n^2 rectangle elements. The numerical experiments are implemented for $n = 4, 6, 8, 10, 12, 14, 16$.

In Table 6 the approach is demonstrated for the essential eigenvalues when $\sigma = 0.2$ and $\sigma = 0$. These results confirm the statement of Theorem 3.1. The sequence $\{\lambda_{1,h}\}$ increases, while $\{\tilde{\lambda}_{1,h}\}$ decreases. Here, the exact eigenvalues for $\sigma = 0.2$ are not known. Concerning the case $\sigma = 0$, the exact smallest eigenvalue is approximately $\lambda_1 = 1295$ (see [19], [30]).

	$\sigma = 0.2$		$\sigma = 0$	
n	$\lambda_{1,h}$	$\tilde{\lambda}_{1,h}$	$\lambda_{1,h}$	$\tilde{\lambda}_{1,h}$
4	1167.392908	1632.978596	1185.550861	1629.661664
6	1215.208358	1539.581756	1227.814307	1542.266216
8	1246.188992	1481.631502	1254.152526	1478.265953
10	1263.205943	1430.888621	1268.538691	1425.992665
12	1270.220003	1395.200465	1274.357984	1371.319310
14	1276.904152	1361.499261	1279.910030	1336.316003
16	1280.793015	1331.814826	1283.199186	1313.671528

Table 6. Results for the essential eigenvalue with $\sigma = 0.2$ and $\sigma = 0$

References

[1] *A. Adini, R. Clough*: Analysis of Plate Bending by the Finite Element Method. NSF Report G. 7337, 1961.

[2] *A. B. Andreev, R. D. Lazarov, M. R. Racheva*: Postprocessing and higher order convergence of mixed finite element approximations of biharmonic eigenvalue problems. J. Comput. Appl. Math. *182* (2005), 333–349.

[3] *A. B. Andreev, M. R. Racheva*: Superconvergent FE postprocessing for eigenfunctions. C. R. Acad. Bulg. Sci. *55* (2002), 17–22.

[4] *A. B. Andreev, M. R. Racheva*: Lower bounds for eigenvalues by nonconforming FEM on convex domain. Application of Mathematics in Technical and Natural Sciences (M. Todorov et al., ed.). Proceedings of the 2nd international conference, Sozopol, Bulgaria, 2010. AIP Conf. Proc. 1301, Amer. Inst. Phys., Melville, 2010, pp. 361–369.

[5] *A. B. Andreev, M. R. Racheva*: Properties and estimates of an integral type nonconforming finite element. Large-Scale Scientific Computing (I. Lirkov et al., ed.). 8th international conference, LSSC 2011, Sozopol, Bulgaria, 2011. Lecture Notes in Computer Science 7116, 2012, pp. 252–532, Springer, Berlin.

[6] *A. B. Andreev, M. R. Racheva*: Lower bounds for eigenvalues and postprocessing by an integral type nonconforming FEM. Sib. Zh. Vychisl. Mat. *15* (2012), 235–249 (In Russian.); Numer. Analysis Appl. *5* (2012), 191–203.

[7] *A. B. Andreev, M. R. Racheva, G. S. Tsanev*: A Nonconforming Finite Element with Integral Type Bubble Function. Proceedings of 5th Annual Meeting of the BG. Section of SIAM'10, 2010, pp. 3–6.

[8] *M. G. Armentano, R. G. Durán*: Asymptotic lower bounds for eigenvalues by nonconforming finite element methods. ETNA, Electron. Trans. Numer. Anal. (electronic only) *17* (2004), 93–101.

[9] *I. Babuška, R. B. Kellogg, J. Pitkäranta*: Direct and inverse error estimates for finite elements with mesh refinements. Numer. Math. *33* (1979), 447–471.

[10] *I. Babuška, J. Osborn*: Eigenvalue problems. Handbook of Numerical Analysis, Vol. II: Finite Element Methods (Part 1) (J.-L. Lions, P. G. Ciarlet, ed.). North-Holland, Amsterdam, 1991, pp. 641–787.

[11] *S. C. Brenner, R. L. Scott*: The Mathematical Theory of Finite Element Methods. Texts in Applied Mathematics 15, Springer, New York, 1994.

[12] *P. G. Ciarlet*: The Finite Element Method for Elliptic Problems. Studies in Mathematics and Its Applications. Vol. 4, North-Holland, Amsterdam, 1978.

[13] *P. G. Ciarlet*: Basic error estimates for elliptic problems. Handbook of Numerical Analysis. Vol. II: Finite Element Methods (Part 1) (P. G. Ciarlet et al., eds.). North-Holland, Amsterdam, 1991.

[14] *M. Crouzeix, P. A. Raviart*: Conforming and nonconforming finite element methods for solving the stationary Stokes equations. I. Rev. Franc. Automat. Inform. Rech. Operat. *7* (1973), 33–75.

[15] *G. E. Forsythe*: Asymptotic lower bounds for the fundamental frequency of convex membranes. Pac. J. Math. *5* (1955), 691–702.

[16] *P. Grisvard*: Singularities in Boundary Problems. MASSON and Springer, Berlin, 1985.

[17] *H. T. Huang, Z. C. Li, Q. Lin*: New expansions of numerical eigenvalues by finite elements. J. Comput. Appl. Math. *217* (2008), 9–27.

[18] *P. Lascaux, P. Lesaint*: Some nonconforming finite elements for the plate bending problem. Rev. Franc. Automat. Inform. Rech. Operat. *9*, Analyse numer. R-1 (1975), 9–53.

[19] *Q. Lin, J. F. Lin*: Finite Element Methods: Accuracy and Improvement. Science Press, Beijing, 2006.

[20] *Q. Lin, H. T. Huang, Z. C. Li*: New expansions of numerical eigenvalues for $-\Delta u = \lambda \varrho u$ by nonconforming elements. Math. Comput. *77* (2008), 2061–2084.

[21] *Q. Lin, L. Tobiska, A. Zhou*: Superconvergence and extrapolation of non-conforming low order finite elements applied to the Poisson equation. IMA J. Numer. Anal. *25* (2005), 160–181.

[22] *Q. Lin, H. Xie*: The asymptotic lower bounds of eigenvalue problems by nonconforming finite element methods. Math. Pract. Theory *42* (2012), 219–226. (In Chinese.)

[23] *Q. Lin, H. Xie, F. Luo, Y. Li, Y. Yang*: Stokes eigenvalue approximations from below with nonconforming mixed finite element methods. Math. Pract. Theory *40* (2010), 157–168.

[24] *Q. Lin, H. Xie, J. Xu*: Lower bounds of the discretization for piecewise polynomials. http://arxiv.org/abs/1106.4395 (2011).

[25] *H. P. Liu, N. N. Yan*: Four finite element solutions and comparison of problem for the Poisson equation eigenvalue. Chin. J. Numer. Math. Appl. *27* (2005), 81–91.
[26] *F. Luo, Q. Lin, H. Xie*: Computing the lower and upper bounds of Laplace eigenvalue problem by combining conforming and nonconforming finite element methods. Sci. China, Math. *55* (2012), 1069–1082.
[27] *L. S. D. Morley*: The triangular equilibrium element in the solution of plate bending problems. Aero. Quart. *19* (1968), 149–169.
[28] *S. Nicaise*: A posteriori error estimations of some cell-centered finite volume methods. SIAM J. Numer. Anal. (electronic) *43* (2005), 1481–1503.
[29] *M. R. Racheva, A. B. Andreev*: Superconvergence postprocessing for eigenvalues. Comput. Methods Appl. Math. *2* (2002), 171–185.
[30] *R. Rannacher*: Nonconforming finite element methods for eigenvalue problems in linear plate theory. Numer. Math. *33* (1979), 23–42.
[31] *R. Rannacher, S. Turek*: Simple nonconforming quadrilateral Stokes element. Numer. Methods Partial Differ. Equations *8* (1992), 97–111.
[32] *Z.-C. Shi*: On the error estimates of Morley element. Math. Numer. Sin. *12* (1990), 113–118 (In Chinese.); translation in Chinese J. Numer. Math. Appl. *12* (1990), 102–108.
[33] *G. Strang, G. J. Fix*: An Analysis of the Finite Element Method. Prentice-Hall Series in Automatic Computation, Prentice-Hall, Englewood Cliffs, 1973.
[34] *M. Wang, Z.-C. Shi, J. Xu*: Some n-rectangle nonconforming elements for fourth order elliptic equations. J. Comput. Math. *25* (2007), 408–420.
[35] *L. Wang, Y. Wu, X. Xie*: Uniformly stable rectangular elements for fourth order elliptic singular perturbation problems. Numer. Methods Partial Differ. Equations *29* (2013), 721–737.
[36] *J. Xu, A. Zhou*: A two-grid discretization scheme for eigenvalue problems. Math. Comput. *70* (2001), 17–25.
[37] *Y. D. Yang*: A posteriori error estimates in Adini finite element for eigenvalue problems. J. Comput. Math. *18* (2000), 413–418.
[38] *Y. D. Yang, Z. M. Zhang, F. B. Lin*: Eigenvalue approximation from below using non-conforming finite elements. Sci. China, Math. *53* (2010), 137–150.
[39] *H. Q. Zhang, M. Wang*: The Mathematical Theory of Finite Elements. Science Press, Beijing, 1991.

Authors' address: *Andrey Andreev*, *Milena Racheva*, Technical University of Gabrovo, Gabrovo, Bulgaria, e-mail: `andreev@tugab.bg`, `milena@tugab.bg`.