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EXISTENCE OF SOLUTIONS TO NONLINEAR ADVECTION-DIFFUSION EQUATION APPLIED TO BURGERS' EQUATION USING SINC METHODS

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Abstract. This paper has two objectives. First, we prove the existence of solutions to the general advection-diffusion equation subject to a reasonably smooth initial condition. We investigate the behavior of the solution of these problems for large values of time. Secondly, a numerical scheme using the Sinc-Galerkin method is developed to approximate the solution of a simple model of turbulence, which is a special case of the advection-diffusion equation, known as Burgers' equation. The approximate solution is shown to converge to the exact solution at an exponential rate. A numerical example is given to illustrate the accuracy of the method.

Keywords: Sinc-Galerkin method; advection-diffusion equation; numerical solution

MSC 2010: 35A01, 35K57, 35F05, 65T60

1. Introduction

Nonlinear partial differential equations appear in many branches of physics, engineering and applied mathematics.

We study the behavior as $t \to \infty$ of the solution of the Cauchy problem for the equation

$$u_t + (f'(u))_x = \varepsilon u_{xx}, \quad (x,t) \in \mathbb{R} \times (0,T) \tag{1.1}$$

with the initial and boundary conditions

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{1.2}$$

$$u(-\infty,t) = \gamma(t), \quad u(\infty,t) = \delta(t), \quad t \geqslant 0, \tag{1.3}$$

where ε is the coefficient of the kinematic viscosity, T is the total time, f is the flux function, and $u_0(x), \gamma(t), \delta(t)$ are given functions of the variables. Our first aim of this paper is to prove the existence of solutions for the nonlinear advection-diffusion equation (1.1), and study the behavior of solutions under the condition that the function $u_0(x)$ in equation (1.2) belongs to some class of function $\mathbf{L}_\alpha$ that will be defined later. The solutions to equation (1.1) can approximately describe the flow through a shock wave in a viscous fluid. Equation (1.1) is of some mathematical interest in itself, and has applications in the theory of stochastic processes. This type of equation has been investigated by several authors. Historically, equation (1.1) first appears in a paper by H. Bateman [3], where he mentioned it as worth studying and gave a special solution. Dafermos [5] considered the asymptotic behavior of certain solutions of the initial value problem for the one-dimensional non-homogeneous scalar balance law. Al-Khaled [1] studied the initial value problem for a balance law

$$u_t + (f(u))_x = h(x, t, u, u_x, u_{xx}), \quad (x, t) \in \mathbb{R} \times (0, T)$$

proving two facts regarding the behavior of the solution, if the initial condition belongs to $\mathbf{L}_\alpha$. However, for our purposes we will, mainly refer to the fundamental papers of Venttsel' [13] and Oleĭnik [9]. In those papers, they studied the existence and uniqueness of solutions of the mixed boundary value problem for equation (1.1).

One important application that is of considerable current is the special case where we substitute $f'(u) = u^2/2$ into equation (1.1) to get the advection-diffusion equation

$$u_t + uu_x = \varepsilon u_{xx}, \quad (x, t) \in \mathbb{R} \times (0, T) \tag{1.4}$$

which was used as a simple model of turbulence in an extensive study by Burgers. With regard to the velocity field of a fluid, the essential ingredient of (1.4) in this study is the competition between the dissipative term εu_{xx} the coefficient of which is the kinematic viscosity, and the nonlinear term uu_x. Equation (1.4) appears as a mathematical model for many physical events such as gas dynamics, turbulence, and shock wave theory [6]. Many researchers have used various numerical methods to solve Burgers' equation [8], [4], [7]. Lund [10] uses Sinc-Galerkin method to find a numerical solution of the nonlinear advection-diffusion equation (Burgers' equation). The method results in an iterative scheme of an error of order $\mathcal{O}(\exp(-c/h))$ for some positive constants c and h. In [12], the Burgers' equation is transformed into an equivalent integral equation, and a Sinc-collocation procedure is developed for the integral equation. In this paper, and as a second objective, we will use the Sinc-Galerkin method to study the solution of equation (1.4). The solution is based on using the Sinc method, which builds an approximate solution valid on the entire spatial domain and a small interval in the time domain. The main idea is to replace the differential and integral equations by their Sinc approximations. The ease of implementation coupled with the exponential convergence rate have demonstrated the viability of this method. One avenue that deserves attention is the approximation by Sinc functions that handles singularities in the problem.

The plan of this work is the following. The existence proof of solutions for equation (1.1) will be investigated in Section 2. The Sinc function is briefly described in Section 3. In Section 4, we compute the solution of Burgers' equation, and write an algorithm with the notation in Section 3. In the last section, the scheme is numerically tested on one example.

2. Existence of solutions

We consider the problem (1.1) under the condition that the initial condition $u_0(x)$ in equation (1.2) vanishes as $|x| \to \infty$ (or, equivalently, $u_0(x) \in \mathbf{L}_\alpha$). We shall assume regarding $f(u)$ that it is smooth and satisfies the condition

$$\frac{M^2}{\max\limits_{|u|\leqslant M} |f''(u)| + 1} \to \infty, \quad M \to \infty. \tag{2.1}$$

Under these assumptions, we have:

Theorem 2.1. *For $x \in \mathbb{R}$ there exists a solution of equations* (1.1)–(1.2) *that converges to zero uniformly.*

Proof. Form the scalar product of equation (1.1) with u, and integrating by parts over the infinite strip $\mathcal{R} = \{(x,t) \in \mathbb{R} \times (0,T)\}$, with the use of the identity (see [11]) $\langle u, (f'(u))_x \rangle = (\partial/\partial x)(\langle u, f'(u)\rangle - f(u))$ yields

$$\frac{1}{2}\int_{-\infty}^{\infty} u^2(x,T)\,\mathrm{d}x + \varepsilon \int_0^{\mathrm{T}}\int_{-\infty}^{\infty} \Big(\frac{\partial u}{\partial x}\Big)^2\,\mathrm{d}x\,\mathrm{d}t = \frac{1}{2}\int_{-\infty}^{\infty} u_0^2(x)\,\mathrm{d}x. \tag{2.2}$$

Now, differentiate equation (1.1) with respect to x and multiply the resulting equation by u_x, then integrating by parts over the strip $\mathcal{R}$ yields

$$\begin{aligned}\frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,T)\,\mathrm{d}x &+ \varepsilon \int_0^{\mathrm{T}}\int_{-\infty}^{\infty} \Big(\frac{\partial^2 u}{\partial x^2}\Big)^2\,\mathrm{d}x\,\mathrm{d}t \\ &= \frac{1}{2}\int_{-\infty}^{\infty} (u_0')^2\,\mathrm{d}x + \int_0^{\mathrm{T}}\int_{-\infty}^{\infty} \Big\langle \frac{\partial f'}{\partial x}, \frac{\partial^2 u}{\partial x^2}\Big\rangle\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_0^{\mathrm{T}} \Big\langle \varepsilon\frac{\partial^2 u}{\partial x^2} - \frac{\partial}{\partial x} f'(u), \frac{\partial u}{\partial x}\Big\rangle\Big|_{-\infty}^{\infty}\,\mathrm{d}t.\end{aligned}$$

However,

$$\varepsilon\frac{\partial^2 u}{\partial x^2}-\frac{\partial}{\partial x}f'(u)=\frac{\partial u}{\partial t}$$

and using the fact that the initial condition vanishes as $|x|\to\infty$, we obtain

$$\int_0^{\mathrm{T}}\Big\langle\varepsilon\frac{\partial^2 u}{\partial x^2}-\frac{\partial}{\partial x}f'(u),\frac{\partial u}{\partial x}\Big\rangle\Big|_{-\infty}^{\infty}\,\mathrm{d}t=\int_0^{\mathrm{T}}\Big\langle\frac{\partial u}{\partial t},\frac{\partial u}{\partial x}\Big\rangle\Big|_{-\infty}^{\infty}\,\mathrm{d}t=0.$$

We also note that

$$\Big\langle\frac{\partial f'(u)}{\partial x},\frac{\partial^2 u}{\partial x^2}\Big\rangle\leqslant\frac{1}{2}\varepsilon\Big(\frac{\partial^2 u}{\partial x^2}\Big)^2+\frac{1}{2\varepsilon}\Big(\frac{\partial f'(u)}{\partial x}\Big)^2.$$

However,

$$\Big(\frac{\partial f'(u)}{\partial x}\Big)^2=\Big|f''(u)\frac{\partial u}{\partial x}\Big|^2\leqslant|f''(u)|^2\Big(\frac{\partial u}{\partial x}\Big)^2$$

and so

$$\Big\langle\frac{\partial f'(u)}{\partial x},\frac{\partial^2 u}{\partial x^2}\Big\rangle\leqslant\frac{1}{2}\varepsilon\Big(\frac{\partial^2 u}{\partial x^2}\Big)^2+\frac{1}{2\varepsilon}|f''(u)|^2\Big(\frac{\partial u}{\partial x}\Big)^2. \tag{2.3}$$

For an arbitrary $T_1>0$ such that $T\leqslant T_1$, let $M=\max t\leqslant T_1|u(x,t)|$. Using the above inequality together with equation (2.2), we get

$$\begin{aligned}\int_{-\infty}^{\infty}u_x^2(x,T)\,\mathrm{d}x&+\varepsilon\int_0^{\mathrm{T}}\int_{-\infty}^{\infty}\Big(\frac{\partial^2 u}{\partial x^2}\Big)^2\,\mathrm{d}x\,\mathrm{d}t\\&\leqslant\int_{-\infty}^{\infty}(u_0')^2\,\mathrm{d}x+\frac{1}{2\varepsilon^2}\max_{|u|\leqslant M}|f''(u)|^2\int_{-\infty}^{\infty}u_0^2\,\mathrm{d}x.\end{aligned} \tag{2.4}$$

For $x\in\mathbb{R}$, and if we assume that $u(-\infty,t)=0$, we have

$$\begin{aligned}|u|^2=\langle u,u\rangle&=\int_{-\infty}^{x}\frac{\partial}{\partial x}\langle u,u\rangle\,\mathrm{d}x=2\int_{-\infty}^{x}\Big\langle\frac{\partial u}{\partial x},u\Big\rangle\,\mathrm{d}x\\&\leqslant 2\Big(\int_{-\infty}^{\infty}\Big(\frac{\partial u}{\partial x}\Big)^2\,\mathrm{d}x\Big)^{1/2}\Big(\int_{-\infty}^{\infty}u^2\,\mathrm{d}x\Big)^{1/2}.\end{aligned} \tag{2.5}$$

For $x\in\mathbb{R}$, $T_1\in(0,T)$ and from relations (2.2), (2.4) and (2.5) we have

$$|u(x,T_1)|^2\leqslant\frac{1}{\varepsilon}K_1\max_{|u|\leqslant M}|f''(u)|+K_1 \tag{2.6}$$

for some constant K_1 independent of T_1. Equation (2.6) implies that

$$M^2\leqslant\frac{1}{\varepsilon}K_1\max_{|u|\leqslant M}|f''(u)|+K_1$$

or

$$M^2 \Big/ \Big(\max_{|u| \leqslant M} f''(u) + \varepsilon \Big) \leqslant \frac{1}{\varepsilon} K_1. \tag{2.7}$$

From (2.7) and the assumption in (2.1) it follows that $M \leqslant M_0 < \infty$, where M_0 is independent of T. Thus we have for $x \in \mathbb{R}$ and $t > 0$ that $|u(x,t)| \leqslant M_0$, which shows the existence of a solution. To show that the solution converges to zero uniformly for $x \in \mathbb{R}$ as $t \to \infty$, it is sufficient to show that $\int_{-\infty}^{\infty} u_x^2(x,t)\,\mathrm{d}x$ converges to zero as $t \to \infty$. To do so, since T_1 is arbitrary for $T_1 \leqslant T$, this means

$$\int_0^{\infty} \Big[\int_{-\infty}^{\infty} u_x^2(x,t)\,\mathrm{d}x \Big]\,\mathrm{d}t < \infty. \tag{2.8}$$

Moreover,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big[\int_{-\infty}^{\infty} u_x^2(x,t)\,\mathrm{d}x \Big] &= 2 \int_{-\infty}^{\infty} \Big\langle \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x \partial t} \Big\rangle\,\mathrm{d}x \\ &= 2 \Big\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t} \Big\rangle \Big|_{-\infty}^{\infty} - 2 \int_{-\infty}^{\infty} \Big\langle \frac{\partial^2 u}{\partial x^2}, \frac{\partial u}{\partial t} \Big\rangle\,\mathrm{d}x = -2 \int_{-\infty}^{\infty} \Big\langle \frac{\partial^2 u}{\partial x^2}, \frac{\partial u}{\partial t} \Big\rangle\,\mathrm{d}x, \end{aligned}$$

where we have used the fact that $\partial u / \partial t$ vanishes as $|x| \to \infty$. Using equation (1.1), the above equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[\int_{-\infty}^{\infty} u_x^2(x,t)\,\mathrm{d}x \Big] = -2\varepsilon \int_{-\infty}^{\infty} \Big(\frac{\partial^2 u}{\partial x^2} \Big)^2\,\mathrm{d}x + 2 \int_{-\infty}^{\infty} \Big\langle \frac{\partial^2 u}{\partial x^2}, f''(u) \frac{\partial u}{\partial x} \Big\rangle\,\mathrm{d}x$$

and so using equation (2.3) we have

$$\begin{aligned} &\int_0^{\infty} \Big| \frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{\infty} u_x^2(x,t)\,\mathrm{d}x \Big|\,\mathrm{d}t \\ &\quad < 3\varepsilon \int_0^{\infty}\!\! \int_{-\infty}^{\infty} \Big(\frac{\partial^2 u}{\partial x^2} \Big)^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\varepsilon} \max_{|u| \leqslant M_0} |f''(u)|^2 \int_0^{\infty}\!\! \int_{-\infty}^{\infty} \Big(\frac{\partial u}{\partial x} \Big)^2\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{2.9}$$

Due to equations (2.3) and (2.7) the first term in equation (2.9) is finite, and by equation (2.8) the second term in equation (2.9) is also finite. Therefore, the left-hand side of equation (2.9) is finite, and together with equation (2.8) this shows that

$$\int_{-\infty}^{\infty} u_x^2(x,t)\,\mathrm{d}x \to 0 \quad \text{as } t \to \infty. \tag{2.10}$$

Now going back to equation (2.2) with $T = t$, we have

$$\frac{1}{2} \int_{-\infty}^{\infty} u_0^2(x,t)\,\mathrm{d}x + \varepsilon \int_0^{t}\!\! \int_{-\infty}^{\infty} \Big(\frac{\partial u}{\partial x} \Big)^2\,\mathrm{d}x\,\mathrm{d}t = \frac{1}{2} \int_{-\infty}^{\infty} u_0^2(x)\,\mathrm{d}x.$$

Therefore, by equation (2.10) we have

$$\int_{-\infty}^{\infty} u^2(x,t)\,\mathrm{d}x < \frac{1}{2}\int_{-\infty}^{\infty} u_0^2(x)\,\mathrm{d}x, \quad t \geqslant 0.$$

Substituting this back into equation (2.4) then with the fact $\int_{-\infty}^{\infty} u_x^2(x,t)\,\mathrm{d}x \to 0$ as $t \to \infty$, we conclude that $u(x,t) \to 0$ as $t \to \infty$ uniformly for $x \in \mathbb{R}$. □

3. Sinc function preliminaries

The goal of this section is to recall the notation and definitions of the Sinc function that will be used in this paper. These are discussed in [12], [10], [1]. First we denote the set of all integers, the set of all real numbers, the set of all complex numbers by $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$, respectively. Let f be a function defined on $\mathbb{R}$, and $h > 0$ a step-size. Then the Whittaker cardinal function is defined by $C(f,h,x) = \sum\limits_{k=-\infty}^{\infty} f(kh)S(k,h)(x)$ whenever this series converges, and where

$$S(k,h)(x) = \frac{\sin[\pi(x-kh)/h]}{\pi(x-kh)/h} = \operatorname{sinc}\Big[\frac{\pi(x-kh)/h}{1}\Big]$$

is known as the k-th sinc function. For a positive integer N, define

$$C_N(f,h,x) = \sum_{k=-N}^{N} f(kh)S(k,h)(x). \tag{3.1}$$

Definition 3.1. Let $d > 0$ and let $\mathcal{D}_d$ denote the region $\mathcal{D}_d = \{z = x + \mathrm{i}y\colon |y| < d\}$ in the complex plane $\mathbb{C}$, and φ the conformal map of a simply connected domain $\mathcal{D}$ in the complex plane onto $\mathcal{D}_d$ such that $\varphi(a) = -\infty$, $\varphi(b) = \infty$, where a and b are boundary points of $\mathcal{D}$, i.e., $a, b \in \partial\mathcal{D}$. Let φ denote the inverse map of φ, and let the arc Γ, with endpoints a and b ($a, b \notin \Gamma$), be given by $\Gamma = \varphi(-\infty,\infty)$. For $h > 0$, let the points x_k on Γ be given by $x_k = \varphi(kh)$, $k \in \mathbb{Z}$, and $\varrho(z) = \exp(\varphi(z))$, and let $\mathbf{H}(\mathcal{D})$ denote the family of all functions that are analytic in $\mathcal{D}$, such that $\int_{\partial\mathcal{D}} |f(z)||dz| < \infty$. Corresponding to the number α, let $\mathbf{L}_\alpha(\mathcal{D})$ denote the family of all functions f that are analytic for which there exists a constant C_0 such that

$$|f(z)| \leqslant C_0 \frac{|\varrho(z)|^\alpha}{[1+|\varrho(z)|]^{2\alpha}} \quad \forall\, z \in \mathcal{D}.$$

To approximate $f^{(m)}$ on Γ as indicated by [12], we introduce a nullifier function g. Let g be an analytic function defined on $\mathcal{D}$, and for $k \in \mathbb{Z}$ set

$$S_k(z) = g(z)\operatorname{sinc}\Big[\frac{\varphi(z)-kh}{h}\Big] = g(z)S(k,h)\circ\varphi(z), \quad z \in \mathcal{D}.$$

If x is on the arc Γ, we obtain the following theorem.

Theorem 3.1. *Let $\varphi' f \in \mathbf{H}(\mathcal{D})$, $\sup_{-\pi/h \leqslant t \leqslant \pi/h} |(\mathrm{d}/\mathrm{d}x)^n g(x) \exp(\mathrm{i}t\varphi(x))| \leqslant C_1 h^{-n}$ for $n = 0, 1, 2, \ldots, m$ with C_1 a constant depending only on m, φ, and g. If $f/g \in \mathbf{L}_\alpha(\mathcal{D})$, α is a positive constant, then taking $h = \sqrt{\pi d/(\alpha N)}$ it follows that*

$$\sup_{x\in\Gamma} \left| f^{(n)}(x) - \sum_{j=-N}^{N} \frac{f(x_j)}{g(x_j)} S_j^{(n)}(x) \right| \leqslant C_2 N^{(n+1)/2} \exp\bigl(-\sqrt{\pi d \alpha N}\bigr)$$

for $n = 0, 1, \ldots, m$ with C_2 a constant depending only on m, φ, g, d, α, and f.

The approximation of the m-th derivative of f in Theorem 3.1 is simply the m-th derivative of $C_N(f/g, h, x)/g$ in (3.1). The weight function g is chosen relative to the order of the derivative that is to be approximated. For instance, to approximate the m-th derivative, the choice $g(x) = 1/(\varphi'(x))^m$ is often sufficient. So the approximation of f' by sinc expansion is given by

$$f'(x) \approx \sum_{j=-N}^{N} \frac{f(x_j)}{g(x_j)} S_j'(x). \tag{3.2}$$

The sinc method requires that the derivatives of the sinc functions are evaluated at the nodes. Technical calculations provide the following results that will be useful in formulating the discrete system [12], [10], and these quantities are delineated by

$$\delta_{jk}^{(q)} = h^q \frac{\mathrm{d}^q}{\mathrm{d}x^q} S_j \circ \varphi(x) \Big|_{x=x_k},$$

where

$$\delta_{jk}^{(0)} = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases} \quad \delta_{jk}^{(1)} = \begin{cases} 0, & j = k, \\ \dfrac{(-1)^{k-j}}{(k-j)}, & j \neq k, \end{cases} \quad \text{and} \quad \delta_{jk}^{(2)} = \begin{cases} \dfrac{-\pi^2}{3}, & j = k, \\ \dfrac{-2(-1)^{k-j}}{(k-j)^2}, & j \neq k. \end{cases}$$

So the approximation in (3.2) at the sinc nodes x_k takes the form

$$f'(x_k) \approx \sum_{j=-N}^{N} \left(\frac{\delta_{jk}^{(1)}}{h} + \delta_{jk}^{(0)} g'(x_j) \right) \frac{f(x_j)}{g(x_j)}. \tag{3.3}$$

The system in (3.3) is more conveniently recorded by defining the vector $\vec{f} = (f_{-N}, \ldots, f_0, \ldots, f_N)^{\mathrm{T}}$. Then define the $m \times m$ ($m = 2N+1$) Toeplitz matrices $I_m^{(q)} = [\delta_{jk}^{(q)}]$, $q = 0, 1, 2$, i.e., the matrix whose jk-entry is given by $\delta_{jk}^{(q)}$, $q = 0, 1, 2$. The system in (3.3) takes the form

$$f' \approx \Bigl(\frac{-1}{h} I_m^{(1)} \mathcal{D}(1/g) + I_m^{(0)} \mathcal{D}(g'/g) \Bigr) f \equiv A_1 f. \tag{3.4}$$

For the present paper the interval Γ in Theorem 3.1 is $(-\infty,\infty)$. Therefore, to approximate the first derivative we take $\varphi(x)=x$ and $g(x)=1/\varphi'(x)$. The square matrix (3.4) becomes $A_1=-h^{-1}I_m^{(1)}$. Then the approximation of the first derivative can be written as

$$f'(x_i)\cong A_1 f(x_i). \tag{3.5}$$

In the same way, we can approximate the second derivative by

$$f''(x_i)\cong A_2 f(x_i), \tag{3.6}$$

where the matrix A_2 is defined by $A_2=-h^{-2}I_m^{(2)}$. Let $\delta_{k-j}^{(-1)}=\frac{1}{2}+\int_0^k(\sin(\pi t)/\pi t)\,\mathrm{d}t$. Then define a matrix whose kj-entry is given by $\delta_{k-j}^{(-1)}$ as $I^{(-1)}=[\delta_{k-j}^{(-1)}]$. In the rest of this section, we shall give a general formula for approximating the integral $\int_a^\nu F(u)\,\mathrm{d}u$, $\nu\in\Gamma$. To this end, we state the following result, which we will use to approximate the integral in equation (1.1).

Theorem 3.2. *Let $F/\Upsilon'\in\mathbf{L}_\alpha(\mathcal{D})$ with $0<\alpha\leqslant 1$, let $\delta_{k-j}^{(-1)}$ be defined as above, let N_t be a positive integer, and let h_t be selected as $h_t=\sqrt{\pi d/(\alpha N_t)}$. Then there exists a positive constant C_4 independent of N_t such that*

$$\left|\int_a^{x_k}F(t)\,\mathrm{d}t-h_t\sum_{j=-N_t}^{N_t}\delta_{k-j}^{(-1)}\frac{F(x_k)}{\Upsilon'(x_k)}\right|\leqslant C_4\exp\bigl(-\sqrt{\pi d\alpha N_t}\bigr).$$

4. Implementation of the method

Part of this section has been published in preliminary form in [2]. To determine the sinc approximation for Burgers' equation (1.4), we require that the initial condition $u_0(x)$ belong to the class $\mathbf{L}_\alpha(\mathcal{D})$. To illustrate the situation where the initial condition $u_0(x)$ is not in $\mathbf{L}_\alpha(\mathcal{D})$, consider equation (1.4) with boundary conditions

$$u(-\infty,t)=\gamma(t),\quad u(\infty,t)=\delta(t),\quad t\geqslant 0, \tag{4.1}$$

and initial condition

$$u(x,0)=u_0(x)=\begin{cases}a, & x\geqslant 0\\ b, & x<0,\end{cases} \tag{4.2}$$

where a and b are constants. Also, let the two conditions $\gamma(0) = u_0(-\infty) = b$ and $\delta(0) = u_0(\infty) = a$ be satisfied. Since the sinc functions composed with various conformal maps $S(k, h_x) \circ \varphi$ are zero at the end points of the interval, and since the boundary conditions in (4.1) are non-homogeneous Dirichlet conditions, the transformation

$$\tilde{u}(x,t) = u(x,t) - w(x,t), \tag{4.3}$$

where

$$w(x,t) = \frac{\gamma(t)\exp(-x) + \delta(t)\exp(x)}{\exp(-x) + \exp(x)}, \tag{4.4}$$

will convert the partial differential equation in (1.4) into a problem with homogeneous Dirichlet conditions, and a non-homogeneous smooth initial condition given by

$$\tilde{u}_0(x) = \tilde{u}(x,0) = u(x,0) - w(x,0). \tag{4.5}$$

Now, substitute the transformation (4.3) into equation (1.4) and drop the tilde to get

$$u_t + [u + w]u_x + w_x u - \varepsilon u_{xx} = f(x,t), \tag{4.6}$$

where $f(x,t) = \varepsilon w_{xx} - w w_x$, and w_x, w_t, w_{xx} can be computed exactly from equation (4.4). Now the initial condition in equation (4.5) is in $\mathbf{L}_\alpha(\mathcal{D})$ for $|x|$ large. Integrating equation (4.6) with respect to t, with the initial condition in equation (4.5), we get

$$\begin{aligned} u(x,t) = \int_0^t &[f(x,\tau) - (u(x,\tau) + w(x,\tau))u_x(x,\tau) \\ &- w_x(x,\tau)u(x,\tau) + \varepsilon u_{xx}(x,\tau)]\,\mathrm{d}\tau + u_0(x). \end{aligned} \tag{4.7}$$

To obtain a direct discretization of equation (4.7), since the domain is $\mathbb{R} \times (0,T)$, the relevant maps are defined as follows: In the space direction, choose the map $\varphi(x) = x$ which maps the infinite strip $\mathcal{D}_d = \{\xi = \zeta + \mathrm{i}\eta \colon |\eta| < d\}$ onto $\mathcal{D}_d$. In the time direction, choose the map $\Upsilon(t) = \log(t/T - t)$ which carries the eye-shaped region $\mathbf{D}_\varepsilon = \{t = x + \mathrm{i}y \colon |\arg(t/(T-t))| < d \leqslant \pi/2\}$ onto the infinite strip $\mathcal{D}_d$. The compositions $S(m, h_x) \circ \varphi(x)$, $m = -N_x, \ldots, N_x$, and $S(m, h_t) \circ \Upsilon(t)$, $k = -N_t, \ldots, N_t$ define the basis elements for $(-\infty, \infty)$ and $(0,T)$, respectively, the mesh sizes h_x and h_t represent the mesh sizes in the infinite strip $\mathcal{D}_d$ for the uniform grid $\{ih_x\}$, $-\infty < i < \infty$ $\{jh_t\}$, $-\infty < j < \infty$. The sinc grid points $x_i \in (-\infty, \infty)$

in $\mathcal{D}_d$ and $t_j \in (0,T)$ in $\mathcal{D}_\varepsilon$ are the inverse images of the equispaced grid points; that is $x_i = \varphi^{-1}(ih_x) = jh_x$, and $t_j = \Upsilon^{-1}(jh_t) = T\exp(jh_t)/(1+\exp(jh_t))$. In equation (4.7) let us carry out the sinc approximation of u_x and u_{xx}. To proceed, use equations (3.5), (3.6) and replace u_x and u_{xx} by $u_x(x,t) \approx \frac{-1}{h_x} I^{(1)}_{m_x} u(x_i,t)$ and $u_{xx}(x,t) \approx \frac{-1}{h_x^2} I^{(2)}_{m_x} u(x_i,t)$, where $m_x = 2N_x + 1$ and the skew-symmetric matrices $I^{(1)}_{m_x}$, $I^{(2)}_{m_x}$ are defined as before. In equation (4.7), evaluate $u(x,t)$, $w(x,t)$, and $f(x,t)$ at the x-nodes, getting the Volterra integral equation

$$(4.8) \qquad \vec{u}(t) = \int_0^t [\vec{f}(\tau) - (\vec{u}(\tau) + \vec{w}(\tau))A_1\vec{u}(\tau) - \vec{w}_x(\tau)u(\tau) + \varepsilon A_2\vec{u}(\tau)]\,\mathrm{d}\tau + \vec{u}^0,$$

where the square matrices A_1, A_2 are given by $A_1 \cong -h_x^{-1} I^{(1)}_{m_x}$ and $A_2 \cong -h_x^{-2} \times I^{(2)}_{m_x}$ and $\vec{f}(t) = [f_{-N_x}(t), \ldots, f_{N_x}(t)]^{\mathrm{T}}$, $\vec{w}(t) = [w_{-N_x}(t), \ldots, w_{N_x}(t)]^{\mathrm{T}}$, $\vec{w}_x(t) = [(w_x)_{-N_x}(t), \ldots, (w_x)_{N_x}(t)]^{\mathrm{T}}$, $\vec{u^0}(t) = [u^0(z_{-N_x})(t), \ldots, u^0(z_{N_x})(t)]^{\mathrm{T}}$, where in general $u_i(t) = u(x_i,t)$. Here the superscript "T" denotes the transpose. We next collocate with respect to the t-variable via the use of the indefinite integration formula (see Theorem 3.2 with the conformal map $\Upsilon(t) = \log(t/(T-t))$. Thus, define the matrix B by $B = h_t I^{(-1)}_{m_t} \mathcal{D}(1/\Upsilon')$, with $\mathcal{D}(1/\Upsilon'(t_j)) = \mathrm{diag}[1/\Upsilon'(t_{-N_t}), \ldots, 1/\Upsilon'(t_{N_t})]$ and with the nodes $t_j = \Upsilon^{-1}(jh_t)$ for $j = -N_t, \ldots, N_t$, where $h_t = \sqrt{\pi d/(\alpha N_t)}$, and $I^{(-1)}_{m_t}$ as defined in the previous section, with $m_t = 2N_t + 1$. Define the matrices F, W, W', and U^0 by $F = [f(x_i,t_j)], W = [w(x_i,t_j)], W' = [w_x(x_i,t_j)]$, and $U^0 = [u^0(x_i,0)]$. Then the solution of equation (4.7) in matrix form is given by the rectangular $m_x \times m_t$ matrix $U = [u_{ij}]$:

$$(4.9) \qquad U = (F - (W+U)\circ A_1 U - W' \circ U + \varepsilon A_2 U)B^{\mathrm{T}} + U^0,$$

where the notation $\circ$ denotes the Hadamard matrix multiplication. Also equation (4.9) can be written as $U = G(U) + \vec{F}$, where $\vec{F} = FB^{\mathrm{T}} + U^0$ and $G(U) = -((W+U)\circ A_1 U - W' \circ U + \varepsilon A_2 U)B^{\mathrm{T}}$. Note that in our discretization we are taking the time nodes as rows, and the space nodes as columns, so the matrix $(F - (W+U)\circ A_1 U - W' \circ U + \varepsilon A_2 U)$ forms the vector nodes for the integral in (4.7). In (4.9) the vector U^0 has the same dimensions as the vector U, and every column of U^0 consists of the same vector u^0. Also, W, W', and F are $m_x \times m_t$ rectangular matrices. For the convergence of the method we proved the following two theorems. The details of the proofs can be found in [1].

Theorem 4.1. *Let the function $u(x,t)$ be as in equation* (4.7) *with the initial condition as in equation* (4.5), *and let the matrix U be defined as in* (4.9). *Then for*

N_x, $N_t > 9/\pi d\alpha$ *there exists a constant* C *independent of* N_x, N_t *such that*

$$\sup_{(x_i,t_j)} \|u(x_i,t_j) - U\| \leqslant CN \exp(-\sqrt{\pi d\alpha N}),$$

where $N = \min\{N_x, N_t\}$.

Theorem 4.2. *Given a constant* $R > 0$, *there is a constant* $T > 0$ *such that if* $\|U_1 - U_0\| \leqslant R/2$, *then the equation* $U = G(U) + \widetilde{F}$ *has a unique solution. Moreover, the iteration scheme* $U_{n+1} = G(U_n) + \widetilde{F}$ *converges to this unique solution.*

5. Numerical example

The example reported here is selected to show the convergence of the scheme. Consider the problem

$$u_t + uu_x = 0.05u_{xx}, \quad (x,t) \in \mathbb{R} \times (0,T) \tag{5.1}$$

with boundary conditions

$$u(-\infty,t) = -(1+t), \quad u(\infty,t) = (1+t), \quad t \geqslant 0 \tag{5.2}$$

and initial condition

$$u(x,0) = u_0(x) = \begin{cases} 1, & x \geqslant 0, \\ -1, & x < 0. \end{cases} \tag{5.3}$$

The true solution is given in Cole [4]. Here, the supremum norm error between the numerical approximation u_{ij} using our approach, and the true solution $u(x_i,t_j)$ at the sinc grid-points is determined and reported as $\|u_{ij} - u(x_i,t_j)\|$, see Table 1. Recall that the asymptotic errors for the approximate solution of the given problem for the spatial direction is $\mathcal{O}(\exp(-\sqrt{\pi d\alpha N_x}))$ while along the time direction it is $\mathcal{O}(\exp(-\sqrt{\pi d\alpha N_t}))$. Once N_x is chosen, balancing the asymptotic error with respect to $\exp(-\sqrt{\pi d\alpha N_x})$ determines the step-sizes $h_x = \sqrt{\pi d/(\alpha N_x)}$, $h_t = \sqrt{\pi d/(\alpha N_t)}$, where the parameters are taken to be $\alpha = 1$, $d = \pi/2$, and $T = 3$.

$N_x = N_t$	$\|u_{ij} - u(x_i,t_j)\|$
4	8.5450×10^{-3}
8	5.7183×10^{-3}
16	1.3617×10^{-3}
32	8.8632×10^{-5}
64	5.3823×10^{-6}

Table 1. Results for Example in Section 5.

Conclusions

The Sinc-Galerkin method appears to be very promising for solving Burgers' equation. For the assumptions considered, the resulting nonlinear system of algebraic equations was solved efficiently by fixed-point iteration. The example presented demonstrates the accuracy of the method, which is an improvement over current methods such as finite elements and finite difference methods. This feature shows the method to be attractive for numerical solutions to Burgers' equation.

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