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Lower bounds for simultaneous Diophantine approximation constants[1]

Werner Georg Nowak

Abstract. After a brief exposition of the state-of-art of research on the (Euclidean) simultaneous Diophantine approximation constants θ_s, new lower bounds are deduced for θ_6 and θ_7.

1 Introduction

For a fixed positive integer s, the (Euclidean) *simultaneous Diophantine approximation constant* θ_s is defined as the supremum of all constants c such that, for every point $\mathbf{a}$ in $\mathbb{R}^s \setminus \mathbb{Q}^s$, there exist infinitely many $(s+1)$-tuples $(\mathbf{p}, q) \in \mathbb{Z}^s \times \mathbb{N}^*$ with

$$\left| \mathbf{a} - \frac{1}{q}\mathbf{p} \right| \le \frac{1}{q \sqrt[s]{cq}}\,, \tag{1}$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^s$.

This notion generalizes a question whose answer is known as Hurwitz' classic theorem. This involves the special case $s = 1$ and tells us that $\theta_1 = \sqrt{5}$; see, e.g., Niven and Zuckerman [11, p. 189 and p. 221].

By some very deep analysis, Davenport and Mahler [6] were able to prove that $\theta_2 = \frac{1}{2}\sqrt{23}$.

For $s \ge 3$, the exact values of θ_s are unknown, and only more or less precise bounds have been established.

We remark parenthetically that the problem becomes even considerably more difficult if one replaces in (1) the Euclidean norm by the maximum norm: The constants arising, say $\theta_s^{(\infty)}$, are unknown for all $s \ge 2$, the only general successful approach being due to Spohn [16] who combined the calculus of variation with a classic method of Blichfeldt [2] to estimate $\theta_s^{(\infty)}$ from below.

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2 Survey of methods and known results

The usual approach to estimate θ_s is based on tools from the *geometry of numbers*; cf. throughout the monograph by Gruber and Lekkerkerker [7].

We briefly recall a few basic concepts of this theory: For a star-body $\mathcal{B}$ in $\mathbb{R}^s$, a lattice $\Lambda = A\mathbb{Z}^s$ (A a nonsingular real $(s \times s)$-matrix) is called *admissible* if its only point in the interior of $\mathcal{B}$ is the origin. The *critical determinant* $\Delta(\mathcal{B})$ of $\mathcal{B}$ is then defined as the infimum of the *lattice constants* $d(\Lambda) = |\det A|$, taken over all lattices Λ admissible for $\mathcal{B}$.

By a celebrated result of Davenport [5] (see also [7, p. 480, Theorem 4]), θ_s is equal to the critical determinant $\Delta(\mathcal{B}_{s+1})$ of the $(s+1)$-dimensional star body of points $(x_0, x_1, \dots, x_s) \in \mathbb{R}^{s+1}$

$$\mathcal{B}_{s+1}: \quad |x_0| \left(\sum_{j=1}^{s} x_j^2 \right)^{s/2} \le 1 \,. \tag{2}$$

For $s = 3, 4, 5$, the sharpest known lower[2] estimates for θ_s have been obtained by a method which is based on inequalities relating the critical determinants of star bodies in different dimensions. In its essence it goes back to Mordell [8], [9], [10], and Armitage [1]. Combining these tools with the known critical determinant $\Delta(\mathcal{P}) = \frac{1}{2}$ of the three-dimensional double paraboloid

$$\mathcal{P}: \quad x^2 + y^2 + |z| \le 1 \,, \tag{3}$$

it has been proved [13], [14] that

$$\theta_3 \ge 1.879\dots, \quad \theta_4 \ge 1.3225\dots, \quad \theta_5 \ge 0.876\,. \tag{4}$$

For $s \le 5$, it appears that this is the very limit of present methods. For $s \ge 6$, the only successful approach is due to Prasad [15]. This is based on the simple idea to apply the arithmetic-geometric mean inequality to the left-hand side of (2). In terms of geometry, this amounts to inscribing an $(s+1)$-dimensional ellipsoid

$$\mathcal{E}_{s+1}: \quad \frac{1}{s+1} x_0^2 + \frac{s}{s+1} \sum_{j=1}^{s} x_j^2 \le 1 \tag{5}$$

into $\mathcal{B}_{s+1}$. It follows that

$$\theta_s = \Delta(\mathcal{B}_{s+1}) \ge \Delta(\mathcal{E}_{s+1}) = \frac{(s+1)^{(s+1)/2}}{s^{s/2}} \Delta(\mathcal{S}_{s+1}) \tag{6}$$

where $\mathcal{S}_{s+1}$ is the unit sphere in $\mathbb{R}^{s+1}$. Now the critical determinants of the unit spheres are known up to dimension 8: See [7, p. 410]; in particular, $\Delta(\mathcal{S}_7) = \frac{1}{8}$, $\Delta(\mathcal{S}_8) = \frac{1}{16}$. Hence, using (6), it readily follows that

$$\theta_6 \ge \frac{343}{1728} \sqrt{7}\,, \quad \theta_7 \ge \frac{256}{343} \frac{1}{\sqrt{7}}\,. \tag{7}$$

[2] Obviously, *lower* bounds are the more interesting ones, since they guarantee, for every c less than the bound, the existence of infinitely many solutions of the inequality (1).

We conclude this section by the remark that the question of *upper* bounds for θ_s has been dealt with in [14, section 4]. It amounts to finding certain number fields of degree $s+1$ with small absolute discriminant.

3 Improvement of the estimate (7)

The critical determinants $\Delta(\mathcal{S}_7)$, $\Delta(\mathcal{S}_8)$ once known, the deduction of the lower bounds (7) seems so natural that one might believe that this could be the end-of--the-art for this problem in the cases $s=6,7$. In this little note, however, we will establish a slight refinement.

Theorem 1. *The inequalities*

$$\theta_6 \geq \frac{343}{1728}\sqrt{7}\,(1+\omega_6)\,, \qquad \theta_7 \geq \frac{256}{343}\frac{1}{\sqrt{7}}\,(1+\omega_7)$$

hold true, with certain small constants $\omega_6 > 9\times 10^{-4}$, $\omega_7 > 3\times 10^{-4}$. *I.e., numerically,* $\theta_6 \geq 0.52564$, $\theta_7 \geq 0.28218$.

Of course, this improvement is fairly small, the main interest lying in the method applied. This in turn is inspired by classic work due to Davenport [3], [4], and Žilinskas [17], as well as by an earlier article by the author [12][3].

4 Proof of the theorem

For better readability, we give the details only for $s=6$, the case $s=7$ being completely analogous. In principle, the argument can be extended to $s>7$ as well, but this is of less importance, since $\Delta(\mathcal{S}_{s+1})$ is known for $s\leq 7$ only.

The star body $\mathcal{B}_7$ defined in (2) is *automorphic*, hence there exists a *critical lattice*[4] Λ with a point on the boundary of $\mathcal{B}_7$; cf. [7, p. 305, Theorem 4]. Applying to Λ a suitable automorphism of $\mathcal{B}_7$, if necessary, we can assume this point to be $\mathbf{e}=(1,1,0,0,0,0,0)$. With $\mathbf{x}\in\mathbb{R}^7$, the function

$$G(\mathbf{x}) := \left(\frac{1}{7}x_0^2 + \frac{6}{7}\sum_{j=1}^{6} x_j^2\right)^{1/2},$$

which is the square-root of the left-hand side of (5) for $s=6$, is called the *distance function* of the ellipsoid $\mathcal{E}_7$; it is homogeneous of order 1. Since, according to [7, p. 195, Theorem 3], any $\mathbf{o}$-symmetric ellipsoid has anomaly 1, there exist seven linearly independent lattice points $\mathbf{u}^{(k)}$ of Λ, with $\big(G(\mathbf{u}^{(k)})\big)_{k=1}^{7}$ nondecreasing, and

$$\Delta(\mathcal{E}_7)\prod_{k=1}^{7} G(\mathbf{u}^{(k)}) \leq d(\Lambda) = \Delta(\mathcal{B}_7) = \theta_6\,. \tag{8}$$

[3] Carrying out the numerical details on the basis of the argument developed in that latter paper, one would get only $\omega_6 > 6\times 10^{-5}$, $\omega_7 > 1.5\times 10^{-5}$.

[4] I.e., Λ is admissible for $\mathcal{B}_7$, and $d(\Lambda)=\Delta(\mathcal{B}_7)$.

We pick $\mathbf{u} = (u_0, u_1, \dots, u_6) \in \{\mathbf{u}^{(1)}, \mathbf{u}^{(2)}\}$ in such a way that $\mathbf{u} \neq \pm\mathbf{e}$. Then $\mathbf{u} \pm \mathbf{e}$ are nontrivial lattice points of Λ. Since Λ is admissible for $\mathcal{B}_7$, a look back to (2) shows that

$$|u_0| \left(\sum_{j=1}^{6} u_j^2 \right)^3 \geq 1 \,, \qquad |u_0 \pm 1| \left((u_1 \pm 1)^2 + \sum_{j=2}^{6} u_j^2 \right)^3 \geq 1 \,. \tag{9}$$

Since $\mathcal{E}_7 \subset \mathcal{B}_7$, it follows that $G(\mathbf{u}^{(1)}) \geq 1$, hence (8) implies that

$$\frac{\theta_6}{\Delta(\mathcal{E}_7)} \geq (G(\mathbf{u}))^6 \,. \tag{10}$$

To prove the Theorem, it remains to minimize $G(\mathbf{u})$ under the constraints (9). We put $S = \sum_{j=2}^{6} u_j^2$ for short, and may assume, w.l.o.g., that $u_0 > 0$. Hence we have to deal with a minimization problem in three variables only, namely u_0, u_1 and S. In fact,

$$M := \min_{(9)} G^2(\mathbf{u}) = \min_{(12)} \left(\frac{1}{7} u_0^2 + \frac{6}{7} \left(u_1^2 + S \right) \right) , \tag{11}$$

with

$$u_0 \left(u_1^2 + S \right)^3 \geq 1 \,, \qquad |u_0 \pm 1| \left((u_1 \pm 1)^2 + S \right)^3 \geq 1 \,. \tag{12}$$

Solving (12), we infer that

$$u_1^2 + S \geq \max\left(u_0^{-1/3}, (u_0 + 1)^{-1/3} - 1 - 2u_1, |u_0 - 1|^{-1/3} - 1 + 2u_1 \right) .$$

Hence, in view of (11),

$$7M = \min_{u_0 > 0,\, u_1} \Bigg(\max\Big(u_0^2 + 6u_0^{-1/3},\ u_0^2 + 6(u_0 + 1)^{-1/3} - 6 - 12u_1, \\ u_0^2 + 6\,|u_0 - 1|^{-1/3} - 6 + 12u_1 \Big) \Bigg) .$$

Keeping u_0 fixed for the moment and seeking the minimum with respect to u_1, we observe that the maximum of the last two expressions becomes minimal when they are equal. This obviously happens for

$$u_1 = \frac{1}{4} \left((u_0 + 1)^{-1/3} - |u_0 - 1|^{-1/3} \right) .$$

Consequently,

$$7M = \min_{u_0 > 0} \left(\max\left(u_0^2 + 6u_0^{-1/3},\ u_0^2 + 3(u_0 + 1)^{-1/3} + 3\,|u_0 - 1|^{-1/3} - 6 \right) \right) . \tag{13}$$

In order to solve this ultimate minimization problem, the figure below is very helpful.

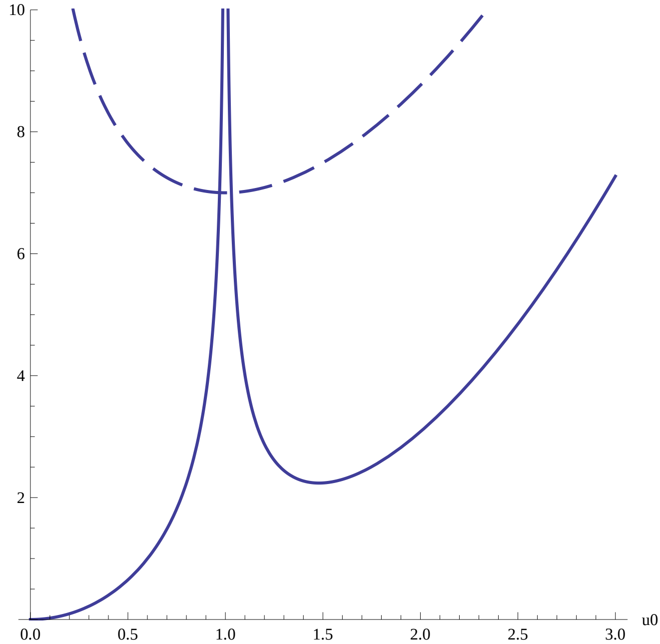

Figure 1: The graphs of $f_1 \colon u_0 \mapsto u_0^2 + 6u_0^{-1/3}$ (dashed) and $f_2 \colon u_0 \mapsto u_0^2 + 3(u_0+1)^{-1/3} + 3\,|u_0-1|^{-1/3} - 6$

In fact, let f_1, f_2 be defined as in the graphics, then an easy calculus exercise shows that f_1 decreases on $]0,1[$ and increases on $]1,\infty[$, and that f_2 increases on $]0,1[$. Furthermore, $f_1 - f_2$ increases on $]1,\infty[$, since there

$$\frac{\mathrm{d}}{\mathrm{d}u_0}\Big(f_1(u_0) - f_2(u_0)\Big) = \frac{1}{(u_0+1)^{4/3}} + \frac{1}{(u_0-1)^{4/3}} - \frac{2}{u_0^{4/3}} > 0\,,$$

on applying the mean inequality to the first two fractions. It readily follows that the equation $f_1(u_0) = f_2(u_0)$ has exactly two solutions, $u_0^{(1)} < 1$ and $u_0^{(2)} > 1$, say, and that, recalling (13),

$$7M = \min\Big(f_1(u_0^{(1)}), f_1(u_0^{(2)})\Big)\,.$$

Carrying out the numerics, we get $u_0^{(1)} = 0.97012\ldots$, $u_0^{(2)} = 1.030799\ldots$, hence,

$$7M = \min(7.002111\ldots, 7.00218\ldots) \geq 7.002111\,.$$

Going back to (10) and (11), we finally infer that

$$\frac{\theta_6}{\Delta(\mathcal{E}_7)} \geq \left(\frac{7.002111}{7}\right)^3 \geq 1.0009\,,$$

which completes the proof of the Theorem. $\square$

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